Generalized Splines

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What is a Spline?

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- Splines are used in engineering to represent objects.

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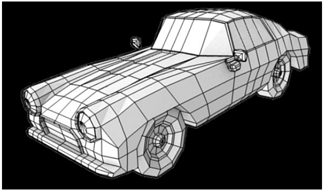

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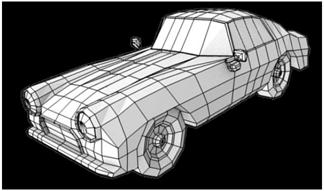

- We will use splines to graphically represent systems of congruences.

What is a System of Congruences?

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- We say $x \equiv y \mod a$ if $x - y$ is a multiple of n.

Example

$$3 \equiv 13 \mod 5 \text{ because } 13 - 3 = 10 = 2 \cdot 5$$

What is a System of Congruences?

- We say $x \equiv y \mod a$ if $x - y$ is a multiple of n.

Example

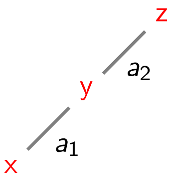
$$3 \equiv 13 \mod 5 \text{ because } 13 - 3 = 10 = 2 \cdot 5$$

- A spline is a graphical representation of a system of congruences. We label the nodes of the graph with the variables and the edges with the moduli. Below is the spline for the above congruence.

Paths with 3 Vertices

System of Congruences

$$x \equiv y \mod a_1$$
$$y \equiv z \mod a_2$$



Why is this useful?

System of Congruences

$$
\begin{array}{lll}
x_1 \equiv x_2 \mod a_1 & x_2 \equiv x_3 \mod a_2 & x_3 \equiv x_4 \mod a_3 \\
x_4 \equiv x_5 \mod a_4 & x_5 \equiv x_6 \mod a_5 & x_6 \equiv x_1 \mod a_6 \\
x_5 \equiv x_1 \mod a_7 & x_5 \equiv x_2 \mod a_8 & x_5 \equiv x_3 \mod a_9 \\
x_4 \equiv x_2 \mod a_{10} & x_4 \equiv x_1 \mod a_{11} & x_4 \equiv x_6 \mod a_{12}
\end{array}
$$

Why is this useful?

System of Congruences

$$x_1 \equiv x_2 \mod a_1 \quad x_2 \equiv x_3 \mod a_2 \quad x_3 \equiv x_4 \mod a_3$$
$$x_4 \equiv x_5 \mod a_4 \quad x_5 \equiv x_6 \mod a_5 \quad x_6 \equiv x_1 \mod a_6$$
$$x_5 \equiv x_1 \mod a_7 \quad x_5 \equiv x_2 \mod a_8 \quad x_5 \equiv x_3 \mod a_9$$
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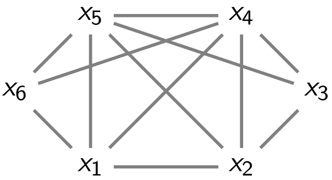


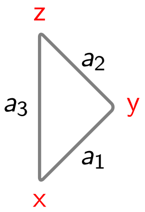
Main Questions

- Can we list all possible splines of a certain shape?
- Can we find a basis for the set of triangular splines?
- What can we find out about splines other than cycles?

General Triangle

System of Congruences

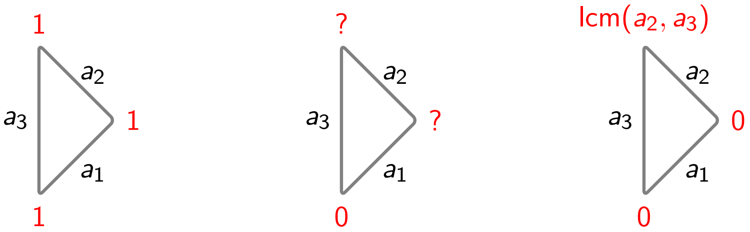
$$x \equiv y \mod a_1 \qquad y \equiv z \mod a_2 \qquad z \equiv x \mod a_3$$



Triangle Solutions

System of Congruences

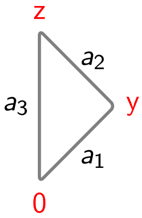
$$x \equiv y \mod a_1 \qquad y \equiv z \mod a_2 \qquad z \equiv x \mod a_3$$



Existence of Solution as Application of CRT

System of Congruences

$$0 \equiv y \mod a_1 \qquad y \equiv z \mod a_2 \qquad z \equiv 0 \mod a_3$$



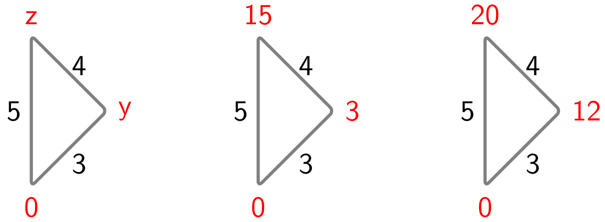
Theorem

There is a minimal value for y such that a spline of this form exists.

Existence of Solution as Application of CRT

System of Congruences

$$0 \equiv y \mod 3 \qquad y \equiv z \mod 4 \qquad z \equiv 0 \mod 5$$



Chinese Remainder Theorem

System of Congruences

$$x \equiv a_1 \mod n_1$$
$$x \equiv a_2 \mod n_2$$

There exists a solution x if the integers n_1, n_2 are coprime.
If $\gcd(n_1, n_2) \neq 1$ a solution x exists if and only if $a_1 \equiv a_2 \mod \gcd(n_1, n_2)$.

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Existence of Solution as Application of CRT

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$$y \equiv 0 \mod \gcd(a_2, a_3)$$

- Thus $y = k \gcd(a_2, a_3)$ for some $k \in \mathbb{Z}$.

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- Recall $y = k \gcd(a_2, a_3)$ for some $k \in \mathbb{Z}$.
- We want to minimize $y \ldots$

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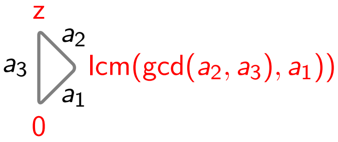
- Recall $y = k \gcd(a_2, a_3)$ for some $k \in \mathbb{Z}$.
- We want to minimize $y \ldots$
- Let $y = \operatorname{lcm}(\gcd(a_2, a_3), a_1)$.

Existence of Solution as Application of CRT



Next we check that a spline exists:

- We must have a solution to the system of congruences

$$z \equiv \operatorname{lcm}(\gcd(a_2, a_3), a_1)) \mod a_2$$
$$z \equiv 0 \mod a_3$$

- By the Chinese Remainder Theorem, we have a solution z if and only if

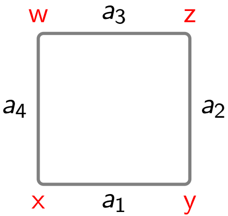
$$\operatorname{lcm}(\gcd(a_2, a_3), a_1)) \equiv 0 \mod \gcd(a_2, a_3).$$

General Square

System of Congruences

$$x \equiv y \mod a_1 \qquad y \equiv z \mod a_2$$

$$z \equiv w \mod a_3 \qquad w \equiv x \mod a_4$$



Square Solutions

System of Congruences

$$x \equiv y \mod a_1 \qquad y \equiv z \mod a_2$$

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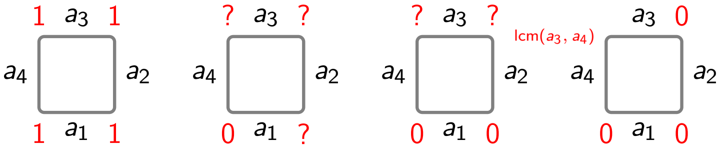


Square Solutions

System of Congruences

$$0 \equiv 0 \mod a_1 \qquad 0 \equiv z \mod a_2$$
$$z \equiv w \mod a_3 \qquad w \equiv 0 \mod a_4$$

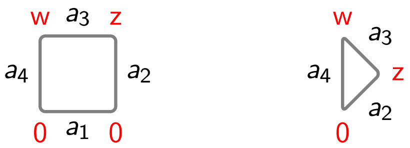
Rewrite this as...

Square Solutions

System of Congruences

$$0 \equiv z \mod a_2 \qquad z \equiv w \mod a_3 \qquad w \equiv 0 \mod a_4$$

This is essentially the same as a triangle.



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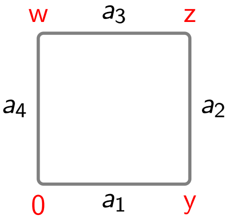

Bases: Can we find a basis for the set of triangular splines?

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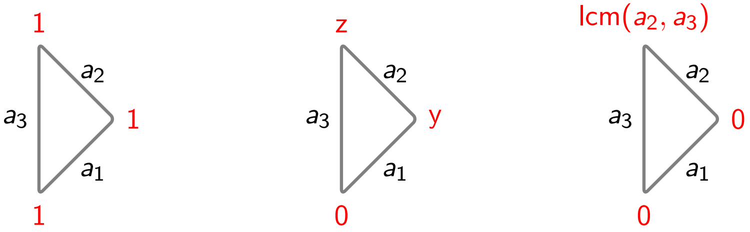
Bases: What Solutions Have We Mentioned So Far?



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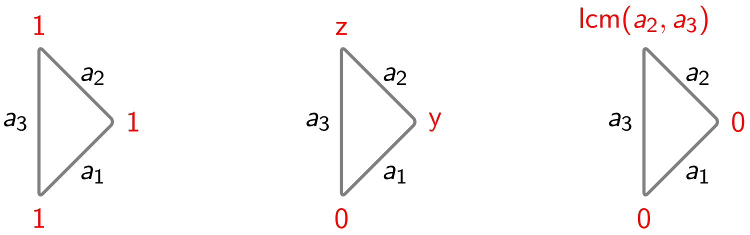

In the center triangle, y is the minimal solution stated in the theorem we mentioned earlier. $y = \text{lcm}(a_1, \gcd(a_2, a_3))$

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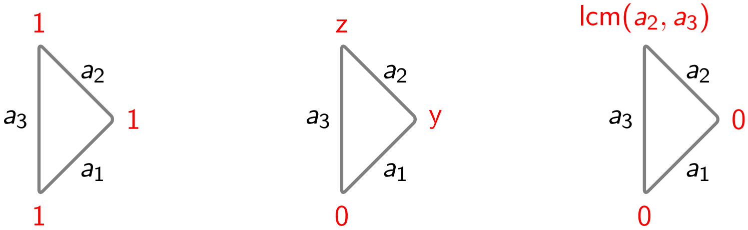

In the center triangle, y is the minimal solution stated in the theorem we mentioned earlier. $y = \text{lcm}(a_1, \text{gcd}(a_2, a_3))$
We can write each of these solutions as a vector:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} z \\ y \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \text{lcm}(a_2, a_3) \\ 0 \\ 0 \end{pmatrix}$$

Bases: Do these vectors form a basis?

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Do these vectors form a basis?

- Goal 1: Show that every linear combination of these vectors is a solution.

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Do these vectors form a basis?

- Goal 1: Show that every linear combination of these vectors is a solution.
- Goal 2: Show that every solution can be written as a linear combination of these vectors.

Goal 1: Show that every linear combination of these vectors a solution

$$\alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

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$$\alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} z \\ y \\ 0 \end{pmatrix}$$

Goal 1: Show that every linear combination of these vectors a solution

$$\alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} z \\ y \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} \operatorname{lcm}(a_2, a_3) \\ 0 \\ 0 \end{pmatrix}$$

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$$= \begin{pmatrix} \alpha + \beta z + \gamma \operatorname{lcm}(a_2, a_3) \\ \alpha + \beta y \\ \alpha \end{pmatrix}$$

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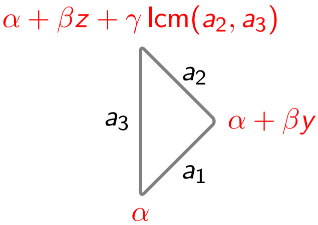

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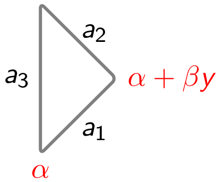
Is this a solution?

$\alpha + \beta z + \gamma \, \text{lcm}(a_2, a_3)$

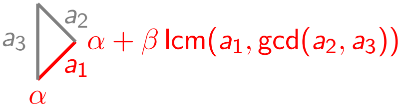


Remember: $y = \text{lcm}(a_1, \text{gcd}(a_2, a_3))$

Goal 1: Show that every linear combination of these vectors a solution

Check edge a_1:

$\alpha + \beta z + \gamma \operatorname{lcm}(a_2, a_3)$



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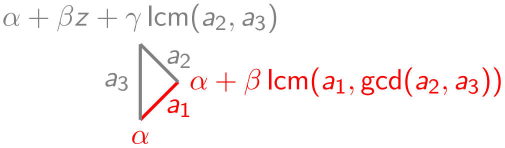

Need to show: $\alpha + \beta \operatorname{lcm}(\gcd(a_2, a_3), a_1) \equiv \alpha \mod a_1$

Goal 1: Show that every linear combination of these vectors a solution

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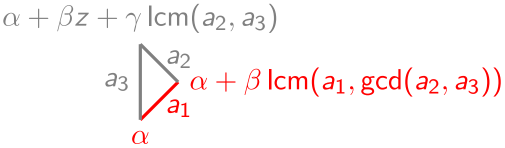

Need to show: $\alpha + \beta \operatorname{lcm}(\gcd(a_2, a_3), a_1) \equiv \alpha \mod a_1$

We can write $\operatorname{lcm}(\gcd(a_2, a_3), a_1)$ as some multiple k of a_1. Then we have

$$\alpha + \beta k a_1$$

Goal 1: Show that every linear combination of these vectors a solution

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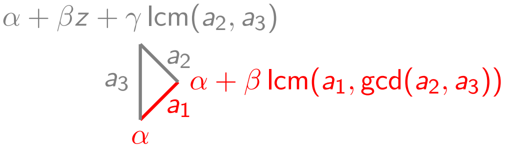

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We can write $\operatorname{lcm}(\gcd(a_2, a_3), a_1)$ as some multiple k of a_1. Then we have

$$\alpha + \beta k a_1 = \alpha + n a_1 \text{ for some integer } n.$$

Goal 1: Show that every linear combination of these vectors a solution

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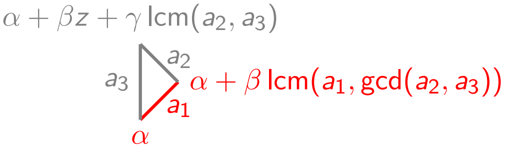

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$$\alpha + \beta k a_1 = \alpha + n a_1 \text{ for some integer } n.$$

Thus $\alpha + \beta \operatorname{lcm}(\gcd(a_2, a_3), a_1) \equiv \alpha \mod a_1$.

Goal 1: Show that every linear combination of these vectors a solution

We showed that the first edge, a_1, is satisfied.

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We showed that the first edge, a_1, is satisfied.
In a similar fashion we can show that edges a_2 and a_3 are also satisfied.

Theorem

Every linear combination of

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} z \\ y \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \text{lcm}(a_2, a_3) \\ 0 \\ 0 \end{pmatrix}$$

is a solution to a triangular spline.

Bases: Work in Progress

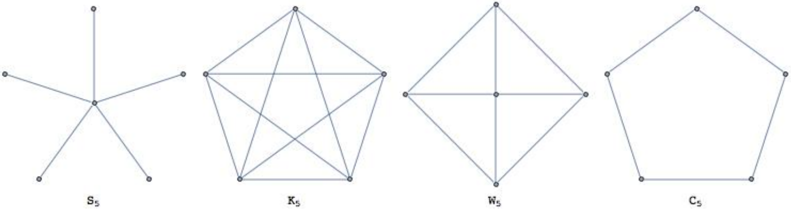
- Goal 1 Complete: We showed every linear combination of the vectors is a solution.

Bases: Work in Progress

- Goal 1 Complete: We showed every linear combination of the vectors is a solution.
- Goal 2: Every solution can be written as a linear combination of the vectors. We are still working to prove this.

Notation

- Let S_n = the star graph with n vertices of degree 1
- K_n = the complete graph on n vertices
- W_n = the wheel graph on n vertices
- C_n = the cycle graph on n vertices.



Chinese Remainder Theorem: Star Graphs

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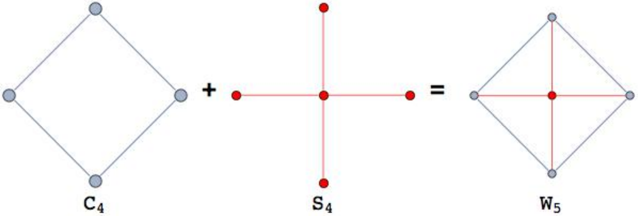
There exists a solution x if the integers n_1, n_2 are coprime.

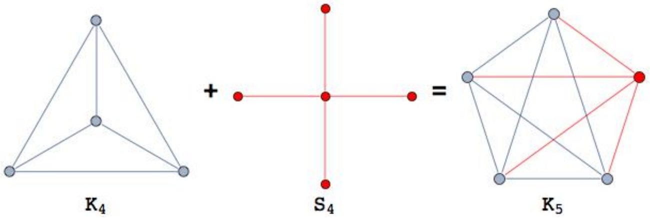
If $\gcd(n_1, n_2) \neq 1$ a solution x exists if and only if $a_1 \equiv a_2 \mod \gcd(n_1, n_2)$.

Wheels: Relationship to star graphs

- $W_n = C_{n-1} + S_{n-1}$
- 

- Corollary: A sufficient condition for W_n to have non-trivial solutions is for the labels of the edges adjacent to the "center vertex" (i.e. the vertex of degree n - 1) to be pairwise relatively prime.

Complete Graphs: Relationship to star graphs

- $K_n = K_{n-1} + S_{n-1}$
- $K_n = C_3 + \sum_{i=3}^{n-1} S_i$
- 

- Corollary: A sufficient condition for K_n to have non-trivial solutions is for n-3 of the vertices to have the following condition on their adjacent edges: all n-1 edges adjacent to the vertex v_i have labels that are pairwise relatively prime.

Thanks!

Thank you to

- Our advisor Professor Julianna Tymoczko
- The Smith College Department of Mathematics and Statistics
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