A Generalization of Pascal's Triangle

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Pascal Triangles

$n = 0$:					1				
$n = 1$:				1		1			
$n = 2$:			1		2		1		
$n = 3$:		1		3		3		1	
$n = 4$:	1		4		6		4		1

Pascal Triangles

$$\binom{0}{0}$$
$$\binom{1}{0}\binom{1}{1}$$
$$\binom{2}{0}\binom{2}{1}\binom{2}{2}$$
$$\binom{3}{0}\binom{3}{1}\binom{3}{2}\binom{3}{3}$$
$$\binom{4}{0}\binom{4}{1}\binom{4}{2}\binom{4}{3}\binom{4}{4}$$

Any entry in Pascal's Triangle can be defined by the binomial coefficients.

$$T_{n,k} = \binom{n}{k}$$

Properties

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Generalization

a				
b	a			
b	a+b	a		
b	a+2b	2a+b	a	
b	a+3b	3a+3b	3a+b	a

Generalization

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In the blue, the coefficients of $b = \binom{n}{k}$, shifted down one row.
In the red, the coefficients of $a = \binom{n}{k}$, shifted down and to the the right one row and column.

Shifting a value down one row is the same as $\binom{n-1}{k}$.
Shifting a value to the right is the same as $\binom{n}{k-1}$. Combining these gets us this:

$$T_{n,k} = a\binom{n-1}{k-1} + b\binom{n-1}{k} \text{ for } n, k > 0$$

$$T_{n,0} = b \text{ for } n > 0$$

$$T_{0,0} = a$$

Testing

a				
b	a			
b	a+b	a		
b	a+2b	2a+b	a	
b	a+3b	3a+3b	3a+b	a

$$T_{4,2} = 3a + 3b$$

$$T_{4,2} = a\binom{4-1}{2-1} + b\binom{4-1}{2}$$

$$\binom{3}{1} = 3 \text{ and } \binom{3}{2} = 3$$

$$\therefore\ T_{4,2} = 3a + 3b$$

Testing the general case

When a and $b = 1$, $T_{n,k}$ should be equal to $\binom{n}{k}$
Known:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

by definition of Pascal's Triangle.

$$T_{n,k} = a\binom{n-1}{k-1} + b\binom{n-1}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \text{ for } a, b = 1$$

Therefore:
For $a, b = 1$, $T_{n,k} = \binom{n}{k}$

Mathematica evaluation

```
a = a;
b = b;
p[n_,k_] := a Binomial[n - 1, k - 1] + b Binomial[n - 1, k];
p[0,0] = a;
Column[Table[p[n,k],{n,0,10},{k,0,n}] Left]
Do[Print[Sum[p[n-i,i],0,Floor[n/2]]]{n,0,5}]
```

generates the following:

```
{a}
{b, a}
{b, a + b, a}
{b, a + 2 b, 2 a + b, a}
{b, a + 3 b, 3 a + 3 b, 3 a + b, a}
{b, a + 4 b, 4 a + 6 b, 6 a + 4 b, 4 a + b, a}
```

Replacing *a* and *b* with numbers will generate the appropriate values instead of the generalization.

Generalizations of figurative numbers - Triangular Numbers

Known:

$$\triangle_{n-1} = \frac{n(n-1)}{2}$$

$$\triangle_{n-1} = \binom{n}{2} \text{ for } n \geq 2$$

Generalized:

$$T_{n,2} = a\binom{n-1}{1} + a\binom{n-1}{2} = a(n-1) + b\binom{n-1}{2}$$

$$a(n) + b\binom{n}{2} = \triangle_{n_{a,b}} \text{ for } n \geq 2$$

Generalizations of figurative numbers - Tetrahedral Numbers

Known: $h_n = \binom{n+3}{3}$

$$h_n = \binom{n+3}{3} \text{ for}$$

$$h_n = \frac{n(n+1)(n+2)}{6}$$

Generalized:

$$T_{n+3,3} = a\binom{n+2}{2} + b\binom{n+2}{3}$$

Future avenues of research

- Diagonal sums
- Properties of generalized figurative numbers
- Generalized proof