Markov Chains in the Game of Monopoly

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Markov Chains

Markov Chain: random process containing a sequence of variables $X_1, X_2, X_3, \ldots, X_r$ such that given the present state, the future state is conditionally independent of past states.

$$p(X_{t+1} = j | X_t = i_t)$$

Markov Chains

Examples:

- Games of chance
- Drunkard's walk
- Google PageRank
- Asset pricing models
- Baseball analysis

State of Economy Example

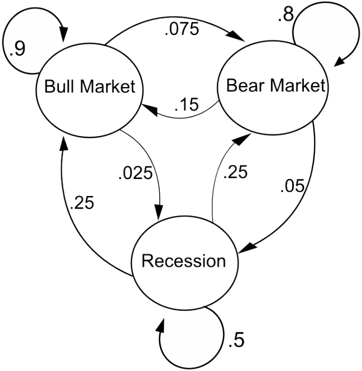

Figure: Directed Graph

$$P = \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}.$$

Figure: Transition Matrix

Long Term Markov Chain Behavior

Transition Matrix:

		To			
		1	2	...	n
From	1	$a_{1,1}$	$a_{1,2}$	$\cdots$	$a_{1,n}$
	2	$a_{2,1}$	$a_{2,2}$	$\cdots$	$a_{2,n}$
	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	n	$a_{n,1}$	$a_{n,2}$	$\cdots$	$a_{n,n}$

Long Term Markov Chain Behavior

Define p as the probability state distribution of ith row vector, with transition matrix, A. Then at time $t = 1$,

$$pA = p_1$$

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Taking subsequent iterations, the Markov chain over time develops to the following

$$(pA)A = pA^2,\ pA^3,\ pA^4$$

State of Economy Example

For example if at time t we are in a bear market, then 3 time periods later at time $t+3$ the distribution is,

$$pA^3 = p_3$$

$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} .9 & .075 & .025 \\ .15 & .8 & .05 \\ .25 & .25 & .5 \end{bmatrix}^3 = \begin{bmatrix} .3575 & .56825 & .07425 \end{bmatrix}$$

Long Term Markov Chain Behavior

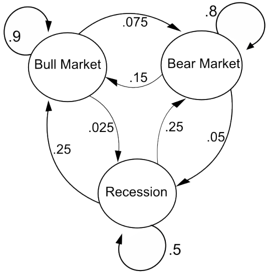
To determine stationary state distributions, we must find a probability distribution p which satisfies the condition

$$pA = p$$

$$\begin{bmatrix} p(1) & p(2) & \cdots & p(n) \end{bmatrix} \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} = \begin{bmatrix} p(1) & p(2) & \cdots & p(n) \end{bmatrix}$$

Long Term Markov Chain Behavior

However, there is an easier way to determine stationary probability distributions. Let's reverse our thinking and consider the probability of being in a certain state at $t+1$.



- $p(1) = .9p(1) + .15p(2) + .25p(3)$,
- $p(2) = .075p(1) + .8p(2) + .25p(3)$,
- $p(3) = .025p(1) + .05p(2) + .5p(3)$,
- with the condition, $p(1) + p(2) + p(3) = 1$

Four Square Circuit

		To			
		1	2	3	4
From	1	1/6	3/6	2/6	0
	2	1/6	3/6	2/6	0
	3	2/6	3/6	1/6	0
	4	2/6	3/6	1/6	0

Four Square Circuit

After the first throw, the probabilities of landing on each square are:

$$p_1(1) = \frac{1}{6} \qquad p_1(2) = \frac{1}{2} \qquad p_1(3) = \frac{1}{3} \qquad p_1(4) = 0$$

After two throws, the probabilities of landing on each square are:

$$p_2(1) = \frac{2}{9} \qquad p_2(2) = \frac{1}{2} \qquad p_2(3) = \frac{5}{18} \qquad p_2(4) = 0$$

Four Square Circuit

Let $p_t(n)$ represent the probability of landing on square n after t die rolls.

- $p_0(1) = 1, \qquad p_0(2) = p_0(3) = p_0(4) = 0.$
- $p_{t+1}(1) = \frac{1}{6}p_t(1) + \frac{1}{6}p_t(2) + \frac{2}{6}p_t(3) + \frac{2}{6}p_t(4),$
- $p_{t+1}(2) = \frac{3}{6}p_t(1) + \frac{3}{6}p_t(2) + \frac{3}{6}p_t(3) + \frac{3}{6}p_t(4),$
- $p_{t+1}(3) = \frac{2}{6}p_t(1) + \frac{2}{6}p_t(2) + \frac{1}{6}p_t(3) + \frac{1}{6}p_t(4),$
- $p_{t+1}(4) = 0.$

Four Square Circuit

$$p(1) = \frac{1}{6}p(1) + \frac{1}{6}p(2) + \frac{2}{6}p(3)$$

$$p(2) = \frac{3}{6}p(1) + \frac{3}{6}p(2) + \frac{3}{6}p(3)$$

$$p(3) = \frac{2}{6}p(1) + \frac{2}{6}p(2) + \frac{1}{6}p(3)$$

$$p(4) = 0$$

with the condition,

$$p(1) + p(2) + p(3) + p(4) = 1$$

Four Square Circuit

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ \frac{5}{6} & -\frac{1}{6} & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{5}{6} & 0 & 0 \end{array}\right) \xrightarrow{\text{row reduce echelon form}} \left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{3}{14} & \frac{3}{14} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{2}{7} & \frac{2}{7} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$p(1) = \frac{3}{14} \qquad p(2) = \frac{1}{2} \qquad p(3) = \frac{2}{7} \qquad p(4) = 0$$

Application to Monopoly

Modifications

- 40 squares
- Doubles Rule
- Community Chest and Chance Cards

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Markov Chain with $3 \times 40 = 120$ states

Stable Probabilties

Square	Property	Probability
0	Go	0.02914
1	Mediterranean Avenue	0.02007
2	Community Chest	0.01775
3	Baltic Avenue	0.02037
4	Income Tax	0.02193
5	Reading Railroad	0.02801
6	Oriental Avenue	0.02132
7	Chance	0.00815
8	Vermont Avenue	0.02187
9	Connecticut Avenue	0.02168
10	Just Visiting (Jail)	0.02139
11	St. Charles Place	0.02556
12	Electric Company	0.02614
13	States Avenue	0.02174
14	Virginia Avenue	0.02426
15	Pennsylvania Railroad	0.02635
16	St. James Place	0.02680
17	Community Chest	0.02296
18	Tennessee Avenue	0.02821
19	New York Avenue	0.02812

Square	Property	Probability
20	Free Parking	0.02825
21	Kentucky Avenue	0.02614
22	Chance	0.01045
23	Indiana Avenue	0.02567
24	Illinois Avenue	0.02993
25	B&O Railroad	0.02893
26	Atlantic Avenue	0.02537
27	Ventnor Avenue	0.02519
28	Water Works	0.02651
29	Marvin Gardens	0.02438
30	Go To Jail	0.09457
31	Pacific Avenue	0.02524
32	North Carolina Avenue	0.02472
33	Community Chest	0.02228
34	Pennsylvania Avenue	0.02353
35	Short Line Railroad	0.02291
36	Chance	0.00816
37	Park Place	0.02060
38	Luxury Tax	0.02052
39	Boardwalk	0.02483

Monopoly Strategy

Considerations

- Rent Earnings
- Probability of Landing on Property
- Development Costs

Monopoly Strategy

Analyze by probability of landing on a square for a single turn, not a roll.

$$p(1) = \frac{30}{36}, \qquad p(2) = \frac{6}{36}\left(\frac{30}{36}\right), \qquad p(3) = \frac{6}{26}\left(\frac{6}{36}\right)(1)$$

$$E[X] = 1\left(\frac{30}{36}\right) + 2\left(\frac{6}{36} \cdot \frac{30}{36}\right) + 3\left(\frac{6}{36} \cdot \frac{6}{36} \cdot 1\right) = \frac{43}{36} = 1.19\overline{4}$$

Monopoly Strategy

Consider the following inequality.

$$Revenue \geq Cost$$

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$$p(n) \cdot R \cdot E[X] \cdot Turn \geq Cost$$

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$$p(n) \cdot R \cdot E[X] \cdot Turn \geq Cost$$

$$Turn = \left(\frac{Cost}{p(n) \cdot R \cdot E[X]} \right)$$

Monopoly Strategy

Property	Prob.	E[Rent]	Cost	Turns	E[Rent]	Cost	Turns	E[Rent]	Cost	Turns	E[Rent]	Cost	Turns	E[Rent]	Cost	Turns	E[Rent]	Cost	Turns
Mediterranean Avenue	0.02007	0.05	60	1252	0.24	170	710	0.72	220	306	2.16	270	126	3.84	320	84	5.99	370	62
Baltic Avenue	0.02037	0.10	60	617	0.49	170	350	1.46	220	151	4.38	270	62	7.79	320	42	10.95	370	34
Reading Railroad	0.02801	1.67	200	120	3.35	400	120	5.02	600	120	6.69	800	120						
Oriental Avenue	0.02132	0.15	100	655	0.76	370	485	2.29	420	184	6.88	470	69	10.19	520	52	14.01	570	41
Vermont Avenue	0.02187	0.16	100	639	0.78	370	473	2.35	420	179	7.05	470	67	10.45	520	50	14.37	570	40
Connecticut Avenue	0.02168	0.21	120	580	1.04	370	358	2.85	420	148	7.77	470	61	11.65	520	45	15.54	570	37
St. Charles Place	0.02556	0.31	140	459	1.53	540	354	4.58	640	140	13.74	740	54	19.08	840	45	22.90	940	42
States Avenue	0.02174	0.26	140	540	1.30	540	416	3.90	640	165	11.69	740	64	16.23	840	52	19.48	940	49
Virginia Avenue	0.02426	0.35	160	461	1.74	540	311	5.22	640	123	14.49	740	52	20.28	840	42	26.08	940	37
Pennsylvania Railroad	0.02635	1.57	200	128	3.15	400	128	4.72	600	128	6.29	800	128						
St. James Place	0.02680	0.45	180	402	2.24	660	295	6.40	760	119	17.61	860	49	24.01	960	40	30.41	1060	35
Tennessee Avenue	0.02821	0.47	180	382	2.36	660	280	6.74	760	113	18.53	860	47	25.27	960	38	32.01	1060	34
New York Avenue	0.02812	0.54	200	373	2.69	660	246	7.39	760	103	20.15	860	43	26.87	960	36	33.59	1060	32
Kentucky Avenue	0.02614	0.56	220	392	2.81	830	296	7.81	980	126	21.86	1130	52	27.32	1280	47	32.78	1430	44
Indiana Avenue	0.02567	0.55	220	399	2.76	830	301	7.67	980	128	21.46	1130	53	26.83	1280	48	32.19	1430	45
Illinois Avenue	0.02993	0.71	240	336	3.57	830	233	10.72	980	92	26.81	1130	43	33.07	1280	39	39.32	1430	37
B&O Railroad	0.02893	1.73	200	116	3.46	400	116	5.18	600	116	6.91	800	116						
Atlantic Avenue	0.02537	0.67	260	390	3.33	950	285	10.00	1100	110	24.24	1250	52	29.55	1400	48	34.85	1550	45
Ventnor Avenue	0.02519	0.66	260	393	3.31	950	288	9.93	1100	111	24.07	1250	52	29.34	1400	48	34.60	1550	45
Marvin Gardens	0.02438	0.70	280	401	3.49	950	272	10.48	1100	105	24.75	1250	51	29.85	1400	47	34.94	1550	45
Pacific Avenue	0.02524	0.78	300	383	3.92	1120	286	11.76	1320	113	27.13	1520	57	33.16	1720	52	38.44	1920	50
North Carolina Avenue	0.02472	0.77	300	391	3.84	1120	292	11.52	1320	115	26.57	1520	58	32.48	1720	53	37.65	1920	52
Pennsylvania Avenue	0.02353	0.79	320	407	4.22	1120	266	12.65	1320	105	28.11	1520	55	33.73	1720	51	39.35	1920	49
Short Line Railroad	0.02291	1.37	200	147	2.74	400	147	4.10	600	147	5.47	800	147						
Park Place	0.02060	0.86	350	407	4.31	950	221	12.30	1150	94	27.07	1350	50	31.99	1550	49	36.91	1750	48
Boardwalk	0.02483	1.48	400	270	5.93	950	161	17.79	1150	65	41.52	1350	33	50.42	1550	31	59.32	1750	30

Monopoly Strategy

Color	0	E.Rent	Cost	Turn	1	E.Rent	Cost	Turn	2	E.Rent	Cost	Turn	3	E.Rent	Cost	Turn	4	E.Rent	Cost	Turn	5	E.Rent	Cost	Turn
Purple		0.15	120	827		0.73	220	303		2.18	320	147		6.54	420	65		11.62	520	45		16.94	620	37
Light Blue		0.52	320	620		2.58	470	182		7.49	620	83		21.70	770	36		32.29	920	29		43.91	1070	25
Maroon		0.91	440	483		4.56	740	163		13.69	1040	76		39.91	1340	34		55.59	1640	30		68.45	1940	29
Orange		1.46	560	385		7.29	860	119		20.53	1160	57		56.29	1260	23		76.15	1560	21		96.01	1860	20
Red		1.83	680	372		9.14	1130	124		26.20	1580	61		70.13	2030	29		87.22	2480	29		104.30	2930	29
Yellow		2.03	800	395		10.14	1250	124		30.41	1700	56		73.07	2150	30		88.73	2600	30		104.39	3050	30
Green		2.34	920	394		11.97	1520	127		35.92	2120	60		81.81	2720	34		99.37	3320	34		115.43	3920	34
Dark Blue		2.34	750	320		10.24	1150	113		30.10	1550	52		68.59	1950	29		82.41	2350	29		96.22	2750	29
Railroad		1.73	200	116		6.80	400	59		14.92	600	41		25.37	800	32								

Monopoly Strategy

Color	Investment	Turn
Orange	Hotel	20
Light Blue	Hotel	25
Dark Blue	3 House	29
Maroon	3 House	29
Red	3 House	29
Yellow	3 House	30
Railroad	All 4	32
Green	3 Houses	34
Purple	Hotel	37

Jorg Bewersdorff, *Luck, Logic and White Lies: The Mathematics of Games*, A K Peters (2005), 106-120.

J. Laurie Snell *Finite Markov Chains and their Applications*, The American Mathematical Monthly (1959), 66 (2), 99-104.