Green Chicken Contest Solutions - 2013

1. MATHew noticed that his age, that of his three children, and that of his mother MATHilda are all divisors of 2013. What is the sum of their ages? (All the children have different ages.)

Solution: $2013 = 3 \cdot 11 \cdot 61$. So Mathew is 33, his children are 1, 3, and 11, and his mother is 61. The sum of their ages is $61 + 33 + 11 + 3 + 1 = 109$.

2. What are the two times between noon (12:00 P.M.) and 1:00 P.M. that the hour and minute hands of a clock are perpendicular to each other?

Solution: The hour and minute hands line up at noon. While the minute hand moves m minutes on the clock, the hour hand moves m/12 minutes. They will be perpendicular when $(m/12) + 15 = m$. So $m = (15)(12)/11 = 180/11$ minutes after noon. (This is 16 and 4/11 minutes after noon or at 12:16:21 and 9/11 secs.) Since the hands lined up at noon, they must have been perpendicular at 180/11 minutes before noon as well. So the time elapsed from when they are perpendicular to the next must be $2(180)/11 = 360/11$ minutes. So they are perpendicular again at $360/11 + 180/11 = 540/11$ minutes after noon (or 49 and 1/11 minute after noon, i.e. at 12:49:05 and 5/11 secs.)
Alternatively, the hands are perpendicular once again when $m + 15 = 60 + m/12$ which again leads to $m = 540/11$.

3. Let **N** denote the positive integers. A function f:**N** → **N** satisfies
(i) $f(ab) = f(a)f(b)$ whenever $\gcd(a, b) = 1$, and
(ii) $f(p + q) = f(p) + f(q)$ whenever p and q are primes.
Find f(33).

Solution: Note that f is never zero. $f(1)f(2) = f(2) \Rightarrow f(1) = 1$. Now $f(6) = f(2 \cdot 3) = f(2)f(3)$ and $f(6) = f(3+3) = f(3) + f(3) = 2 \cdot f(3)$. So $f(2) = 2$. Also $f(4) = f(2+2) = 2 \cdot f(2) = 4$. Next, $f(5) = f(2) + f(3) = 2 + f(3)$. Also, $f(7) = f(2) + f(5) = 2 + f(5)$ and $f(7) = f(4) + f(3) = 4 + f(3)$. But $f(12) = f(3) \cdot f(4) = 4 \cdot f(3)$ and $f(12) = f(5) + f(7) = (2 + f(3)) + (4 + f(3)) = 6 + 2 \cdot f(3)$. Thus, $6 + 2 \cdot f(3) = 4 \cdot f(3) \Rightarrow f(3) = 3$. Hence, $f(5) = 2 + f(3) = 5$ and $f(6) = 2 \cdot f(3) = 6$. Now $f(11) = f(5) + f(6) = 5 + 6 = 11$. Finally, $f(33) = f(3) \cdot f(11) = 3 \cdot 11 = 33$.

4. Consider a 100 x 100 checkerboard consisting of 10,000 unit squares.
(a) Show that if the middle 2x2 squares are removed, then the remaining board can be tiled with a sufficient number of 3x1 sized-tiles.
(b) Show that if, instead, a 2x2 square is removed from the lower left corner, then the remaining board cannot be tiled in this way.

Solution: (a) The corners of the checkerboard correspond to points of the integer lattice $\{(m, n): 0 \le m, n \le 100.\}$ Number the squares of the board by the coordinates of their top right corner. We have removed the four squares (50. 50), (50, 51), (51, 50), and (51,51). The remaining board can be partitioned by the lines $x = 48$, $x = 52$, $y = 48$, and $y = 52$ into four corner blocks measuring 48 x 48, four side blocks measuring 4 x 48, and a 4 x 4 annulus with its center four squares removed. Since $3|48$, there are many ways to tile the eight solid pieces. Finally, four contiguous 3 x 1 tiles laid perpendicular to one another fills in the final piece.

(b) Tiles can be placed horizontally or vertically. Horizontal tiles cover tiles (m-1, n), (m, n), and (m+1, n) for appropriate m, n. The sum of both their x-coordinates and y-coordinates are divisible by 3. Similarly, vertical tiles cover tiles (m, n-1), (m, n), and (m, n+1). Again the sum of their x-coordinates and y-coordinates are divisible by 3. So the sum of the x-coordinates (or y-coordinates) of a set of tiles will always be divisible by 3. The sum of the x-coordinates of all the squares with no tiles removed is $100(1 + 2 + ... + 100) = 50(100)(101) = 505{,}000$ which is not divisible by 3. When we remove the bottom four squares, we subtract $1 + 1 + 2 + 2$ from this amount (which is divisible by 3). So the sum of the x-coordinates (504,994) is not divisible by 3. Hence it cannot be tiled with 3x1 tiles.

5. Define the Fibonacci sequence by $F_1 = 1$, $F_2 = 1$, and $F_{n+2} = F_n + F_{n+1}$ for $n \ge 1$. Show that there is a Fibonacci number that is divisible by 1000.

Solution: In fact, we can show that for any $n \in \mathbf{N}$, there are infinitely many Fibonacci numbers divisible by n. First, we extend the Fibonacci sequence to include $F_0 = 0$. Consider the set of ordered pairs $\{(F_1, F_2), (F_2, F_3), (F_3, F_4), ...\}$ (mod n). There are only n^2 possible distinct ordered pairs (mod n). Hence there exists an $i \ge 1$ and $m \ge 1$ such that $F_i \equiv F_{i+m}$ (mod n) and $F_{i+1} \equiv F_{i+m+1}$ (mod n). Then $F_{i-1} \equiv F_{i+1} - F_i \equiv F_{i+m+1} - F_{i+m} \equiv F_{i+m-1}$ (mod n). Similarly, $F_{i+2} \equiv F_{i+1} + F_i \equiv F_{i+m+1} + F_{i+m} \equiv F_{i+m+2}$ (mod n). By repetition of this process, we get $F_j \equiv F_{j+m}$ (mod n) for all $j \ge 1$. But $F_0 = F_2 - F_1 = 0 \equiv F_{m+2} - F_{m+1} \equiv F_m \equiv F_{2m} \equiv F_{3m} \equiv ...$ (mod n). So $n|F_{km}$ for all $k \ge 1$. In our problem, let $k = 1$ and $n = 1000$.

6. Define a positive integer to be <u>balanced</u> if the number of its decimal digits equals the number of its distinct prime divisors. (For example, 12, 21, 105 are all balanced, but 25 and 210 are not.) Show that there are only finitely many balanced numbers.

Solution: For $n \ge 16$, consider $P_n = p_1p_2 \cdots p_n$ where $p_1, p_2, ..., p_n$ are the first n primes. Then $P_{16} = (2 \cdot 53) \cdot (3 \cdot 47) \cdot (5 \cdot 43) \cdot (7 \cdot 41) \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 > 100^4 \cdot 10^8 = 10^{16}$. Since $p_k > 10$ for all $k > 4$, P_n has more than n digits for all $n \ge 16$. Now if x has n digits and is balanced with $n \ge 16$, then $x \ge P_n \ge 10^n$, a contradiction since 10^n has n+1 digits. So x has at most 15 digits; thus there are only finitely many balanced numbers.