Game of Cards and Dynamical Systems

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Some Parameters:

- x : the position of the card
- b : the number of columns of cards
- p : the size of the largest column
- N : the number of cards in the deck

Dynamical Systems

First Trick:

- $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = a + \left\lceil \frac{x}{b} \right\rceil$
  $(a = p)$

Dynamical Systems

First Trick:

- $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = a + \left\lceil \frac{x}{b} \right\rceil$
  $(a = p)$

Second Trick:

- $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = a - \left\lceil \frac{x}{b} \right\rceil$
  $(a = N + 1 - p)$

Consider the dynamical system $(\mathbb{Z}, f)$ with $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = a + \left\lceil \frac{x}{b} \right\rceil$:

- If $a \not\equiv 0 \bmod (b-1)$ then
$$\bar{x} = \left\lceil \frac{ab}{b-1} \right\rceil$$
is the only fixed point, and every trajectory converges monotonically to $\bar{x}$.
- If $a \equiv 0 \bmod (b-1)$ then
$$\bar{x}_1 = \frac{ab}{b-1}, \qquad \bar{x}_2 = \bar{x}_1 + 1$$
are the only fixed points, and every trajectory converges monotonically to the nearest fixed point.

Proof:

Let

$$a = \mu(b-1) + \rho, \qquad 0 \le \rho < b-1$$

and assume that

$$x = mb + r, \qquad 0 \le r < b, \qquad m \in \mathbb{Z},$$

is a fixed point. Then $f(x) = x$, so

$$mb + r = \mu(b-1) + \rho + m + \left\lceil \frac{r}{b} \right\rceil$$

or

$$(m-1)(b-1) = \rho - r + \left\lceil \frac{r}{b} \right\rceil .$$

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From these we get: $\left\lceil \frac{r}{b} \right\rceil = 1$, $r - \left\lceil \frac{r}{b} \right\rceil = \rho$, **and** $m = \mu$.

Case 1: $a \not\equiv 0 \bmod (b-1)$

If $a \not\equiv 0 \bmod (b-1)$, then $\rho > 0$. Then $r \neq 0, 1$ and $r = \rho + 1$, so we get

$$
\begin{aligned}
x &= mb + r \\
&= \mu b + \rho + 1 \\
&= a + \mu + 1 \\
&= a + \frac{a}{b-1} + \frac{b-1-\rho}{b-1} \\
&= a + \left\lceil \frac{a}{b-1} \right\rceil \\
&= \left\lceil \frac{ab}{b-1} \right\rceil = \bar{x}.
\end{aligned}
$$

Case 2: $a \equiv 0 \bmod (b-1)$

If $a \equiv 0 \bmod (b-1)$, then $\rho = 0$. Then $r = 0$ or $r = 1$. With $r = 0$, we get

$$x = mb + r = \mu b = a + \mu = a + \frac{a}{b-1} = \frac{ab}{b-1} = \bar{x}_1.$$

Case 2: $a \equiv 0 \bmod (b-1)$

If $a \equiv 0 \bmod (b-1)$, then $\rho = 0$. Then $r = 0$ or $r = 1$.
With $r = 0$, we get

$$x = mb + r = \mu b = a + \mu = a + \frac{a}{b-1} = \frac{ab}{b-1} = \bar{x}_1.$$

With $r = 1$, we get

$$x = \mu b + 1 = \bar{x}_1 + 1 = \bar{x}_2.$$

Monotonic Convergence

We also want to show that these trajectories monotonically approach the fixed points we just solved for.

- Monotonicity comes from the construction of the function.
- Convergence follows because the function is bounded by the constraints of the card trick.

Consider the dynamical system $(\mathbb{Z}, f)$ with $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = a - \left\lceil \frac{x}{b} \right\rceil$.

- If $a \not\equiv 1 \mod (b+1)$, then the function has exactly one fixed point
$$\bar{x} = \left\lfloor \frac{ab}{b+1} \right\rfloor.$$
- If $a \equiv 1 \mod (b+1)$, then the function has an attracting two-cycle
$$\bar{x}_1 = \frac{b(a-1)}{b+1}, \qquad \bar{x}_2 = \bar{x}_1 + 1$$
and no fixed points.

Proof:

I'm not actually going to do this proof.

Comparing the Tricks

First Trick

- monotonic convergence
- uses ceiling function
- $a = p$
- when $a \equiv 0 \mod (b-1)$, two fixed points
- with 14 cards, $\bar{x} = 8$
- with 32 cards, $\bar{x} = 17$

Second Trick

- no monotonic convergence
- uses floor function
- $a = N + 1 - p$
- when $a \equiv 1 \mod (b+1)$, attracting two-cycle
- with 14 cards, $\bar{x} = 7$
- with 32 cards, $\bar{x} = 16$

Consider the dynamical system $(\mathbb{Z}, f)$ with $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = a + \left\lceil \frac{x}{b} \right\rceil$:

Let $B = \frac{ab}{b-1}$. Then

- If $b - 1 \nmid a$ and $x > \lceil B \rceil$,

$$\tau(x) = \left\lceil \log\left(\frac{x - B}{\lceil B \rceil - B}\right) \Big/ \log b \right\rceil.$$

- If $b - 1 \nmid a$ and $x < \lceil B \rceil$,

$$\tau(x) = \left\lfloor \log\left(\frac{B - x}{1 - (\lceil B \rceil - B)}\right) \Big/ \log b \right\rfloor + 1.$$

Consider the dynamical system $(\mathbb{Z}, f)$ with $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = a + \left\lceil \frac{x}{b} \right\rceil$:

(continued...)

- If $b - 1 \mid a$ and $x > B + 1$,

$$\tau(x) = \left\lceil \frac{\log(x - B)}{\log b} \right\rceil.$$

- If $b - 1 \mid a$ and $x < B$,

$$\tau(x) = \left\lfloor \frac{\log(B - x)}{\log b} \right\rfloor + 1.$$

Proof:

When $x > 17 = \left\lceil \frac{33}{2} \right\rceil$,

$$\begin{aligned}
\tau_1(x) &= \left\lceil \log\left(\frac{x-B}{\lceil B\rceil - B}\right) \Big/ \log b \right\rceil \\
&= \left\lceil \log\left(\frac{x-\frac{33}{2}}{17-\frac{33}{2}}\right) \Big/ \log 3 \right\rceil \\
&= \left\lceil \log_3\left(\frac{x-\frac{33}{2}}{\frac{1}{2}}\right) \right\rceil \\
&= \lceil \log_3(2x-33) \rceil .
\end{aligned}$$

Proof:

(continued...)

When $x < 17$,

$$\begin{aligned}
\tau_2(x) &= \left\lfloor \log\left(\frac{B-x}{1-(\lceil B\rceil - B)}\right) \Big/ \log b \right\rfloor + 1 \\
&= \left\lfloor \log\left(\frac{\frac{33}{2}-x}{1-(17-\frac{33}{2})}\right) \Big/ \log 3 \right\rfloor + 1 \\
&= \left\lfloor \log_3\left(\frac{\frac{33}{2}-x}{\frac{1}{2}}\right) \right\rfloor + 1 \\
&= \lfloor \log_3(33-2x) \rfloor + 1.
\end{aligned}$$

Proof:

(continued...)

$$\begin{aligned}\tau_1(32) &= \lceil \log_3(2(32) - 33) \rceil \\ &= \lceil \log_3(31) \rceil \\ \tau &= 4.\end{aligned}$$

$$\begin{aligned}\tau_2(1) &= \lfloor \log_3(33 - 2(1)) \rfloor + 1 \\ &= \lfloor \log_3(31) \rfloor + 1 \\ \tau &= 4.\end{aligned}$$

Consider the dynamical system $(\mathbb{Z}, f)$ with $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = a - \left\lceil \frac{x}{b} \right\rceil$.

No known formula.

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No known formula.

Calculations imply $\tau = 4$.

In Conclusion...

Two card tricks that seem similar are driven by very different dynamical systems.

We saw...

- different types of fixed points
- different uses of the parameters
- different levels of predictability

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Any Questions??