# MINIMAL SURFACES

## SAINT MICHAEL'S COLLEGE

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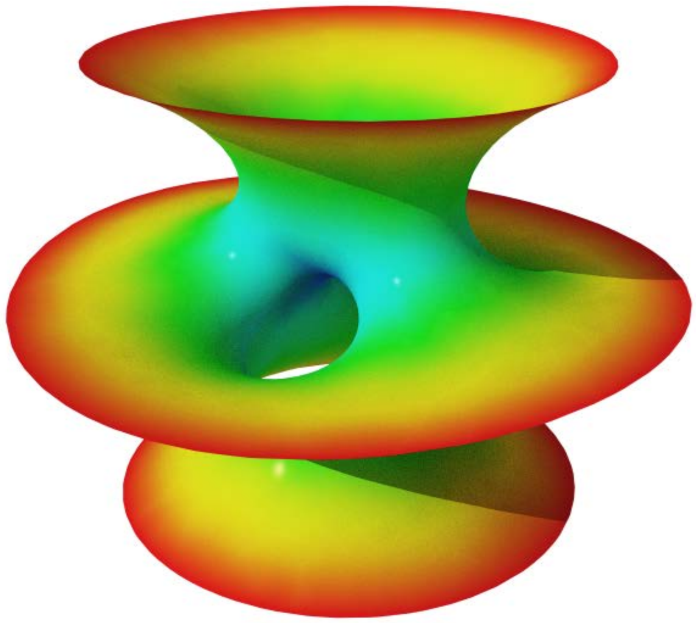
# Overview

- What is a minimal surface?
- Surface Area & Curvature
- History
- Mathematics
- Examples & Applications

http://www.bugman123.com/MinimalSurfaces/Chen-Gackstatter-large.jpg

# What is a Minimal Surface?

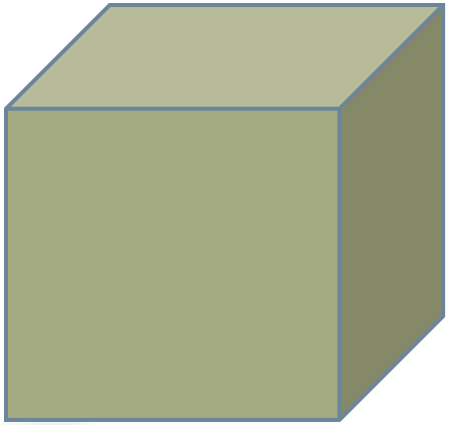
- A surface with mean curvature of zero at all points
- Bounded VS Infinite
- A plane is the most trivial minimal surface



http://commons.wikimedia.org/wiki/File:Costa's_Minimal_Surface.png

# Minimal Surface Area

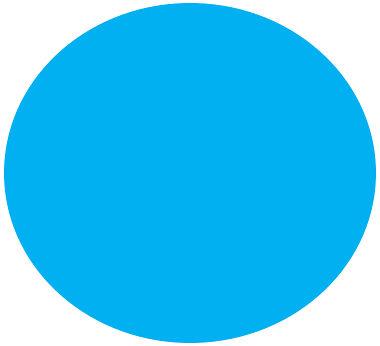
- **Cube side length 2**
  - Volume enclosed= 8
  - Surface Area = 24
- **Sphere**
  - Volume enclosed = 8
  - r=1.24
  - Surface Area = 19.32



http://1.bp.blogspot.com/-fHajrxa3gE4/TdFRVtNB5XI/AAAAAAAAAAo/AAdIrxWhG7Y/s1600/sphere+copy.jpg

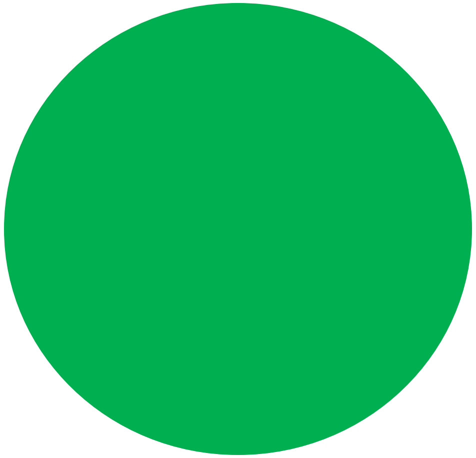
# Curvature ~ rate of change

- More Curvature



$$\kappa = \left\| \frac{dT}{ds} \right\|$$

- Less Curvature



# History

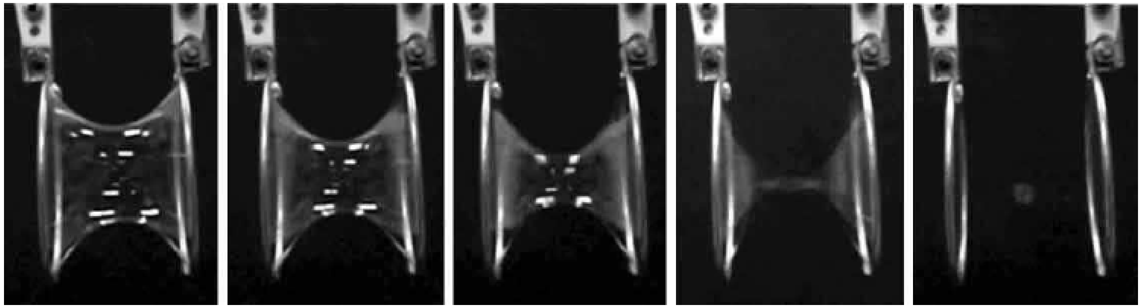
- Joseph-Louis Lagrange first brought forward the idea in 1768
- Monge (1776) discovered mean curvature must equal zero
- Leonhard Euler in 1774 and Jean Baptiste Meusiner in 1776 used Lagrange's equation to find the first non-trivial minimal surface, the catenoid
- Jean Baptiste Meusiner in 1776 discovered the helicoid
- Later surfaces were discovered by mathematicians in the mid nineteenth century

# Soap Bubbles and Minimal Surfaces

- Principle of Least Energy

- A minimal surface is formed between the two boundaries.

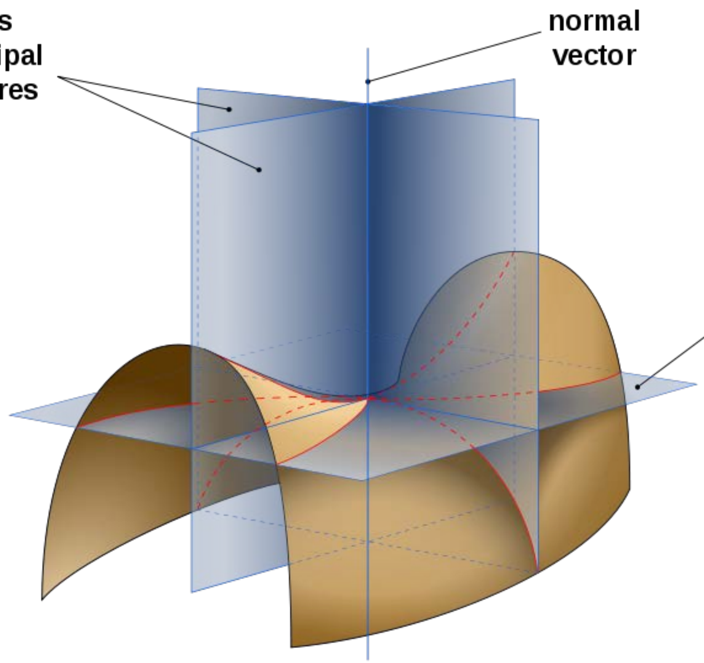
- The sphere is not a minimal surface.



http://arxiv.org/pdf/0711.3256.pdf

# Principal Curvatures

- Measures the amount that surfaces bend at a certain point
- Principal curvatures give the direction of the plane with the maximal and minimal curvatures




http://upload.wikimedia.org/wikipedia/commons/thumb/e/eb/Minimal_surface_curvature_planes-en.svg/800px-Minimal_surface_curvature_planes-en.svg.png

# First Fundamental

- Characterizes a surface $x(u,v)$ principal curvatures

- First fundamental matrix is given by:

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

$$E = x_u \bullet x_u$$

$$F = x_u \bullet x_v$$

$$G = x_v \bullet x_v$$

- Where subscripts denote partial derivatives and • denotes the dot product

# Second Fundamental

- The second fundamental is given by:

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

$$L = x_{uu} \bullet \vec{n} \qquad N = x_{vv} \bullet \vec{n}$$

$$M = x_{uv} \bullet \vec{n} \qquad \vec{n} = \frac{x_u \times x_v}{|x_u \times x_v|}$$

- Using the two fundamentals you can get the mean curvature:

$$\kappa_M = \frac{Trace(I * adj(II))}{2 * Det(I)}$$

# Lagrange's Equation

- Using a similar method, but with vectors rather than matrices, Lagrange found that minimal surfaces must obey:

$$0 = (1 + x_u{}^2)x_{vv} - 2x_u x_v x_{uv} + (1 + x_u{}^2)x_{vv}$$

- Where subscript denotes a partial derivative

# Weierstrass-Enneper Parameterization

- Makes a minimal surface by the following method:

$$x_k(\alpha) = \Re\left(\int_0^{\alpha} \beta(z)\,dz\right) \qquad \beta_1 = \frac{1}{2} f\,(1 - g^2) \qquad \beta_2 = \frac{1}{2} f\,(1 + g^2)\mathbf{i}$$

$$\beta_3 = fg$$

- The surface given by

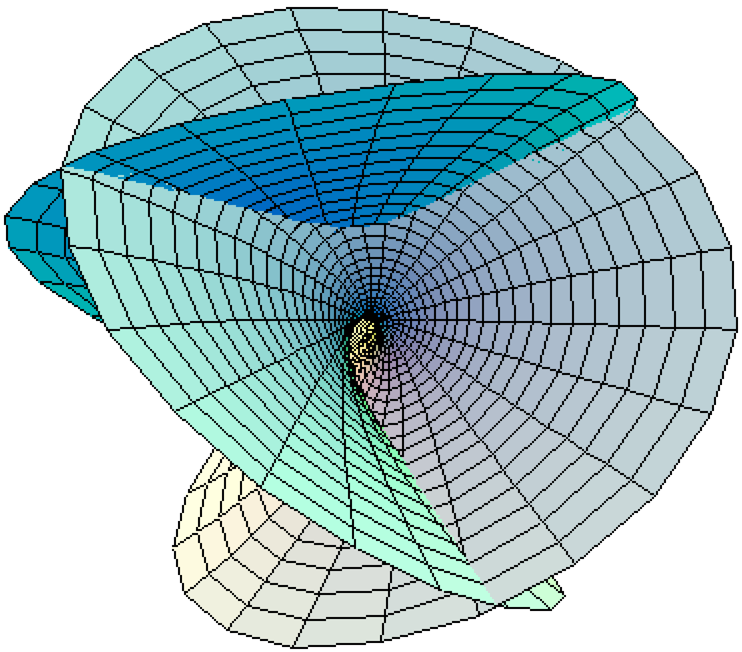
$$\left\langle x_1, x_2, x_3 \right\rangle$$

$$\alpha = u + \mathbf{i}v$$

will be a minimal one

# Weierstrass-Enneper Example

$$f(z) = 1 \qquad g(z) = z$$



$$x_1 = u(1 - u^2)/3 + v^2)/3$$

$$x_2 = -v(1 - v^2/3 + u^2)/3$$

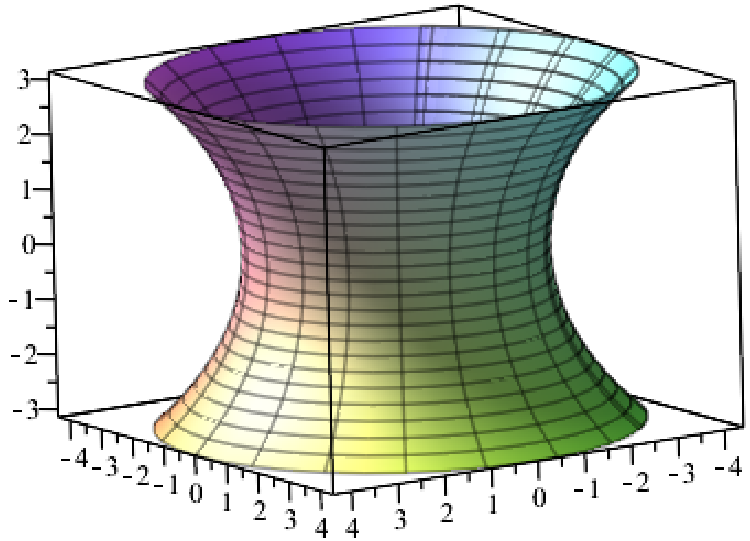
$$x_3 = (u^2 - v^2)/3$$

http://upload.wikimedia.org/wikipedia/commons/d/dc/EnneperSurface.PNG

# Catenoid

- Discovered by Euler (1774) and Meusiner (1776)

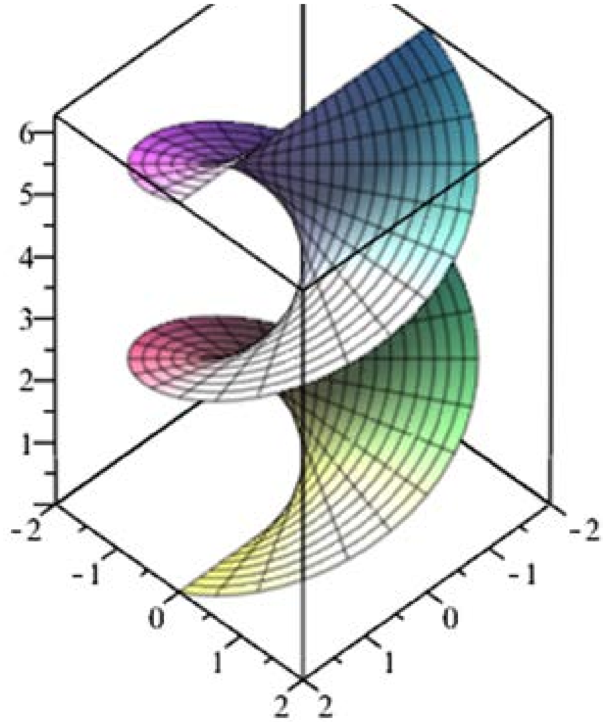
$$[c * \cosh(v/d) * \cos(u), c * \cosh(v/d) * \cos(u), v]$$



# Helicoid

- The third minimal surface found by Meusiner (1776).

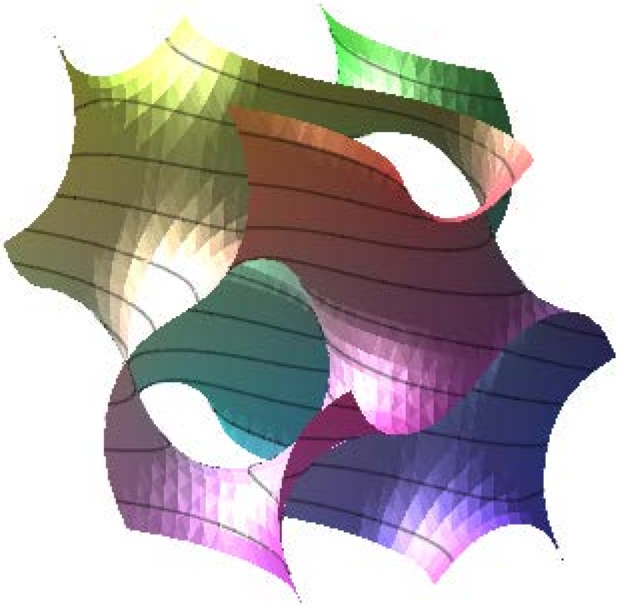
$$[u * \cos(v), u * \sin(v), v]$$



# Gyroid

- Discovered by Alan Schoen in 1970

$$\cos(x) * \sin(y) + \cos(y) * \sin(z) + \cos(z) * \sin(x) = 0$$



# Applications

- Road structures
- Architecture
- Efficiency
- Least Cost

http://m-cdc.de/wp-content/uploads/2010/07/boxel_eingangsbild_2.jpg

# Conclusion


http://www.math.cornell.edu/~mec/Summer2009/Fok/introduction.html

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