Using Origami to Find Rational Approximations of Irrational Roots

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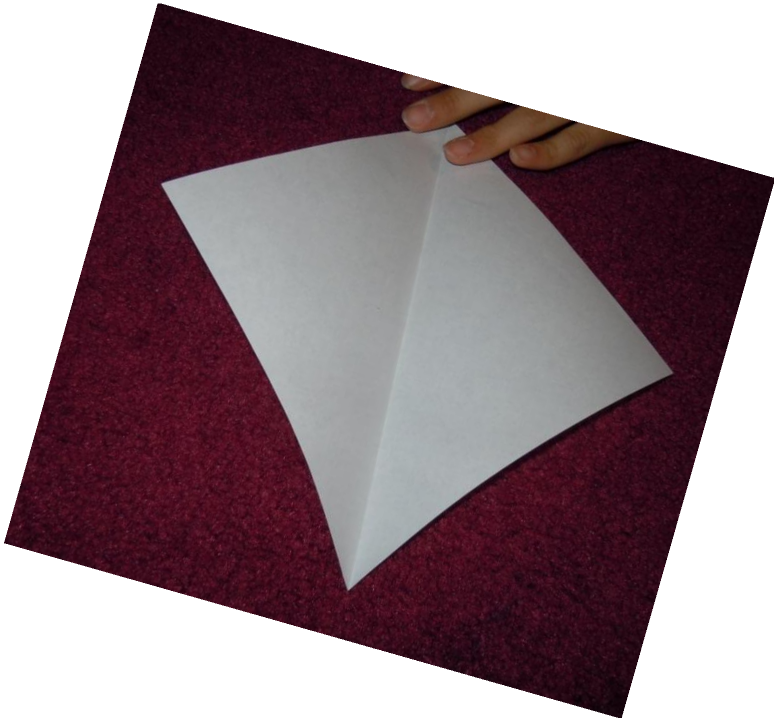
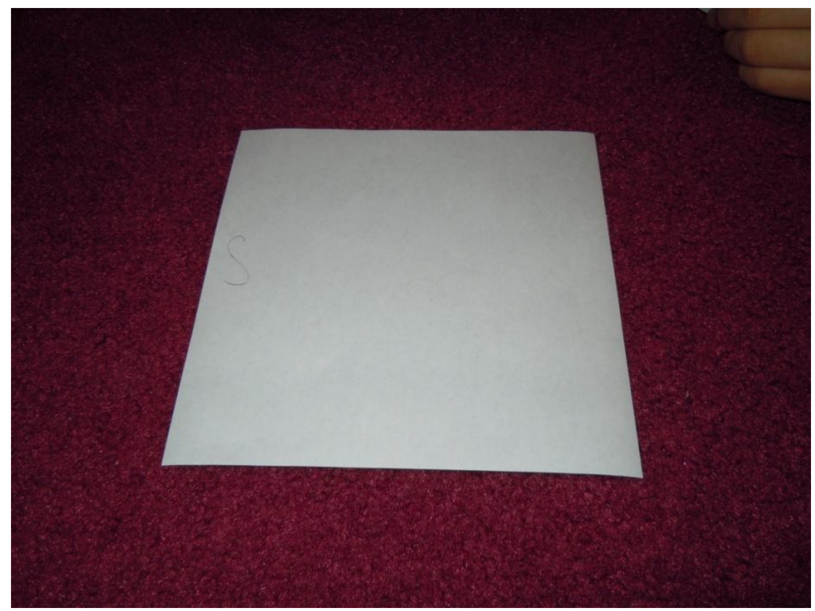
Origami Instructions

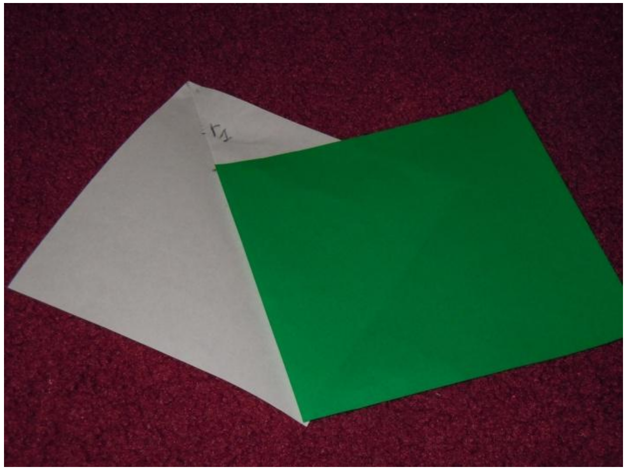




- Folding paper allows us to find rational approximations of the square root of two.

Side Unit fits Diagonal Once with remainder.

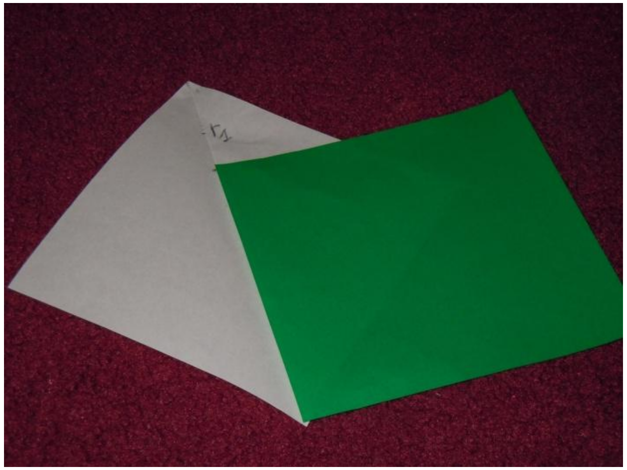
Origami Instructions



Side Unit fits Diagonal Once with remainder.

Remainder fits side unit twice with remainder$_2$ space

Origami Instructions

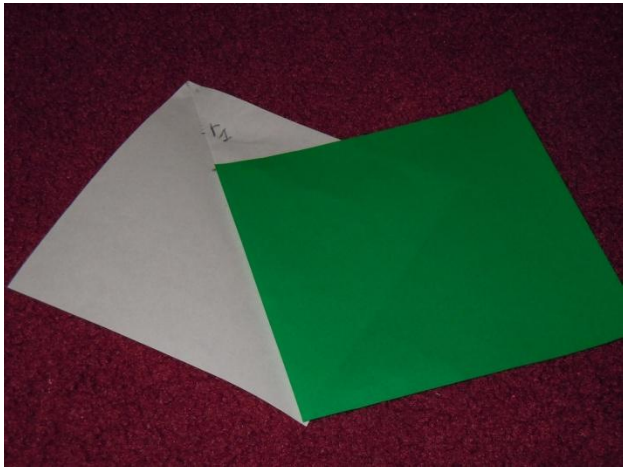


Side Unit fits Diagonal Once with remainder.

Remainder fits side unit twice with $remainder_2$ space

$Remainder_2$ fits Remainder unit twice with $Remainder_3$ left over.

Origami Instructions



Summary of Origami Exercise

- As we keep doing the exercise, you always get the number two after the first step. The next remainder will fit into previous remainder twice again.

Summary of Origami Exercise

- All these approximations of the square root of two come from the continued fractions representation of the square root of two.

Continued Fractions Method

$$\sqrt{2} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

Continued Fractions Method

$$\sqrt{2} = 1 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

Continued Fractions Method

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

Continued Fractions Method

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

Continued Fractions Method

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}$$

There will be twos continuing throughout the fraction after one.

How to Use Continued Fractions to Build Approximations

$$\sqrt{2} \approx 1$$

$$\sqrt{2} \approx 1 + \frac{1}{2} = \frac{3}{2}$$

$$\sqrt{2} \approx 1 + \cfrac{1}{2 + \cfrac{1}{2}} = \frac{7}{5}$$

$$\sqrt{2} \approx 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}} = \frac{17}{12}$$

h_i	k_i
1	1
3	2
7	5
17	12
41	29
99	70
239	169
577	408
1393	985
3363	2378
8119	5741

Rational Approximations Of the Square Root of Two

h_i	k_i
1	1
3	2
7	5
17	12
41	29
99	70
239	169
577	408
1393	985
3363	2378
8119	5741
19601	13860
47321	33461
114243	80782
275807	195025
665857	470832
1607521	1136689
3880899	2744210
9369319	6625109
22619537	15994428

Denoting Parts of the Square Root of Two Continued Fraction

$$\sqrt{2} = 1 + \boxed{\frac{1}{2 + \boxed{\frac{1}{2 + \frac{1}{2 + \frac{1}{\ddots}}}}}}$$

Let Big Box equal x

What does the big box converge to?

Algebra that Continued Fractions Converges to the Square Root of Two

$$x = \frac{1}{2+x}$$

$$2x + x^2 = 1$$

$$x^2 + 2x - 1 = 0$$

$$-2 \pm \frac{\sqrt{8}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 + \sqrt{2}$$

$$\sqrt{2} = 1 + (-1 + \sqrt{2})$$

Why does the answer have to be $-1+\sqrt{2}$**?**

Assuming that this continued fraction converges, that is what x converges to. Therefore the continued fraction equals the square root of 2.

The Babylonian Method

Let's try it with the square root of 2
Have x_0 equal 1, our first approximate.

$$\sqrt{S} \approx x_0$$

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{S}{x_n} \right)$$

$$x_0 = 1$$

$$x_1 = \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2}$$

$$x_2 = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{\left(\frac{3}{2}\right)}\right) = \frac{17}{12}$$

$$x_3 = \frac{1}{2}\left(\frac{17}{12} + \frac{2}{\left(\frac{17}{12}\right)}\right) = \frac{577}{408}$$

Algebra that Babylonian Method Converges to the Square Root of Two

$$x = \frac{1}{2}\left(x + \frac{2}{x}\right)$$

$$x = \frac{x}{2} + \frac{1}{x}$$

$$x^2 = \frac{x^2}{2} + 1$$

$$2x^2 = x^2 + 2$$

$$x^2 = 2$$

$$x = \sqrt{2}$$

The reason why x is positive is because if the Babylonian Method converges , the result is what it will converge to.

If x is negative then the result is the negative square root of two.

There is a wonderful blog entry that is focused on the Babylonian Method and how it works with the square root of two.

http://johncarlosbaez.wordpress.com/2011/12/02/babylon-and-the-square-root-of-2/

	h_i	k_i	h_{i+1}/h_i	k_{i+1}/k_i
1	1	1		
2	3	2	3	2
3	7	5	2.333333	2.5
4	17	12	2.428571	2.4
5	41	29	2.411765	2.416666667
6	99	70	2.414634	2.413793103
7	239	169	2.414141	2.414285714
8	577	408	2.414226	2.414201183
9	1393	985	2.414211	2.414215686
10	3363	2378	2.414214	2.414213198
11	8119	5741	2.414213	2.414213625
12	19601	13860	2.414214	2.414213552
13	47321	33461	2.414214	2.414213564
14	114243	80782	2.414214	2.414213562
15	275807	195025	2.414214	2.414213562
16	665857	470832	2.414214	2.414213562
17	1607521	1136689	2.414214	2.414213562
18	3880899	2744210	2.414214	2.414213562
19	9369319	6625109	2.414214	2.414213562
20	22619537	15994428	2.414214	2.414213562
21	54608393	38613965	2.414214	2.414213562
22	1.32E+08	93222358	2.414214	2.414213562
23	3.18E+08	2.25E+08	2.414214	2.414213562
24	7.68E+08	5.43E+08	2.414214	2.414213562
25	1.86E+09	1.31E+09	2.414214	2.414213562
26	4.48E+09	3.17E+09	2.414214	2.414213562
27	1.08E+10	7.65E+09	2.414214	2.414213562
28	2.61E+10	1.85E+10	2.414214	2.414213562
29	6.3E+10	4.46E+10	2.414214	2.414213562
30	1.52E+11	1.08E+11	2.414214	2.414213562
31	3.67E+11	2.6E+11	2.414214	2.414213562
32	8.87E+11	6.27E+11	2.414214	2.414213562

What are the explicit formulas for each the numerator and the denominator of all rational approximations of the square root of two?

There is almost a common ratio between consecutive numerators of the rational approximations. There is almost a common ratio between consecutive denominators of the rational approximations. The ratio is around 1 + sqrt(2) for both of these relationships.

Maybe, h_i and k_i are the sums of two geometric sequences with a common ratio of 1 + sqrt(2)

Rational Approximations Numerator

$$h_1 = c_1\left(1+\sqrt{2}\right) + c_2(r) = 1$$
$$h_2 = c_1\left(3+2\sqrt{2}\right) + c_2(r)^2 = 3$$
$$2h_1 = c_1\left(2+2\sqrt{2}\right) + c_2(2r) = 2$$
$$c_1 = 1 - c_2\left(r^2 - 2r\right)$$
$$5h_1 = \left(1 - c_2\left(r^2-2r\right)\right)\left(5+5\sqrt{2}\right) + c_2(5r) = 5$$
$$h_3 = \left(1 - c_2\left(r^2-2r\right)\right)\left(7+5\sqrt{2}\right) + c_2(r)^3 = 7$$
$$0 = c_2\left(r^3 - 2r^2 - r\right)$$
$$0 = r^2 - 2r - 1$$
$$r = 1 - \sqrt{2}$$

$$h_4 = c_1\left(17+12\sqrt{2}\right) + c_2 r^4 = 17$$
$$6h_2 = c_1\left(18+12\sqrt{2}\right) + 6c_2 r^2 = 18$$
$$c_1 + c_2\left(6r^2 - r^4\right) = 1$$
$$c_1 + c_2 = 1$$

$$h_1 = c_1\left(1+\sqrt{2}\right) + c_2\left(1-\sqrt{2}\right) = 1$$
$$c_1 + c_2 + \sqrt{2}c_1 - \sqrt{2}c_2 = 1$$
$$1 + \sqrt{2}c_1 - \sqrt{2}c_2 = 1$$
$$c_1 - c_2 = 0$$

$$h_i = \frac{1}{2}\left(1+\sqrt{2}\right)^i + \frac{1}{2}\left(1-\sqrt{2}\right)^i$$

=0.5*(1+SQRT(2))^A1+0.5*(1-SQRT(2))^A1

	i	h_i	C	D
1	1	1	1	
2	2	3	2	
3	3	7	5	
4	4	17	12	
5	5	41	29	
6	6	99	70	
7	7	239	169	
8	8	577	408	
9	9	1393	985	
10	10	3363	2378	
11	11	8119	5741	
12	12	19601	13860	
13	13	47321	33461	
14	14	114243	80782	
15	15	275807	195025	
16	16	665857	470832	
17	17	1607521	1136689	
18	18	3880899	2744210	
19	19	9369319	6625109	
20	20	22619537	15994428	
21	21	54608393	38613965	
22	22	1.32E+08	93222358	
23	23	3.18E+08	2.25E+08	
24	24	7.68E+08	5.43E+08	
25	25	1.86E+09	1.31E+09	
26	26	4.48E+09	3.17E+09	
27	27	1.08E+10	7.65E+09	
28	28	2.61E+10	1.85E+10	
29	29	6.3E+10	4.46E+10	
30	30	1.52E+11	1.08E+11	
31	31	3.67E+11	2.6E+11	
32	32	8.87E+11	6.27E+11	

Rational Approximations Denominator

$$k_1 = c_1(1+\sqrt{2}) + c_2 r = 1$$

$$k_2 = c_1(3+2\sqrt{2}) + c_2 r^2 = 2$$

$$2k_1 = c_1(2+2\sqrt{2}) + 2c_2 r = 2$$

$$c_1 + c_2 r^2 - 2c_2 r = 0$$

$$k_3 = c_1(7+5\sqrt{2}) + c_2 r^3 = 5$$

$$5k_1 = c_1(5+5\sqrt{2}) + 5c_2 r = 5$$

$$2c_1 + 2c_2 r^3 - 5c_2 r = 0$$

$$2c_1 + 2c_2 r^2 - 4c_2 r = 0$$

$$r^2 - 2r - 1 = 0$$

$$r = 1 - \sqrt{2}$$

$$c_1(1+\sqrt{2}) + c_2(1-\sqrt{2}) = 1$$

$$c_1 + c_2\left(3-2\sqrt{2}\right) - 2c_2\left(1-\sqrt{2}\right) = 0$$

$$c_1 = -c_2$$

$$-c_2(1+\sqrt{2}) + c_2(1-\sqrt{2}) = 1$$

$$-2\sqrt{2}c_2 = 1$$

$$c_2 = -\frac{\sqrt{2}}{4}$$

$$c_1 = \frac{\sqrt{2}}{4}$$

$$k_i = \frac{\sqrt{2}}{4}\left(1+\sqrt{2}\right)^i - \frac{\sqrt{2}}{4}\left(1-\sqrt{2}\right)^i$$

=SQRT(2)/4*(1+SQRT(2))^(A1)-SQRT(2)/4*(1-SQRT(2))^(A1)

	i	h_i	k_i	D	E	F
1	1	1	1			
2	2	3	2			
3	3	7	5			
4	4	17	12			
5	5	41	29			
6	6	99	70			
7	7	239	169			
8	8	577	408			
9	9	1393	985			
10	10	3363	2378			
11	11	8119	5741			
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27	27	1.08E+10	7.65E+09			
28	28	2.61E+10	1.85E+10			
29	29	6.3E+10	4.46E+10			
30	30	1.52E+11	1.08E+11			
31	31	3.67E+11	2.6E+11			
32	32	8.87E+11	6.27E+11			

Connection between Continued Fractions and Babylonian Method

What happens when the continued fraction approximation is put into the Babylonian Method?

$$(h_i)^2 = \frac{1}{4}\left(1+\sqrt{2}\right)^{2i} + \frac{1}{2}(-1)^i + \frac{1}{4}\left(1-\sqrt{2}\right)^{2i}$$

$$(k_i)^2 = \frac{1}{8}\left(1+\sqrt{2}\right)^{2i} - \frac{1}{4}(-1)^i + \frac{1}{8}\left(1-\sqrt{2}\right)^{2i}$$

$$2(k_i)^2 = \frac{1}{4}\left(1+\sqrt{2}\right)^{2i} - \frac{1}{2}(-1)^i + \frac{1}{4}\left(1-\sqrt{2}\right)^{2i}$$

$$h_i k_i = \frac{\sqrt{2}}{8}\left(1+\sqrt{2}\right)^{2i} + \frac{\sqrt{2}}{8}\left(1-\sqrt{2}\right)^{2i}$$

$$2h_i k_i = \frac{\sqrt{2}}{4}\left(1+\sqrt{2}\right)^{2i} + \frac{\sqrt{2}}{4}\left(1-\sqrt{2}\right)^{2i} = k_{2i}$$

$$(h_i)^2 + 2(k_i)^2 = \frac{1}{2}\left(1+\sqrt{2}\right)^{2i} + \frac{1}{2}\left(1-\sqrt{2}\right)^{2i} = h_{2i}$$

$$\frac{1}{2}\left(\frac{(h_i)^2 + 2(k_i)^2}{h_i k_i}\right) = \frac{h_{2i}}{k_{2i}}$$

$$\frac{1}{2}\left(\frac{h_i}{k_i} + \frac{2k_i}{h_i}\right) = \frac{1}{2}\left(\frac{(h_i)^2 + 2(k_i)^2}{h_i k_i}\right)$$

This is the Babylonian Method by taking a continued fraction approximation and average it with twice its reciprocal.

$$\frac{h_{2i}}{k_{2i}}$$

The result is the continued fraction approximation of the square root of two, it's just the number of steps of the continued fractions approximation is doubled for the Babylonian Method. You only did one step of the Babylonian Method.

Conclusion

- We did an origami experiment that finds the same rational approximations as the continued fractions method.
- We found that the continued fraction and Babylonian Methods produced the same rational approximations for the square root of two.

The Blog Entry

http://johncarlosbaez.wordpress.com/2011/12/02/babylon-and-the-square-root-of-2/

Conclusion

- We did an origami experiment that finds the same rational approximations as the continued fractions method.
- We found that the continued fraction and Babylonian Methods produced the same rational approximations for the square root of two.
- Could we do something similar with Fibonacci Numbers and the Golden Ratio which has the square root of five in it?

The Blog Entry
http://johncarlosbaez.wordpress.com/2011/12/02/babylon-and-the-square-root-of-2/