Quadratic Forms and their Application to Whale Localization

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Introduction

Topics:

1. Quadratic Forms
2. Principle Axis Theorem
3. An Example
4. Three Component Analysis of Ellipsoids
5. Ocean Bottom Seismometers
6. A Three Component Analysis Example

Quadratic Forms as Polynomials and Square Matrices

- An expression in the form

$$ax^2 + by^2 + cxy = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c/2 \\ c/2 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

is called a **quadratic form** in 2 variables; x and y. A quadratic form in n variables is a function $f : \mathbb{R}^n \to \mathbb{R}$ of the form

$$f(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$$

where A is a symmetric $n \times n$ matrix and $\mathbf{x}$ is in $\mathbb{R}^n$.

Ellipses

- An ellipse is in **standard position** to the coordinate axes if its equation can be expressed as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1$$

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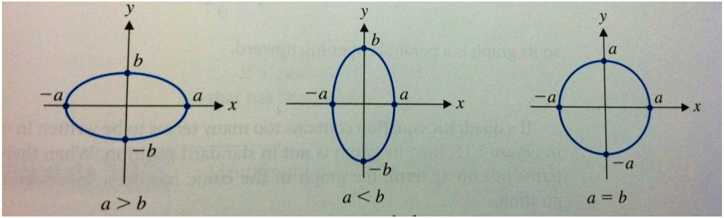

Definitions

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- An orthogonal matrix, Q, **diagonalizes** a symmetric $n \times n$ matrix, A if

$$Q^T A Q = D$$

where D is a diagonal matrix.

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If

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- Note that if the eigenvalues of A are $\lambda_1, \ldots, \lambda_n$ and $\mathbf{y} = [y_1, \ldots, y_n]^T$, then

$$\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T D\mathbf{y} = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2.$$

A Quick Example

- Let's look at the equation $5x^2 + 2y^2 + 4xy = 6$. The left hand side of this equation is a quadratic form that can be written as $\mathbf{x}^T A\mathbf{x} = 6$ where

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- The diagonal matrix of eigenvalues is therefore

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}.$$

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- The corresponding eigenvectors are

$$\mathbf{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} \end{bmatrix}, \mathbf{q}_2 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}.$$

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- We construct Q using these eigenvectors

$$Q = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

to orthogonally diagonalize A as $Q^T AQ = D$.

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- By substituting $\mathbf{x} = Q\mathbf{x}'$ into $\mathbf{x}^T A\mathbf{x} = 6$ we have $\mathbf{x}'^T Q^T AQ\mathbf{x}' = 6$.

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- If
$$\mathbf{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$
then $(x')^2 + 6(y')^2 = 6$ or
$$\frac{(x')^2}{6} + (y')^2 = 1$$
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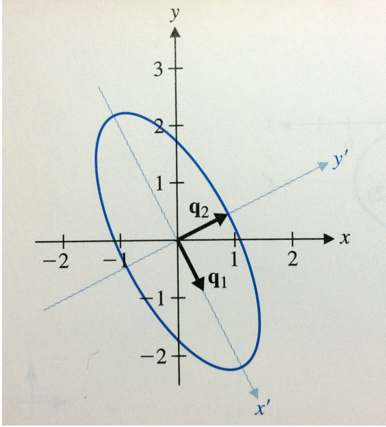
- The eigenvectors of A determine the positions of the $\mathbf{x}'$ and $\mathbf{y}'$ axes.

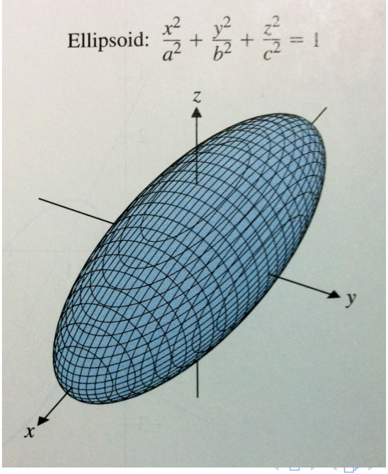
A Quick Example

The graph of $5x^2 + 4xy + 2y^2 = 6$ is:



Three-Dimensional Quadratic Forms

Quadratic forms can also be applied to three-dimensional structures such as an ellipsoid:



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- Current research utilizes a number of ocean-bottom seismometers off of the coast of New Zealand.

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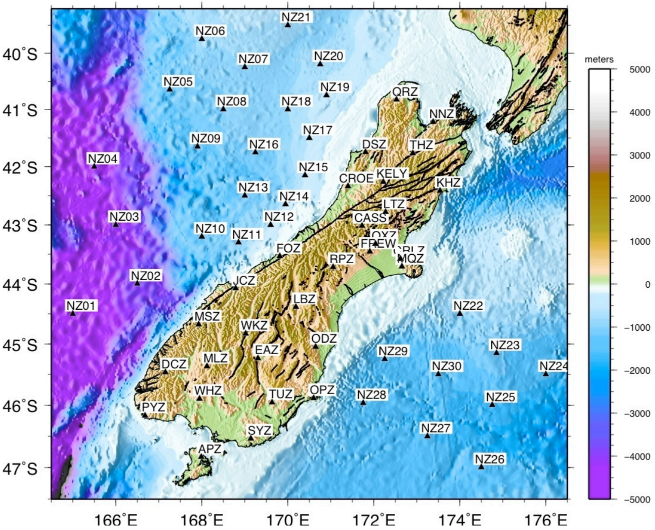

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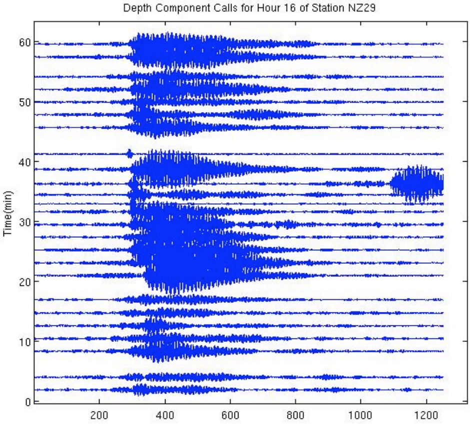

Covariance Matrix

- The covariance matrix, $\mathbf{S}$ is 3×3, real, and symmetric.

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- The terms of **S** are auto- and cross-variances of the three components (north, east, and depth).
- **S** is evaluated as

$$S_{jk} = \frac{\mathbf{X}\mathbf{X}^T}{N} = \left[\frac{1}{N} \sum_{i=1}^{N} x_{ij} x_{ik} \right]$$

where $\mathbf{X} = [x_{ij}], i = 1, \cdots, N$, and $j = 1, 2, 3$. The i^{th} sample of component j is x_{ij}, and N is the number of samples.

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- In many real applications, all three eigenvalues are nonzero and unequal.

Azimuth of a Signal

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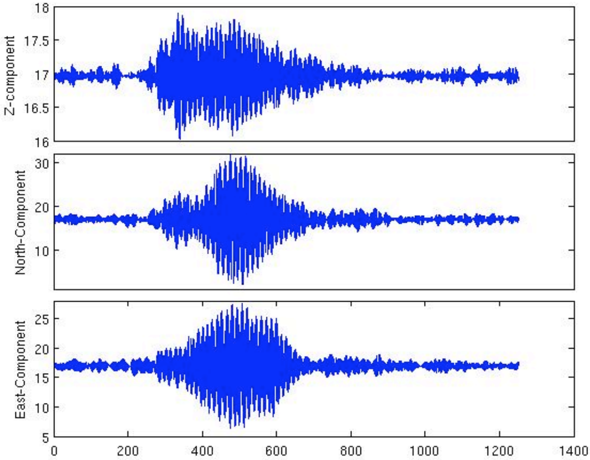
- We can find the azimth for the direction the signal came from using

$$\theta_{azimuth} = tan^{-1}\left(\frac{u_{21}}{u_{31}}\right)$$

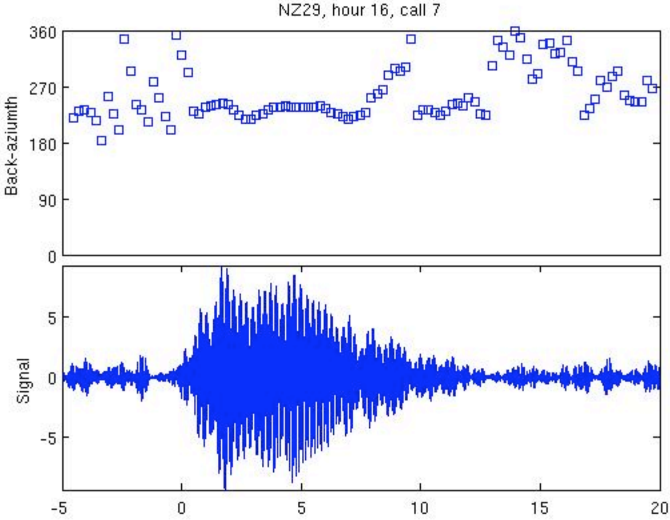
where $\mathbf{u}_{j1}$ are the j^{th} components of eigenvector $\mathbf{u}_1$.

Application to Whale Localization

Three Component Directions of station NZ29, hour 16, call 7



Three Component Analysis of Whale Calls



Questions

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- Thank you for your time. You have been a great audience.

Bibliography

Bibliography

- Jurkevics, Andy. "Polarization Analysis of Three-Component Array Data." Seismological Society of America 78 (1988): 1725-743.
- Poole, David. "Section 5.5 Applications." Linear Algebra: A Modern Introduction. 2nd ed. N.p.: Thomson Brooks/Cole, 2006. 411-26.