Frieze Patterns and Triangulated Polygons

Jean-Francois Marceau

Bishop's University

jmarceau10@ubishops.ca

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Reference

Triangulated Polygons and Frieze Pattern

J.h. Conway and H.S.M. Coxeter
Math. Gaz. 1976

Definition

Frieze Pattern

A *Frieze Pattern* is an array of natural numbers, displayed on shifted lines (like a brick wall) such that the top and bottom lines are composed only of 1's and for each unit diamond:

$$\begin{array}{ccc} & a & \\ b & & c \\ & d & \end{array}$$

the following rule hold:

$$(bc) - (ad) = 1.$$

Examples

Example 1

...	1		1		1		1		1		1		1		1		1		1		1		1	...
	...	1		2		3		1		3		1		4		1		...						
...	3		1		5		2		2		2		3		3		1		...					
	...	2		2		3		3		1		5		2		2		...						
...	1		3		1		4		1		2		3		1		3		...					
	...	1		1		1		1		1		1		1		1		...						

Example 2

	...	1		1		1		1		1		1		1		1		1		...				
		...	1		2		1		2		1		2		...									
	...	1		1		1		1		1		1		1		1		...						

Frieze Pattern Properties

Property 1

Every pattern of order n (i.e. with n lines) admits an horizontal translation of length $n+1$.

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Property 3

Every pattern of order n is completely determined by a sequence of $n+1$ numbers.

Which Secences Are Valid?

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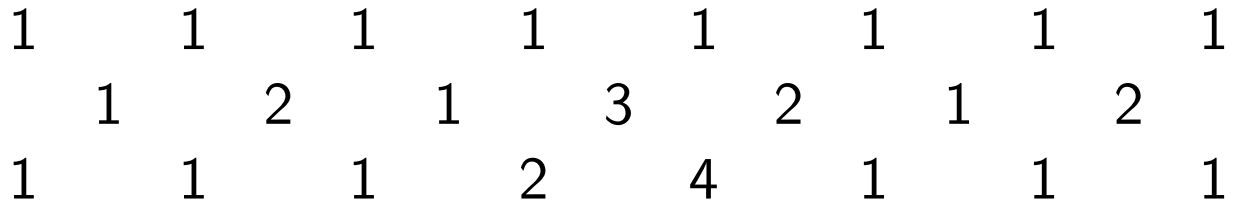

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Try a random sequence, eg: 1,2,1,3,2

1		1		1		1		1		1		1		1
	1		2		1		3		2		1		2	
1		1		1		2		4		1		1		1
	0		0		1		7/3		3/2		0		0	

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1		1		1		1		1		1		1		1					
	1		2		1		3		2		1		2						
1		1		1		2		4		1		1							
	0		0		1		7/3		3/2		0		0						
−1		−1		−1		2/3		5/8		−1		−1							
	?		?		−5/3		−1/4		−13/12		?		?						

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1		1		1		1		1		1		1		1					
	1		2		1		3		2		1		2						
1		1		1		2		4		1		1							
	0		0		1		7/3		3/2		0		0						
−1		−1		−1		2/3		5/8		−1		−1							
	?		?		−5/3		−1/4		−13/12		?		?						

But using 1,2,2,1,3 instead, we get:

	1		1		1		1		1		1		1		1				
		1		2		2		1		3		1		2					
	2		1		3		1		2		2		1		3				
		1		1		1		1		1		1		1					

How to Find a Valid Sequence

Theorem (Conway and Coxeter)

There is a bijection between the valid sequences for frieze patterns and the number of triangles adjacent to the vertices of a triangulated polygon.

Thanks!