

Laws of Logic and Rules of Evidence

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Transitivity of Order

- For any real numbers a, b, c ,
if $a < b$ and $b < c$ then $a < c$.
- Transitivity is an implication. We must know $a < b$ and $b < c$ in order to conclude $a < c$.
- In mathematics: No problem. 😊
We know numbers.
- In life: There's a problem. ☹️

- In life we don't know anything for certain.
- To find the value of a number a , we must take measurements and determine a as best we can from the evidence.
- Transitivity in the real world:
Suppose the evidence shows $a < b$ and the evidence shows $b < c$. Can we conclude, based on the evidence, that $a < c$?

Background

- Let μ_F be the mean height of HC females.
Let μ_M be the mean height of HC males.

Conjecture: $\mu_F < \mu_M$.

- Evidence: Take a sample of HC females and measure each subject's height.

Data: f_1, f_2, \dots, f_n . $\bar{f} = (1/n) \sum f_i \sim \mu_F$

Then take a sample of HC males and measure each subject's height.

Data: m_1, m_2, \dots, m_p . $\bar{m} = (1/p) \sum m_j \sim \mu_M$

- $\bar{m} - \bar{f} \sim \mu_M - \mu_F$

- To investigate our conjecture we will test

$$\begin{array}{ll} & H_o: \mu_F = \mu_M \quad (\text{or } \mu_M - \mu_F = 0) \\ \text{versus} & H_a: \mu_F < \mu_M \quad (\text{or } \mu_M - \mu_F > 0) \end{array}$$

- Test statistic: $\bar{m} - \bar{f}$
- Burning question: How likely is it to get a value of $\bar{m} - \bar{f}$ as extreme as what we observed, or more so, if $\mu_F = \mu_M$?

- Goal: Calculate probabilities for $\bar{m} - \bar{f}$. To do so, we need a probability distribution.
- Starters: Assume each f_i is normally distributed with mean μ_F and standard deviation σ_F .
Assume each m_j is normally distributed with mean μ_M and standard deviation σ_M .
- Further, assume we were reasonably intelligent as to how we chose our subjects, so that the measurements are independent.

- Under our assumptions the random variable

$$Z = \frac{(\bar{m} - \bar{f}) - (\mu_M - \mu_F)}{\sqrt{\frac{\sigma_M^2}{n} + \frac{\sigma_F^2}{p}}}$$

has a standard normal distribution.

- Unfortunately there is a problem. We don't know σ_F or σ_M .
- Solution? We can replace the population standard deviations σ_F and σ_M by the sample standard deviations s_F and s_M .

- Under our assumptions the random variable

$$\frac{(\bar{m} - \bar{f}) - (\mu_M - \mu_F)}{\sqrt{\frac{s_M^2}{n} + \frac{s_F^2}{p}}}$$

has a $\# @ ! ! \& \%$ distribution.

- The distribution of the random variable can be approximated by a t -distribution, if you don't mind fractional degrees of freedom. Or, it can be approximated by a standard t -distribution with integer degrees of freedom if you don't mind a less than optimal approximation. ☹

- Alternatively, we can assume a common variance, so $\sigma_M = \sigma_F = \sigma_C$. Then

$$Z = \frac{(\bar{m} - \bar{f}) - (\mu_M - \mu_F)}{\sqrt{\frac{\sigma_C^2}{n} + \frac{\sigma_C^2}{p}}} = \frac{(\bar{m} - \bar{f}) - (\mu_M - \mu_F)}{\sigma_C \sqrt{\frac{1}{n} + \frac{1}{p}}}$$

has a standard normal distribution.

- The common s.d. can be approximated by a pooled average of the two sample s.d.

$$s_C = \sqrt{\frac{(n-1)s_M^2}{n+p-2} + \frac{(p-1)s_F^2}{n+p-2}}$$

- With the common variance assumption the random variable

$$t = \frac{(\bar{m} - \bar{f}) - (\mu_M - \mu_F)}{s_C \sqrt{\frac{1}{n} + \frac{1}{p}}}$$

has a t -distribution with $n + p - 2$ df.

- All is sweetness and light – except
- The assumption of a common variance raises a conflict with transitivity.

Example

The following are selected random samples generated by Minitab from a normal distribution with a common variance of 1.

- | | | |
|--------------------|---------------------|--------------------|
| • 5.45915 | • 5.21471 | • 6.30382 |
| 2.52951 | ... | 7.98292 |
| 5.45659 | 3.89517 | 4.74109 |
| 1.99805 | 4.92814 | 4.81625 |
| • $n = 4$ | • $n = 100$ | • $n = 4$ |
| • $\bar{a} = 3.86$ | • $\bar{b} = 4.887$ | • $\bar{c} = 5.96$ |
| • $s_a = 1.86$ | • $s_b = 0.92$ | • $s_c = 1.53$ |

- Test $H_0: a = b$

T-Test of difference = 0 (vs <): T-Value = -2.10

P-Value = 0.019 DF = 102 Both use Pooled StDev = 0.9605

Reject H_0 . Evidence supports the claim $a < b$.

- Test $H_0: b = c$

T-Test of difference = 0 (vs <): T-Value = -2.23

P-Value = 0.014 DF = 102 Both use Pooled StDev = 0.9433

Reject H_0 . Evidence supports the claim $b < c$.

- Test $H_0: a = c$

T-Test of difference = 0 (vs <): T-Value = -1.75

P-Value = 0.066 DF = 6 Both use Pooled StDev = 1.7004

Do NOT reject H_0 . Evidence is not strong enough to reject $a = c$.

- The breaking of transitivity comes from the pooling of the standard deviations.
- The standard deviation is a measure of how well we know the location of a quantity. If we know one quantity well (small s.d. and large n) then the common variance assumption carries that knowledge over to the difference – even though knowledge of the second part of the difference is much less precise.

In the example we know b , the middle quantity, very well while our knowledge of a and of c is much less precise.

- Comparing a and b : the estimated difference is $\bar{b} - \bar{a} = 1.026$ with a pooled s.d. of 0.9605. For the given sample sizes, the difference is statistically significant.
- The comparison for b and c is similar.
- Comparing a and c : the estimated difference is $\bar{c} - \bar{a} = 2.10$ with a pooled s.d. of 1.7004. Even though the estimated difference is larger, the pooled s.d. is also much larger and the sample sizes are both small. Consequently the difference is not significantly different from 0.

- So much for real world transitivity.

There is a logic to the rules of evidence, but the logic is not quite as simple as the logic of mathematics.

- It should be noted that ANOVA, the ANalysis Of VAriance, applies to the comparison of n quantities – and is based on the assumption of a common variance, which leads to some interesting outcomes.