Trajectories of Homothety Surface

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Jasmine Osorio
Katherine Phillips
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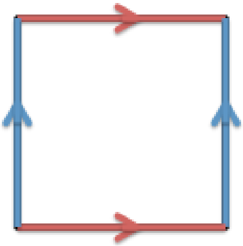
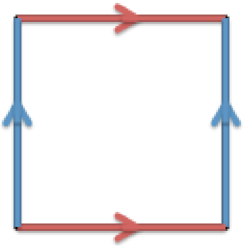
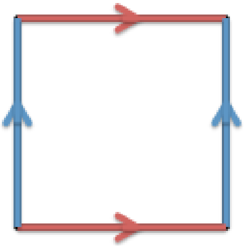
Smith College

6 April 2013





Gluing opposite sides of a square forms a surface called a torus:



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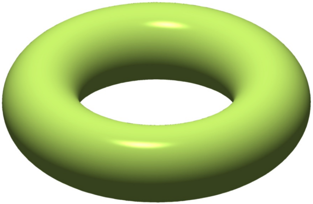

[1]

Definition (Trajectory)

A **trajectory** with a slope m is a path $\gamma(t)$ traced by a point moving constantly on a surface in the direction of m.

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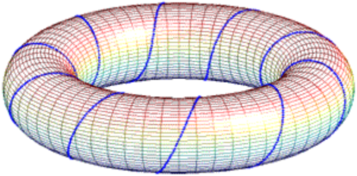
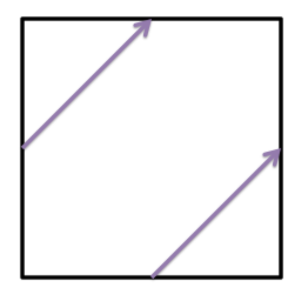


[2]

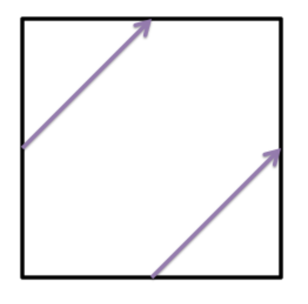
The blue line is an example of a closed trajectory on the torus.

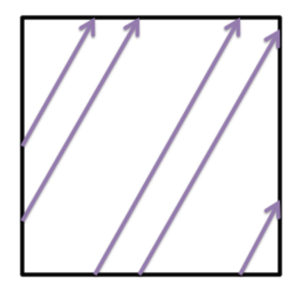
Example



Rational Slope

Example

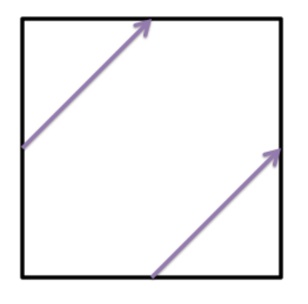


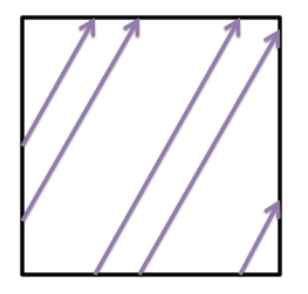


Rational Slope

Irrational slope

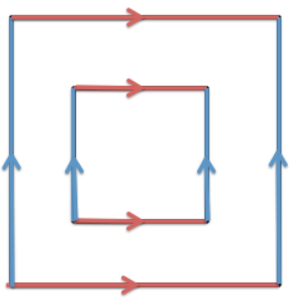
Example



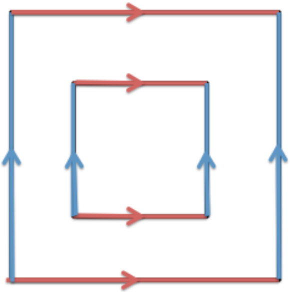


Rational Slope Irrational slope

Observation: Trajectories determined by rational slopes are periodic, while trajectories determined by irrational slopes are not.



Gluing the corresponding sides of an annulus also form a torus:



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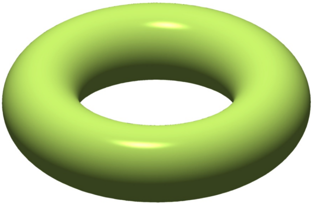
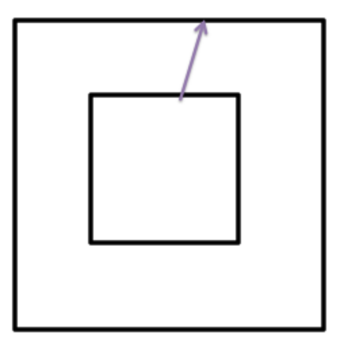
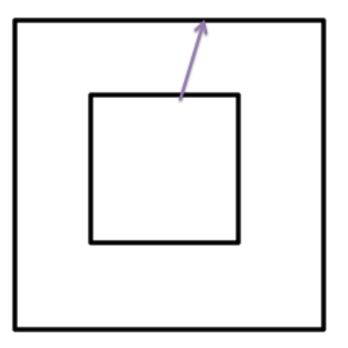


Example

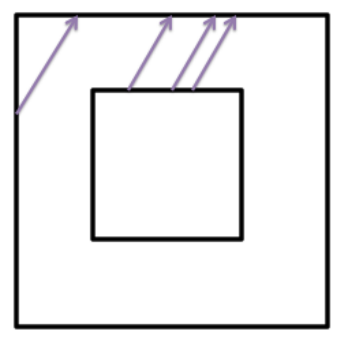


Radial trajectory

Example

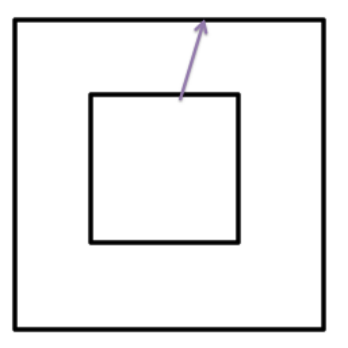


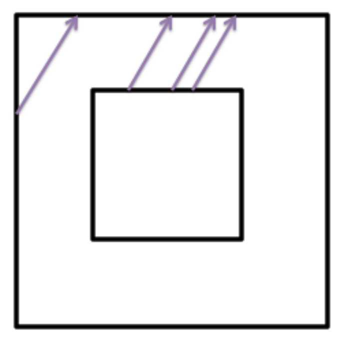
Radial trajectory



Nonradial trajectory

Example





Radial trajectory　　Nonradial trajectory

Observation: Radial trajectories are periodic, while nonradial trajectories are asymptotic to radial trajectories.

Now we will consider a surface formed by making slits on the annular torus and the square torus, and gluing along these slits.

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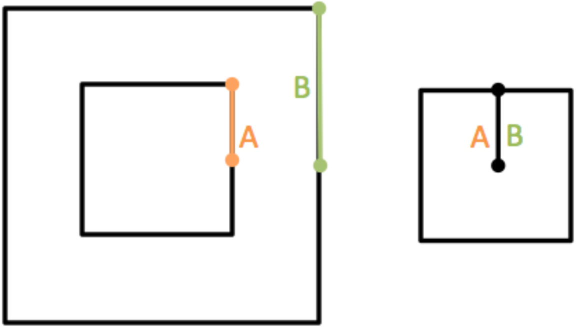

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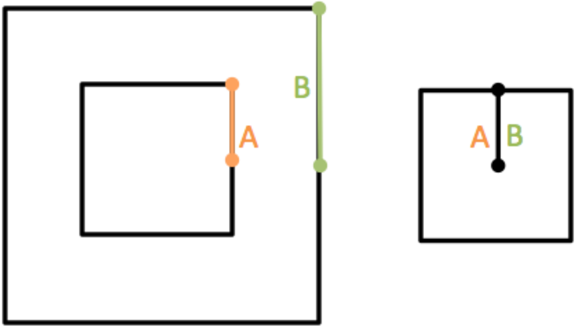

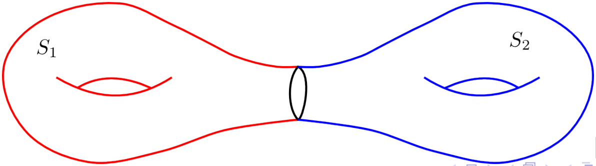

[3]

Definition (Nomadic)

A trajectory is called **nomadic** if it crosses back and forth between the annular torus and square torus indefinitely.

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More precisely, if we let U be the square torus and V be the annular torus, then:

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A trajectory is called **nomadic** if $\forall M > 0, \exists$ times $t_U, t_V > M$ such that $\gamma(t_U) \in U, \gamma(t_V) \in V$.

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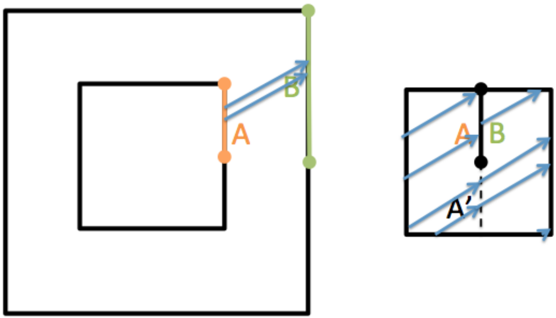
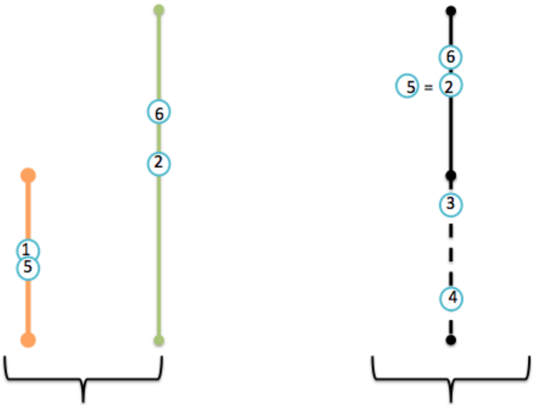
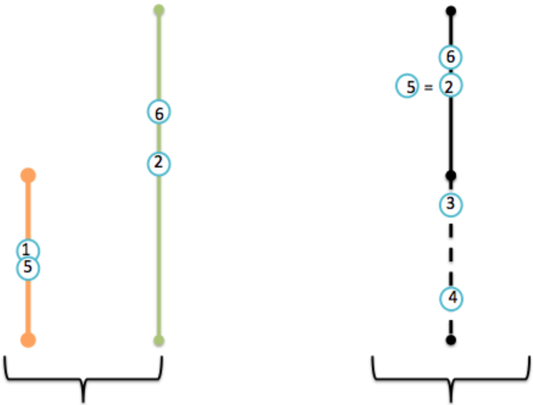


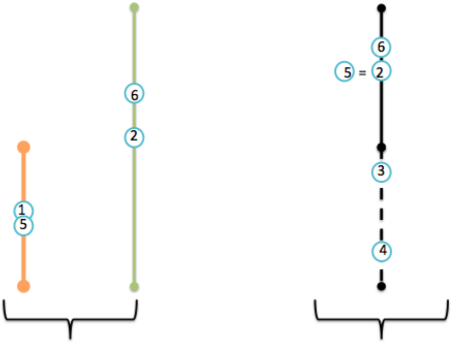
We introduce the segment A' in order to keep track of how many times the trajectory crosses the square torus before returning to the annular torus.


1
5
6
2
6
5 = 2
3
4



An **extended cutting sequence** tracks the order in which the trajectory crosses A, B, and A'.

Example (with slope $m = \frac{1}{3}$): $A, B, A', A', A, B, A', A', A, B, \ldots$.


6
2
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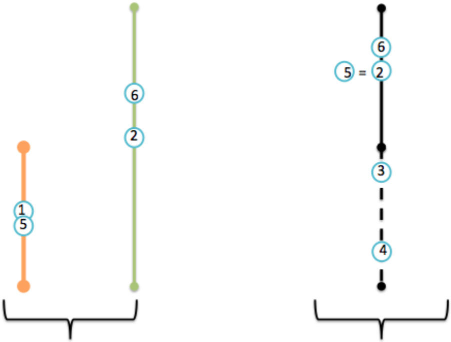
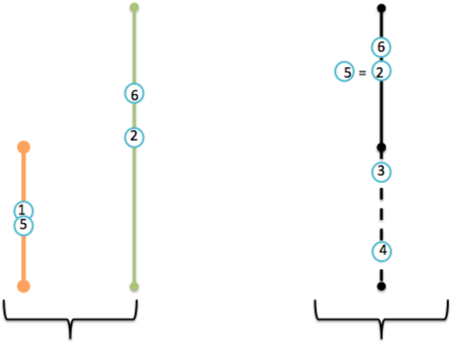


For a more precise characterization of the trajectory, the **trail sequence** gives exact values for where the trajectory crosses A, B, and A', in terms of distance from the top of the slit.



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Example (with slope $m = \frac{1}{3}$): 0.5000, 0.4167, 1.0833, 1.7500, 0.4167, 0.3750, 1.0417, 1.7083, 0.3750, 0.3542, ...

Big Conjecture: All trajectories either are periodic or are asymptotic to a periodic orbit

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- Similar to the behavior we saw on the annular torus
- Description of long term trajectory behavior (of nomadic trajectories)
- Requires description of trajectory behavior on the square torus

Possible Cutting Sequences with a given slope

Conjecture: Given a slope m and a starting point a on A, let n be the number of times a nomadic trajectory crosses the square torus before returning to the annular torus.

For $0 < m < \frac{1}{2}$, find $k \in \mathbb{N}$ such that $\frac{1}{2k} < m < \frac{1}{2k-2}$		For $\frac{1}{2} < m < 1$, find $k \in \mathbb{N}$ such that $\frac{2k-3}{2k-2} < m < \frac{2k-1}{2k}$	
for a in the interval:	$n =$	for a in the interval:	$n =$
$(0, 2-5m)$	1	$(0, 4k-4km-m-2)$	k
$(2-5m, 4-4km-m)$	$k+1$	$(4k-4km-m-2, 4-3m)$	$k+1$
$(4-4km-m, 1)$	k	$(4-3m, 1)$	1

Finally, if $m = \frac{1}{2}$, then $n = 1$ for all $0 \leq a \leq 1$.

This puts restrictions on possible cutting sequences (specifically, the number of consecutive A's that can appear).

Denominators of rational slopes and trail sequence

Slope	Trail sequence
$\frac{2}{7}$	$A = \frac{2}{7}, B = \frac{2}{7}, A = \frac{6}{7}, B = \frac{4}{7}, A' = \frac{8}{7}, A' = \frac{12}{7}, A = \frac{2}{7}$
$\frac{5}{7}$	$A = \frac{5}{7}, B = \frac{5}{7}, A = \frac{1}{7}, B = \frac{3}{7}, A' = \frac{13}{7}, A' = \frac{9}{7}, A = \frac{5}{7}$
$\frac{3}{10}$	$A = \frac{13}{30}, B = \frac{11}{30}, A = \frac{29}{30}, B = \frac{19}{30}, A' = \frac{37}{30}, A' = \frac{11}{5}, A = \frac{13}{30}$
$\frac{7}{10}$	$A = \frac{17}{30}, B = \frac{19}{30}, A = \frac{1}{30}, B = \frac{11}{30}, A' = \frac{53}{30}, A' = \frac{7}{6}, A = \frac{17}{30}$
$\frac{3}{11}$	$A = \frac{5}{33}, B = \frac{7}{33}, A = \frac{25}{33}, B = \frac{17}{33}, A' = \frac{35}{33}, A' = \frac{53}{33}, A = \frac{5}{33}$
$\frac{8}{11}$	$A = \frac{28}{33}, B = \frac{26}{33}, A = \frac{8}{33}, B = \frac{16}{33}, A' = \frac{64}{33}, A' = \frac{46}{33}, A = \frac{28}{33}$

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$\frac{7}{10}$	$A=\frac{17}{30}, B=\frac{19}{30}, A=\frac{1}{30}, B=\frac{11}{30}, A'=\frac{53}{30}, A'=\frac{7}{6}, A=\frac{17}{30}$
$\frac{3}{11}$	$A=\frac{5}{33}, B=\frac{7}{33}, A=\frac{25}{33}, B=\frac{17}{33}, A'=\frac{35}{33}, A'=\frac{53}{33}, A=\frac{5}{33}$
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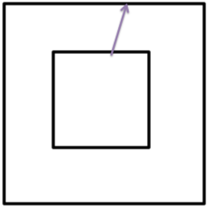
Conjecture: The denominators of the trail sequence will always be a multiple of the denominator of the slope.

All trajectories either are periodic or are asymptotic to a periodic orbit



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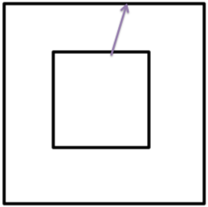

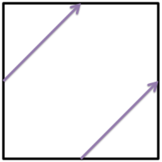

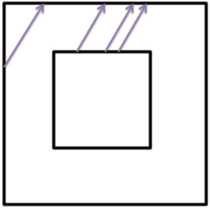

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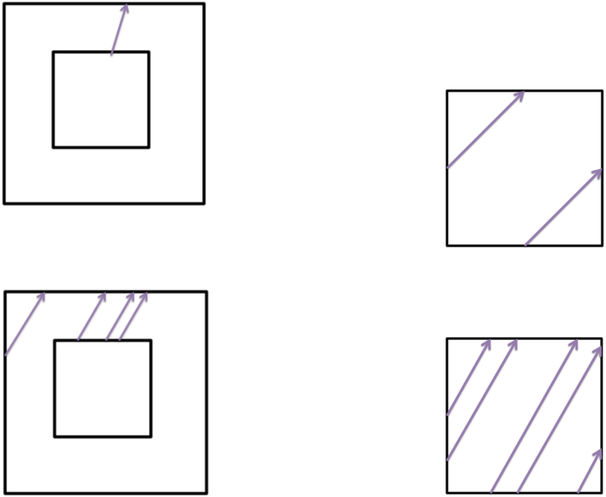

Images Used:

1. Torus: Alexandrov, Oleg. http://commons.wikimedia.org/wiki/File:Torus_illustration.png
2. Trajectory on a Torus: Interdisciplinary Studies of Nonlinear Dynamics. http://www.isnld.com/indexD2.html
3. Genus 2 Surface: Sisto, Alex. http://alexsisto.wordpress.com/2012/02/05/embedding-hyperbolic-spaces-in-products-of-trees/