THE GREEN CHICKEN - NOVEMBER 12, 2016

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Each question is worth 10 points. Turn this page over for Question 7.

Question 1: Greenie the Chicken is hoping to deposit \$2016 in her savings account. Unfortunately, chicken currency only comes in two denominations: \$1701 and \$1871. Is there a way for her to pay the bank \$2016, perhaps with the bank giving her some change (in those same denominations)? If so, how could she do it? If not, show that it's impossible.

Question 2: An integer $p \geq 2$ is called prime if the only positive integers dividing p are 1 and p. Find all prime numbers p of the form $p = N^2 - 2N - 8$, where N is an integer. Prove that you have found all of them.

Question 3: While attending a concert, Greenie was instructed by the performer to "call her, maybe." Greenie can't remember the performer's phone number exactly, but she remembers noticing that it had no repeated digits, was divisible by 3 but not by 6 or by 9, and that it was the largest possible integer satisfying all those properties. What was the phone number?

Question 4: (a) If we expand and collect like terms in $(x+y)^2$, there are three terms (since $(x+y)^2 = x^2 + 2xy + y^2$). After you expand $(w+x+y+z)^{38}$ and collect like terms, how many terms are there?

(b) Greenie picks (uniformly at random) one of the terms you counted up in part (a), deletes the integer coefficient, and plugs in $w = 2$, $x = 3$, $y = 5$, and $z = 7$. What is the probability that the resulting number is divisible by 2016?

Question 5: In order to prove a proposition on Diophantine equations, Greenie was planning to cite Fermat's Last Theorem, which says that there are no integer solutions to $x^n + y^n = z^n$ where $n \geq 3$ and $xyz \neq 0$. However, she noticed a disastrous flaw in the proof of this theorem! Fortunately, she only needed to use a weaker version of Fermat's Last Theorem. Please prove it, so that Greenie can complete her paper:

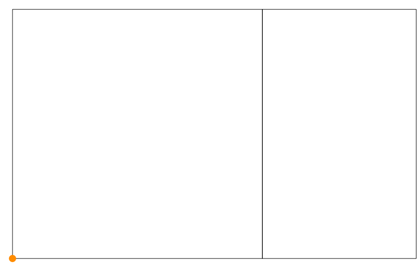
There are no positive integer solutions x, y, z, n to $x^n + y^n = z^n$ where $n \geq z$ and $xyz \neq 0$.

Question 6: Consider differentiable real-valued functions $f(x)$ such that $f'(x) = \sqrt{f(x)}$ for all x, and $f(0) = 0$.

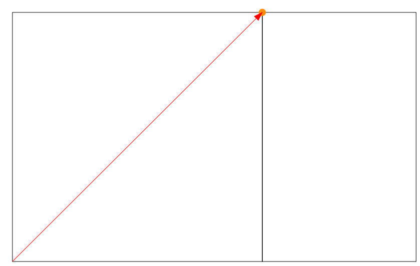
It turns out that there is more than one function satisfying these conditions. Find **THREE** different functions that satisfy these conditions.

Question 7: Greenie's friend Beigey the Turkey lives in a turkey pen that's shaped like a golden rectangle. This means the ratio of the longer side to the shorter side is the golden ratio, $\varphi = \frac{1+\sqrt{5}}{2}$.

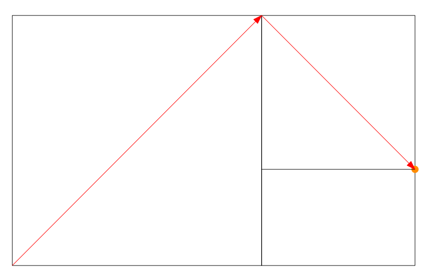
Since Thanksgiving is coming up, Beigey has been plotting how to run away from any farmers that might be chasing him! He starts in one corner of the rectangle, and thinks about splitting the rectangle into a square and a smaller golden rectangle, like so:



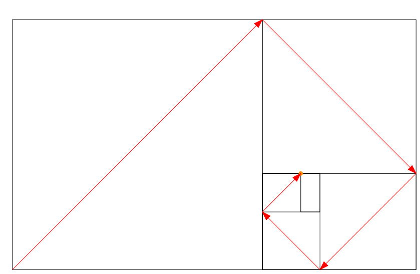
He runs to the opposite corner of the square:



Now he's at the corner of the smaller golden rectangle, and he repeats the process again (splitting it into a square and a smaller golden rectangle, and running to the opposite corner of the square):



He keeps following this process, again and again and again (infinitely many times):



You'd like to be waiting for Beigey at his final location on this spiral to help him escape. Where in the turkey pen should you wait for him?