# Examination of Public Goods in Networks

Qasim Mahmood

Bates College

April 6, 2013

# Presentation Outline

# What are graphical games?

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- Interaction of individuals (agents) with one another in a way that their behaviors and payoffs are affected by other agents
- Graphs provide a more compact representation of multiplayer games where the typical matrix representation becomes undesirable

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**Real life examples of Public Goods**

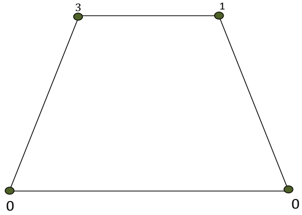
- Cybersecurity: Muller and Schneider (2011) provide a framework for viewing internet security as a public good to better understand the inherent risks in cyberspace
- Innovation: Bramoulle and Kranton (2006) classify innovation by firms as a public good and study how provision of public goods affects the incentive of firms to innovate

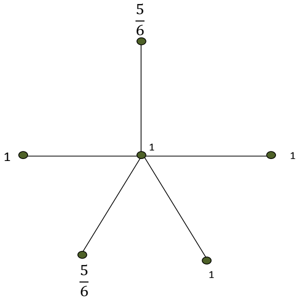
# Presentation Outline

# Model for Undirected Networks (Bramoulle and Kranton 2006)

- Let there be $n$ agents in a network given by $N = \{1, \ldots, n\}$ which are represented by $n$ vertices of a graph $\mathbf{g}$
- The effort of each agent is given by $e_i \in [0, \infty)$
- The effort profile is the $n$-tuple, $\mathbf{e} = (e_1, \ldots, e_n)$
- If agent i's payoff is affected by agent j's effort and vice versa, i and j are joined by an edge and the matrix representing $\mathbf{g}$ has entry $g_{ij} = 1$
- Agents who are directly affected as a result of i's efforts are called i's neighbors, i.e. $N_i = \{j \in N : g_{ij} = 1\}$
- The neighborhood of i is defined as $\{i\} \cup N_i$

# Some Examples



# Agent's Payoff Function (Bramoulle and Kranton 2006)

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$$U_i(\mathbf{e};\mathbf{g}) = b\left(e_i + \sum_{j\in N_i} e_j\right) - ce_i$$

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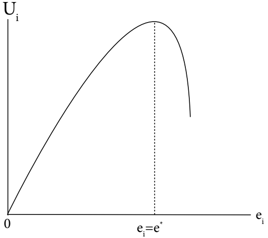

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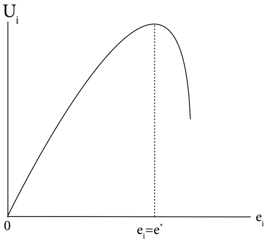


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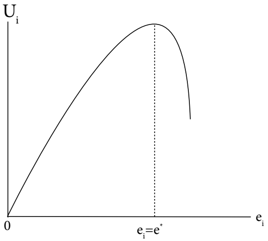


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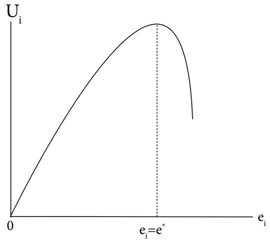


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- Agents' payoff is maximized at an effort level $e^*$ where Marginal Benefit equals Marginal Cost i.e. $b^{'}(e^*) = c$

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- Hybrid: A profile where every agent $i$ exerts an effort $e_i$ such that $0 \leq e_i \leq e^*$

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**Definition of Nash Equilibrium**

An effort profile $\mathbf{e}$ is a Nash Equilibrium if and only if for every agent $i$ either (1) $\overline{e_i} \geq e^*$ and $e_i = 0$ or (2) $\overline{e_i} \leq e^*$ and $e_i = e^* - \overline{e_i}$ (Bramoulle and Kranton 2006).

# Equilibria on complete graphs

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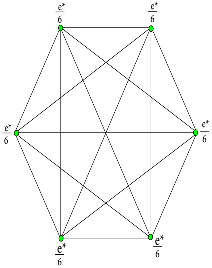


Distributed Equilibrium

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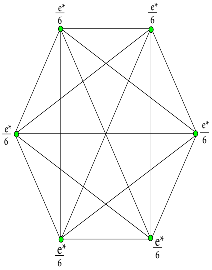


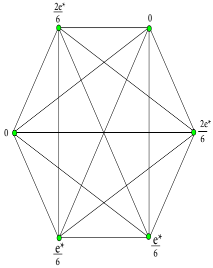


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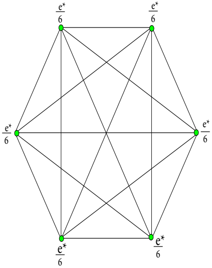


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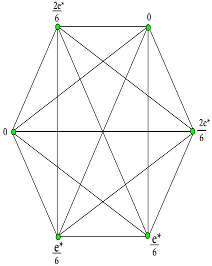


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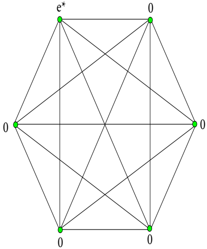


Specialized Equilibrium

# Maximal Independent Sets

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**Order of maximal independent sets**

Given a graph **g**, a *maximal independent set of order r* is defined as a maximal independent set $I$ such that every agent $i \notin I$ is adjacent to at least $r$ agents in $I$ and $r$ is the maximum number of agents in $I$ who are adjacent to every agent who is not in $I$ (Bramoulle and Kranton 2006).

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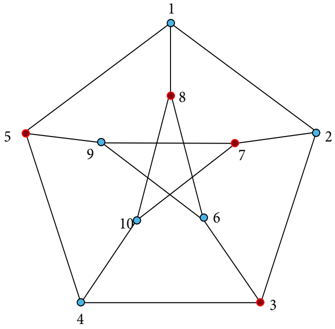
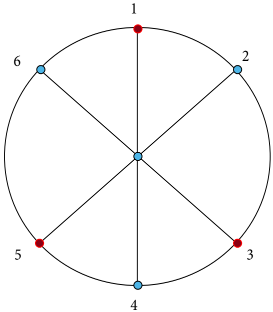
# Examples of Maximal Independent Sets




Peterson Graph




Random Graph

# Maximal Independent Sets Continued

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**Theorem**

*Let $I$ be a maximal independent set in a graph $\mathbf{g}$. Then every agent $i$ is either in $I$ or is adjacent to an agent $j$ who belongs to $I$ (Bramoulle and Kranton 2006).*

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- **Proof by contradiction:**
- Assume there exists an agent $i$ such that this agent is neither in $I$ nor is adjacent to any agent $j \in I$
- This implies that the set $I \cup \{i\}$ is an independent set.
- Therefore, $|I \cup \{i\}| > |I|$ which contradicts our assumption that $I$ is a maximal independent set

# Existence of Specialized Equilibria

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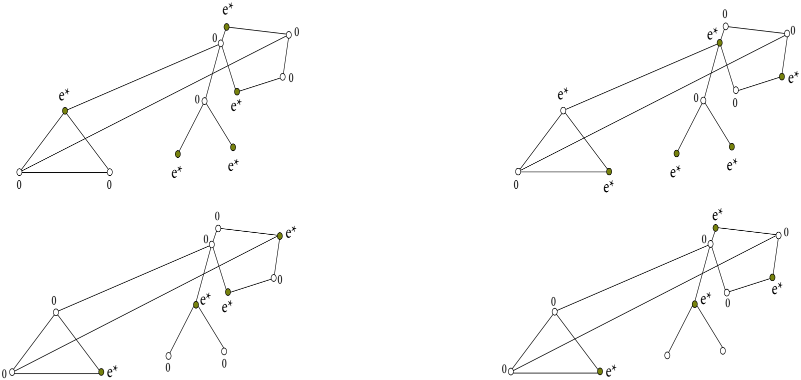
**Theorem**

*A specialized profile is a Nash Equilibrium if and only if its set of specialists is a maximal independent set of the structure* $\mathbf{g}$ *(Bramoulle and Kranton 2006).*

# Maximal Independent Sets and Specialized Equilibria



The same graph can have multiple maximal independent sets and hence multiple specialized equilibria

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- Researchers have concluded that finding approximate optimal equilibria is not difficult (Asta, Pin and Ramezanpour (2010))

Thank You For Listening!