The Discrete Hodge Star Operator

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Abstract

Exterior Calculus expresses the familiar vector calculus operators gradient, divergence and curl in a coordinate free notation in terms of the exterior derivative, the wedge product and the Hodge star operator. Discrete versions of these operators are important for numerical solutions of partial differential equations. While the Stokes theorem provides a clear guideline for constructing the discrete exterior derivative, the problem of the discrete Hodge star operator is more subtle.

Outline

Motivation: Solve PDEs

- Heat equation: $u_{,t} = \alpha\Delta u$
- Wave: $u_{,tt} = \Delta u$
- Laplace: $\Delta u = 0$ and Poisson : $\Delta u = f$
- Fluid flow:
  - Darcy: $v + \frac{k}{\mu}\nabla p = 0, \quad \nabla \cdot v = 0$
  - Euler: $\frac{\partial v}{\partial t} + v \cdot \nabla v = -\frac{\nabla p}{\rho} + g$
  - Navier-Stokes: $\rho(\frac{\partial v}{\partial t} + v \cdot \nabla v) = -\nabla p + \nabla \cdot T + f$
- Elasticity:
  - Euler-Bernoulli Beam: $\rho A u_{,tt} + [EIu_{xx}]_{xx} = p(x,t)$
  - Normal Kirchhoff Plate: $\rho h u_{,tt} + D\nabla^4 u = q(x,y,t)$
- Electromagnetism - Maxwell:

  $\nabla \cdot \vec{B} = 0, \quad \nabla \cdot \vec{E} = \rho, \quad \nabla \times \vec{E} = -\dot{\vec{B}}, \quad \nabla \times \vec{B} = \vec{j} + \dot{\vec{E}}$
- Quantum mechanics - Schroedinger: $-\frac{\hbar^2}{2m}\Delta\Psi + V\Psi = i\hbar\Psi_t$

Gradient: steepest ascent, heat seeking bug

- If $f(x,y)$ is the height of the hill at x,y, then ∇f is the steepest ascent direction (locally)
- Let $f(t,x,y,z)$ be temperature at time t and position x,y,z
- Gradient ∇f
  - points in the direction of the fastest temperature increase
  - $\|\nabla f\|$ tells how fast is the fastest increase
  - in Cartesian coordinates $\nabla f = \frac{\partial f}{\partial x}\frac{\partial}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial}{\partial z}$

$$\nabla f = \begin{bmatrix} f_{,x} \\ f_{,y} \\ f_{,z} \end{bmatrix}, \quad f_{,x} = \frac{\partial f}{\partial x}, \quad \frac{\partial}{\partial x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \partial_x$$

- Heat seeking bug travels in the ∇f direction
- Differential of f - the best linear approximation to f
  - $df : \mathbb{R}^n \to \mathbb{R}$ is a linear functional
  - $df(v) = \nabla f \cdot v$ - directional derivative, if $\|v\| = 1$
  - $df = f_{,x}dx + f_{,y}dy + f_{,z}dz = [f_{,x}, f_{,y}, f_{,z}], \quad dx = [1,0,0]$
  - $df = [\nabla f]^T$
  - $\{dx, dy, dz\}$ is the dual basis to $\{\partial_x, \partial_y, \partial_z\}$

Duality between ∇f and df

- Two dual points of view: ∇f and df
- More precisely:
  - At a fixed point $p \in M = \mathbb{R}^3$: $\nabla f|_p \in V = T_p M = \mathbb{R}^3$
  - Dual point of view: $df|_p \in V' = T_p^* M = (\mathbb{R}^3)'$
  - As the point $p \in M$ varies: $\nabla f \in TM$, the tangent bundle
  - Dual point of view: $df \in T^* M$, the cotangent bundle
- The vector space V and its dual V' are isomorphic:
  - natural isomorphism between V and V'
  - flat: $\flat : V \to V'$, $\partial_x^\flat = dx$, etc.
  - sharp: $\sharp : V' \to V$, $dx^\flat = \partial_x$, etc.
  - extend by linearity
  - more generally, flat: $\flat : V \to V'$, $\vec{u}^\flat(\vec{v}) = \vec{u} \cdot \vec{v}$
  - sharp: $\sharp : V' \to V$, $\sharp = \flat^{-1}$

∇f and df compete on a manifold M: df wins

- In Cartesian coordinates in Euclidean space, it's a tie
  - ∇f and df equally convenient
  - because the length element is $ds = \sqrt{dx^2 + dy^2}$
- Consider polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$.
  - df **is** coordinate independent $df = f_{,r} dr + f_{,\theta} d\theta$
  - ∇f is **not** coordinate independent $\nabla f \neq f_{,r}\partial_r + f_{,\theta}\partial_\theta$
  - length element is **not** $ds^2 = dr^2 + d\theta^2$
  - $dx = x_{,r} dr + x_{,\theta} d\theta = \cos\theta dr - r\sin\theta d\theta$
  - $dy = y_{,r} dr + y_{,\theta} d\theta = \sin\theta dr + r\cos\theta d\theta$
  - $ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2$
- Introduce metric tensor g to measure distances
  - $ds^2 = \sum_{ij} g_{ij} dx^i dx^j$.
  - Einstein summation convention $ds^2 = g_{ij} dx^i dx^j$.
  - $g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$,

Grad is Dead

- Metric tensor g and its inverse g^{-1}
  $g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}, \quad g^{ij} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{bmatrix},$
- Generalize flat: $\flat : TM \to T^*M$
  from $\vec{u}^\flat(\vec{v}) = \vec{u} \cdot \vec{v}$ to $\vec{u}^\flat(\vec{v}) = g(\vec{u}, \vec{v})$
- In components:
  - Vector $\vec{u} = u^k \partial_k$
  - Metric tensor $g = g_{ij} dx^i dx^j, \quad dx^i(\partial_j) = \delta^i_j$
  - Co-vector or 1-form $\vec{u}^\flat = g(u, \cdot)$
    $\vec{u}^\flat = g_{ij} dx^i(u) dx^j = g_{ij} dx^i(u^k \partial_k) dx^j = g_{ij} u^k \delta^i_k dx^j = g_{ij} u^i dx^j$
  - Standard practice to write $\vec{u}^\flat = u_j dx^j$
  - By comparison $u_j = g_{ij} u^i$
- $\sharp = \flat^{-1}$, so $u^i = g^{ij} u_j$
- $\nabla f = (df)^\sharp$
  - $(\nabla f)^i = g^{ij} f_{,j}$
  - $(\nabla f)^r = g^{rj} f_{,j} = g^{rr} f_{,r} + 0 = f_{,r}$
  - $(\nabla f)^\theta = g^{\theta j} f_{,j} = 0 + g^{\theta\theta} f_{,\theta} = \frac{1}{r} f_{,\theta}$

Div and Curl are Dead

- Grad $\nabla f = (df)^\sharp = (\nabla f)^r \partial_r + (\nabla)^\theta \partial_\theta = f_{,r}\partial_r + \frac{1}{r}\partial_\theta$
- Curl $\nabla \times F = [*(dF^\flat)]^\sharp$
- This defines the exterior derivative d
  - Exterior, because takes 1-form into a 2-form
  - Anti-symmetric, because cross product anti-symmetric
  - Need a symbol to denote 2-forms: $\wedge$ wedge product
  - Extend by linearity and associativity: exterior algebra
- This introduces the Hodge $*$
- Div $\nabla \cdot F = *d(*F^\flat)$
- Laplace $\nabla^2 f = *d(*df)$

My Cup of Tea

- Consider a cup of hot tea, which is cooling down
  - $u(x, y, x, t)$ - temperature at x, y, z at time t
- Heat flows from the tea into the surrounding volume
  - from hot to cold, so in the $-\nabla u$ direction
  - Fourier's Law of thermal conduction: $\vec{q} = -k\nabla u$
  - $\vec{q}$ the heat flux density
  - k thermal conductivity
- Divergence tells how fast heat is flowing out: $\nabla \cdot \vec{q}$
- Energy (Heat) Conservation: tea cools down because heat flows out
  - $(\rho c_p u)_t = -\nabla \cdot \vec{q} = -\nabla \cdot (-k\nabla u)$
  - ρ density
  - c_p thermal capacity at constant pressure
- Heat Equation: $u_t = \alpha \nabla^2 u, \quad \alpha = \frac{k}{\rho c_p}$

Stokes Theorem

- Consider a body B of tea, not necessarily all the tea
- If B is warmer than the surroundings, heat flows through the boundary ∂B from B into the surroundings B^C:
  $\Phi = \oint_{\partial B} \vec{q} \cdot \vec{d}S$
- The rate of change of thermal energy (heat) inside a body B equals the heat flux into B minus the heat flux out of B through the boundary ∂B
- Let B be infinitesimal, with volume dV
- Heat inside B: $E = \rho c_p u dV$

Hodge $*$ Example

- $V = \mathbb{R}^2$:

$$\begin{aligned} *1 &= e_1 \wedge e_2 \\ *e_1 &= e_2 \\ *e_2 &= -e_1 \\ *(e_1 \wedge e_2) &= 1 \end{aligned}$$

- $V = \mathbb{R}^3$:

$$\begin{aligned} *1 &= e_1 \wedge e_2 \wedge e_3 \\ *e_1 &= e_2 \wedge e_3 \\ \dots &= \dots \\ *(e_1 \wedge e_2) &= e_3 \\ \dots &= \dots \\ *(e_1 \wedge e_2 \wedge e_3) &= 1 \end{aligned}$$

Hodge $*$

- V oriented inner product space: $* : \bigwedge^p V \to \bigwedge^{(n-p)} V$
- natural isomorphism
- Let $\{e_1, e_2, \ldots, e_n\}$ be a basis for V
- Let $\sigma = e_1 \wedge e_2 \wedge \cdots \wedge e_n$ be a chosen orientation
- Hodge $*$: $u \wedge v = (*u, v)\sigma$
- Intuition: complementary $n - p$ vector to the given p vector
  - orthogonal
  - consistent with the orientation

$*$ on a Manifold

- $*\alpha^p = \alpha^*_J dx^J$
- $\alpha^*_J = \sqrt{|g|}\alpha^K \epsilon_{KJ}$
- In full: $\alpha^*_{j_1,\ldots,j_{n-p}} = \sqrt{|g|} \sum_{k_1<\cdots<k_p} \alpha^{k_1,\ldots,k_p} \epsilon_{k_1,\ldots,k_p,j_1,\ldots,j_{n-p}}$

Finite Differences

- Finite Differences: time permitting, work on board

Finite Elements

- Finite Elements: time permitting, work on board

Discrete d

- d is uniquely determined by the Stokes theorem: the transpose of the discrete boundary operator

M^{DEC}: DEC Hodge *

- Marsden, Hirani, Desbrun, et al.: $\frac{1}{|*\sigma|}\int_{*\sigma} *\alpha = \frac{1}{|\sigma|}\int_{\sigma} \alpha$
- $[M^{DEC}]_{ij} = \frac{|*\sigma_i^k|}{|\sigma_i^k|}\delta_{ij}$
- Advantage: M^{DEC} diagonal
- Disadvantage: M^{DEC} not positive definite, because circumcenter may be outside the simplex
- Example: standard 2-simplex; vertices: $(0,0)$, $(1,0)$, $(0,1)$

$$M_1^{DEC} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

M^{Whit}: FEEC Hodge *

- Whitney 0-forms are barycentric coordinates:
- Whitney 1-forms: $\eta_k = \lambda_i d\lambda_j - \lambda_j d\lambda_i$
- $[M^{Whit}]_{ij} = \int \eta_i \eta_j dA$
- Advantage: M^{Whit} is positive definite, because barycenter is always inside the simplex
- Disadvantage: M^{Whit} not diagonal, $(M^{Whit})^{-1}$ not sparse
- Example: standard 2-simplex; vertices: $(0,0)$, $(1,0)$, $(0,1)$

$$M_1^{Whit} = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & 0 \\ \frac{1}{6} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix}$$

Summary

- Exterior Calculus is a coordinate independent language and consequently superior in curvilinear coordinates and on curved surfaces.
- Discrete Exterior calculus has the potential to revolutionize the numerical PDE field.
- The discrete exterior derivative d is unique and determined by the discrete Stokes theorem.
- The discrete Hodge * operator is an open question and a topic of current research.

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Thank You!

THANK YOU!!!