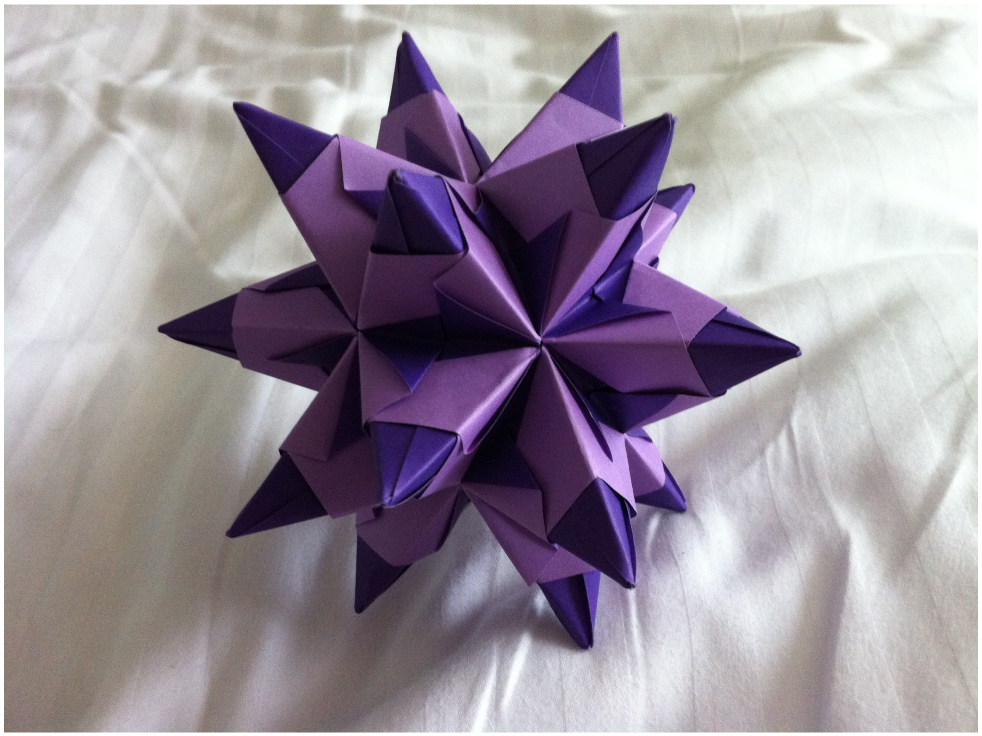
What lurks beneath the folds?

The mathematics of folding on a grid

Ben MacDonald, Massachusetts Academy of Math and Science at WPI
Professor Cristina Ballantine, College of the Holy Cross

History



Some Terminology

- Stack
- Point of View of the Folder
- Fold
- Step
- Cell Sequence
- Fold Sequence
- Family

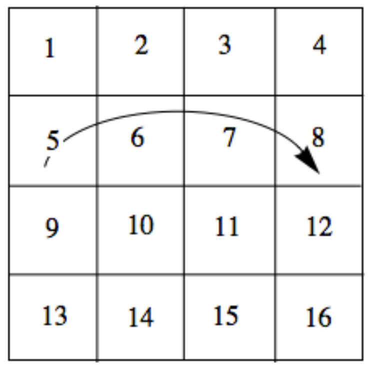
$2^{n-i} \times 2^{n-i} \times 4^i$ for $n = 2$ and $i = 0$

1α	2α	3α	4α
5α	6α	7α	8α
9α	10α	11α	12α
13α	14α	15α	16α

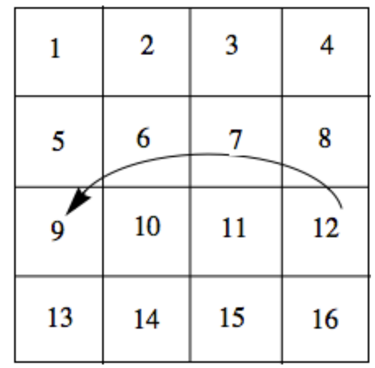
Influence of Grid Size

n	Permutations $(4^n)!$	Sequences 32^n	Distinct Sequences 16^n	Families 2^{4n-2}
0	1	1	1	0.25
1	24	32	16	4
2	2.09E+13	1024	256	64
3	1.27E+89	32768	4096	1024

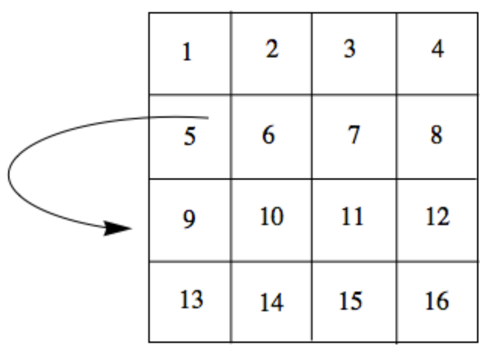
Vertical Folds



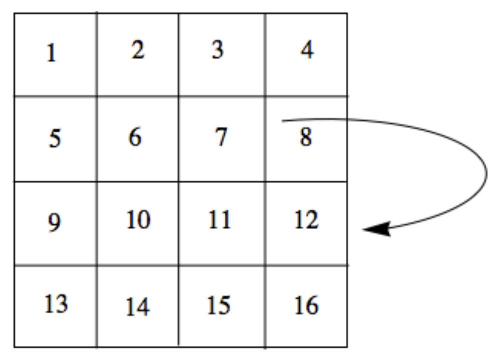
L+



R+

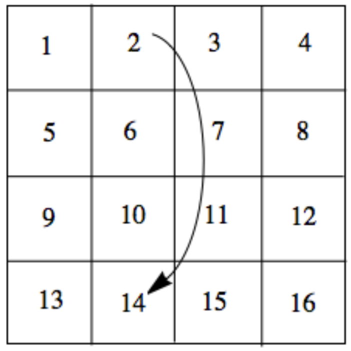


L–

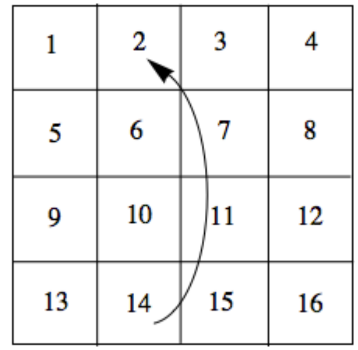


B–

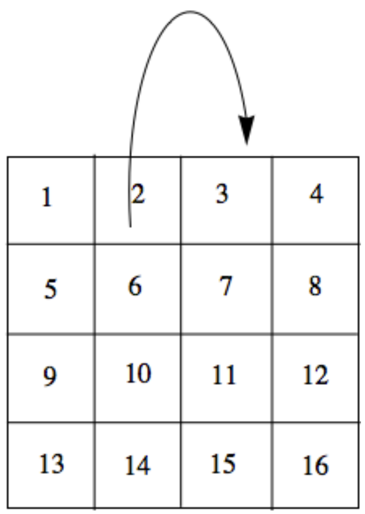
Horizontal Folds



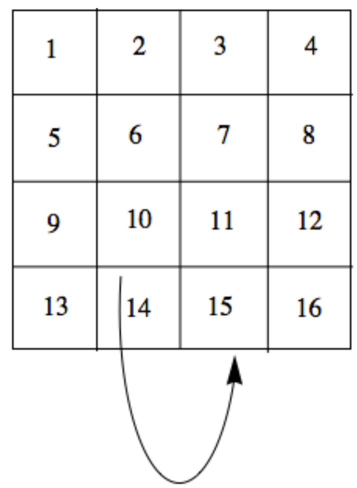
T+



B+



T−



B−

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- Family

Fold Sequences by Family for $n = 1$

n = 1		
Fold Sequence	Fold Sequence	Final Cell Sequence
L+T+	R+B–	2β, 1α, 3β, 4α
L+B+	R+T–	4β, 3α, 1β, 2α
L–T–	R–B+	4α, 3β, 1α, 2β
L–B–	R–T+	2α, 1β, 3α, 4β
L+B–	R+T+	1β, 2α, 4β, 3α
L+T–	R+B+	3β, 4α, 2β, 1α
L–B+	R–T–	3α, 4β, 2α, 1β
L–T+	R–B–	1α, 2β, 4α, 3β
T+L+	B+L–	3β, 2α, 1β, 3α
T+R+	B+R–	4β, 1α, 2β, 4α
T–R+	B–L–	4α, 2β, 1α, 3β
T–L+	B–R–	3α, 1β, 2α, 4β
T+L–	B+R+	2β, 4α, 3β, 1α
T+R–	B+L+	1β, 3α, 4β, 2α
T–R–	B–L+	1α, 3β, 4α, 2β
T–L–	B–R+	2α, 4β, 3α, 1β

Transformations

- Sequence Preserving Transformations
- Sequence Reversing Transformations
- α/β Reversing Transformations
- Numerical Reversing Transformations
- Composition Rules

Sequence Preserving Transformations

n = 1		
Fold Sequence	Fold Sequence	Final Cell Sequence
L+T+	R+B–	2β, 1α, 3β, 4α

$$\text{L+T+} \xleftrightarrow{\text{CSCC}} \text{R+B–}$$

Sequence Reversing Transformations

n = 1		
Fold Sequence	**Fold Sequence**	**Final Cell Sequence**
L+T+	R+B–	2β, 1α, 3β, 4α
L–T–	R–B+	4α, 3β, 1α, 2β

$$L{+}T{+} \xleftrightarrow{\text{SCSC}} L{-}T{-}$$

α/β Reversing Transformations

n = 1		
Fold Sequence	Fold Sequence	Final Cell Sequence
L+T+	R+B–	2β, 1α, 3β, 4α
L–B–	R–T+	2α, 1β, 3α, 4β

L+T+ ←SCCC→ L–B–

Numerical Reversing Transformations

n = 1		
Fold Sequence	**Fold Sequence**	**Final Cell Sequence**
L+T+	R+B–	2β, 1α, 3β, 4α
L+B+	R+T–	4β, 3α, 1β, 2α

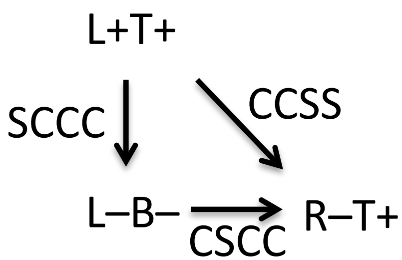
SSCS

L+T+ ⟷ L+B+

Composing Transformations

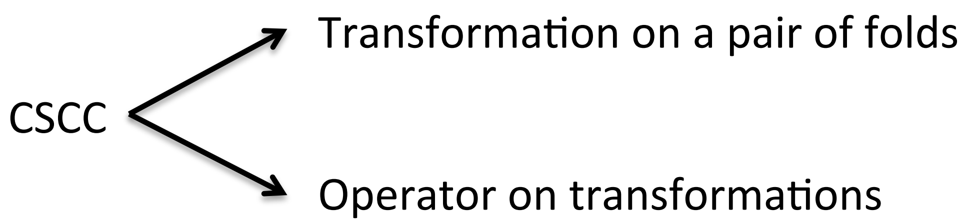
CSCC ∘ SCCC = CCSS

$$\begin{array}{rcccc} & C & S & C & C \\ + & S & C & C & C \\ \hline & C & C & S & S \end{array}$$



∘ is commutative!

∘ binary addition



Applications

- Stacked computer chips
- Solar panels (space)
- Encryption
- Air bags
- Threading/computers

References

- Legendre, S. (2013). Foldings and Meanderings. arXiv:1302.2025v1 [math.CO] 8 Feb 2013. Retrieved February 10, 2013 from http://arxiv.org/pdf/1302.2025v1.pdf
- Hull, Thomas (2006). Project Origami:Activities for Exploring Mathematics. A K Peters, Ltd. ISBN 978-1-56881-258-8.
- Robert Lang's website: langorigami.com
- Pictures:
  - http://spaceports.blogspot.com/2007_11_01_archive.html
  - http://www.wired.com/dangerroom/2013/02/icecool/

Properties

P1. The grid, or piece of paper *A,* is a stack with dimensions $2^{n-i} \times 2^{n-i} \times 4^i$.

P2. For any given *n,* the number of folds performed to transform the piece of paper into a final stack will be $2n$.

P3. When a grid has been completely folded into a final stack, $i = n$.

Properties

P4. For all *n*, there exist eight possible first folds and four possible second folds for each step.

P5. Applying one step (which is two folds, one across each axis) to *A* will increment the index *i* by one and restore the front face of the grid to a square.

Properties

P6. The fold sequence is a sequence of length 2*n* where each pair of elements consists of one element from the set {L+, L–, R+, R–} and another element from the set {T+, T–, B+, B–}. It doesn't matter which set is selected from first.

P7. The same sequence can result from multiple folding sequences.

Theorems

T1. For any *n*, folding the grid allows for a total of 32^n sequences to be generated.

T2. The α and β for any final cell sequence will always be alternating.

T3. A final stack can start with either α or β and the very first fold determines the α/β sequence.

Theorems

T4. For any n, there are 2^n ways to generate a distinct sequence. This also means that there are 16^n distinct sequences.

T5. For any n, there exist 2^{4n-2} possible families.

T6. For n=1, every member of a family starts with either {L, R} or {T, B}. In other words, all fold sequences of a family are generated by an initial fold that is always over the same axis.

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Theorems

T7. For *n*=1, pairs of fold sequences related by the transformation CSCC generate identical final cell sequences.

T8. For *n*=1, pairs of fold sequences related by the transformation SCCC generate two final cell sequences that have the same cell number ordering but have swapped α and β. In other words, the cell number ordering is preserved whereas the α/β sequence is reversed.

Theorems

T9. For n=1, pairs of fold sequences related by the transformation SSCS generate two final cell sequences that have the same α/β sequence but the cell number ordering is reversed. In other words, α/β sequence is preserved whereas the cell number ordering is reversed.

T10. For n=1, pairs of fold sequences related by the transformation SCSC generate two final cell sequences that have the reversed α/β sequence and the cell number ordering is reversed. In other words, the α/β sequence and the cell numbering order are reversed.

Theorems

T11. The results of applying multiple transformations to a fold sequence follow the addition properties described on the next slide.