Ramsey Number and Other Knot Invariants

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Knots

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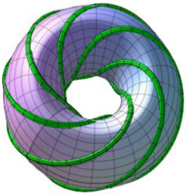
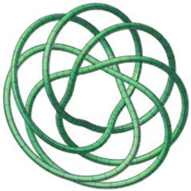
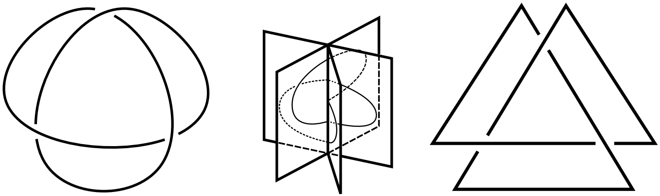
We will deal particularly with the class of torus knots $T_{p-1,p}$.





Knot Invariants

The Trefoil Knot, $T_{2,3}$



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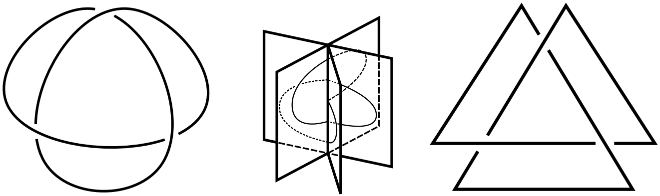

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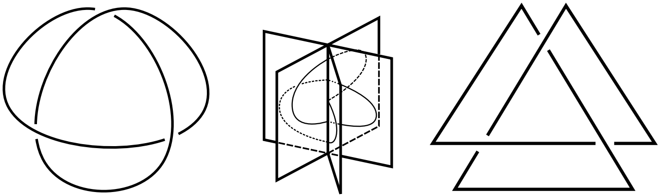

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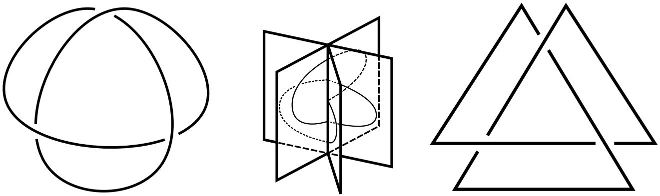

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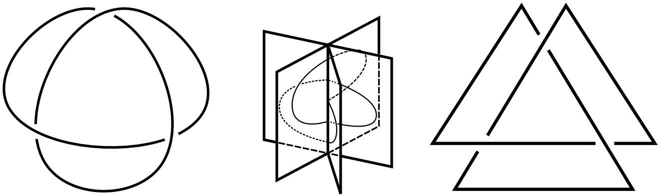

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- *Crossing Number:* Fewest number of crossings in any projection of the knot. $c(T_{2,3}) = 3$.

Graph Theory

A **graph** is a set of vertices connected by edges. A **complete graph** on n vertices is a set of n vertices such that each pair of vertices is connected by an edge. A **Hamiltonian cycle** of a graph is a cycle which visits every vertex exactly once.

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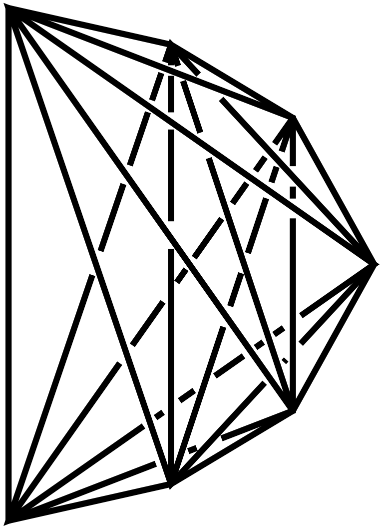
Definition

A **linear spatial embedding** of K_n is a copy of K_n in 3-space such that every edge is straight and no two edges intersect one another.

Cyclic Polytope



Ramsey Number

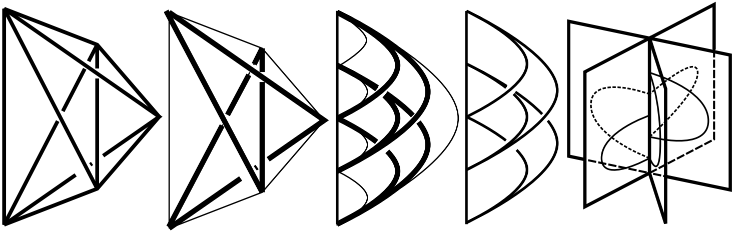
Definition

The **Ramsey number** of a knot is the smallest n such that any linear spatial embedding of K_n contains the knot K.

Negami [5] proved $R(K)$ is finite for any knot K.

It is known $R(unknot) = 3$.
Ramírez Alfonsín [1] proved $R(T_{2,3}) = 7$.

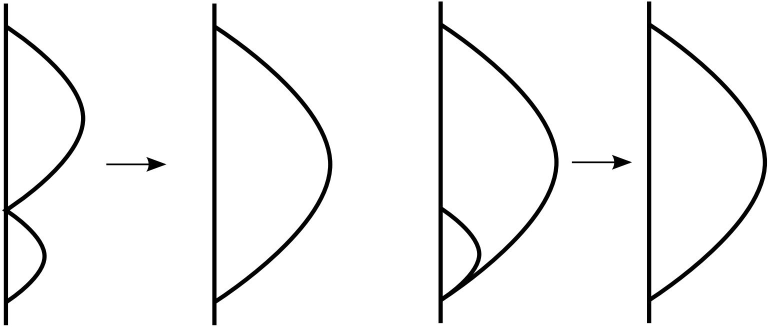
Constructing Arc Presentations



Reducing Arc Presentations

Based on moves defined by Cromwell [2].

Arc Merge:



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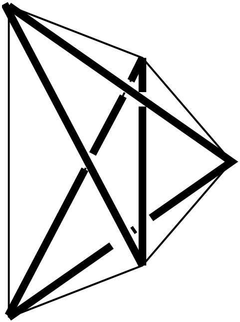

Reducing Arc Presentations

Lemma

For a knot K, corresponding to a Hamiltonian cycle of C_n, $\alpha(K) + b(H) \leq n$.

Proof.

Note that a maximum can be reduced by an arc merge, so we can always perform $b(H)$ arc merges, showing $\alpha(K) \leq n - b(H)$, or $\alpha(K) + b(H) \leq n$. □

Application to Ramsey Number

Theorem

For any knot K, $\alpha(K) + br(K) \leq R(K)$.

Proof.

Note that since for any knotted cycle of C_n $b(H) \geq br(K)$, our previous result implies that for K corresponding to H of C_n, $\alpha(K) + br(K) \leq n$. Since K lives on $C_{R(K)}$ by definition of Ramsey number, it follows that $\alpha(K) + br(K) \leq R(K)$. □

Application to Ramsey Number

Corollary

$R(T_{p-1,p}) \geq 3p - 2.$

Proof.

Our theorem gives us that $R(T_{p-1,p}) \geq \alpha(T_{p-1,p}) + br(T_{p-1,p})$. An application of work by Schubert [6] shows $br(T_{p-1,p}) = p - 1$, and an application of work by Matsuda [4] shows us $\alpha(T_{p-1,p}) = 2p - 1$. Combining these results we find $R(T_{p-1,p}) \geq 3p - 2$. □

Application to Ramsey Number

Corollary

$R(T_{p-1,p}) - s(T_{p-1,p}) \geq p - 2.$

Proof.

We just showed $R(T_{p-1,p}) \geq 3p - 2$. An application of a theorem by Jin [3] gives us that $s(T_{p-1,p}) = 2p$. Combining these results, $R(T_{p-1,p}) - s(T_{p-1,p}) \geq p - 2$. □

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1. Can we find other classes of knots for which the difference between the Ramsey number and stick number grows without bound?

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2. We have shown that the difference between Ramsey number and stick number of $T_{p-1,p}$ is at least linear. Is it, in actuality, approximately linear? Quadratic? How can we best model this difference?
3. Do our reductions on arc presentations create a normal form from which we can gather more information about the knot?

References

J.L. Ramírez Alfonsín, *Spatial graphs and oriented matroids: the trefoil*, Discrete and Computational Geometry **22** (1999), no. 1, 149–158.

Peter Cromwell, *Knots and links*, Cambridge University Press, 2004.

Gyo Taek Jin, *Polygon indices and superbridge indices of torus knots and links*, Journal of Knot Theory and Its Ramifications **6** (1997), no. 2, 281–289.

Hiroshi Matsuda, *Links in an open book decomposition and the standard contact structure*, Preceedings of the American Mathematical Society **134** (2006), no. 12, 3697–3702.

Seiya Negami, *Ramsey theorems for knots, links and spatial graphs*, Transactions of the American Mathematical Society **324** (1991), no. 2, 527–541.

Horst Schubert, *Über eine numerische knoteninvariante*, Mathematische Zeitschrift **61** (1954), 245–288.