An Introduction to Self-Similar and Combinatorial Tiling Part II

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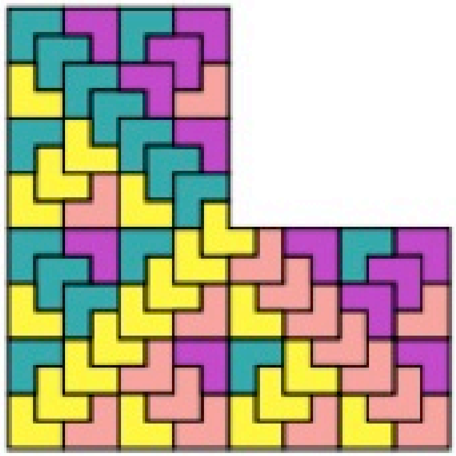
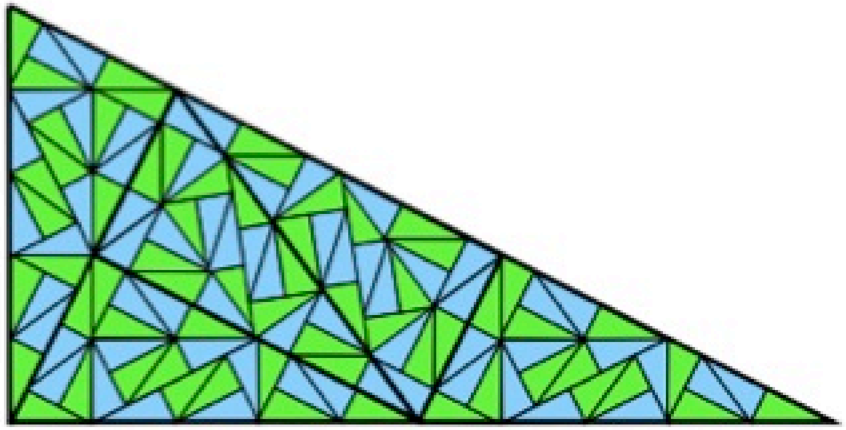
Vassar College

Self-Similar vs. Combinatorial Tiling

- Substitution/Inflate and Subdivide Rule

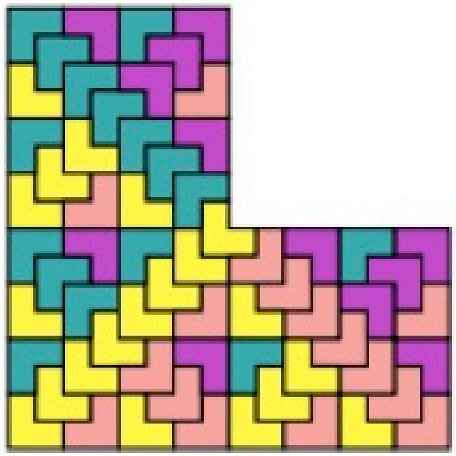


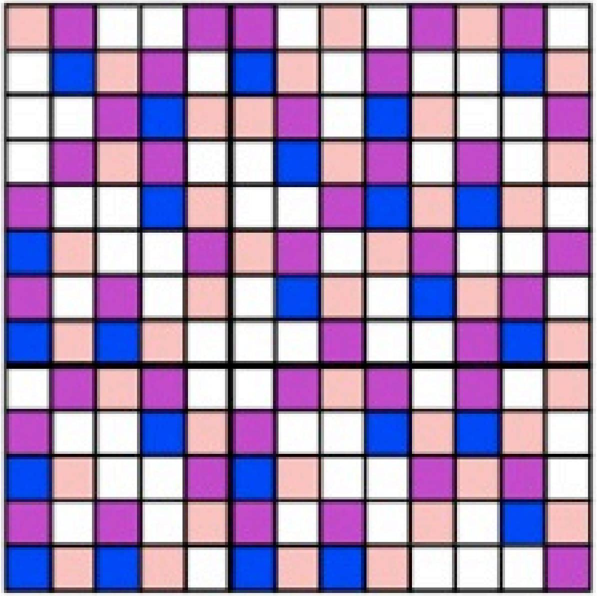


Self-Similar Tiling

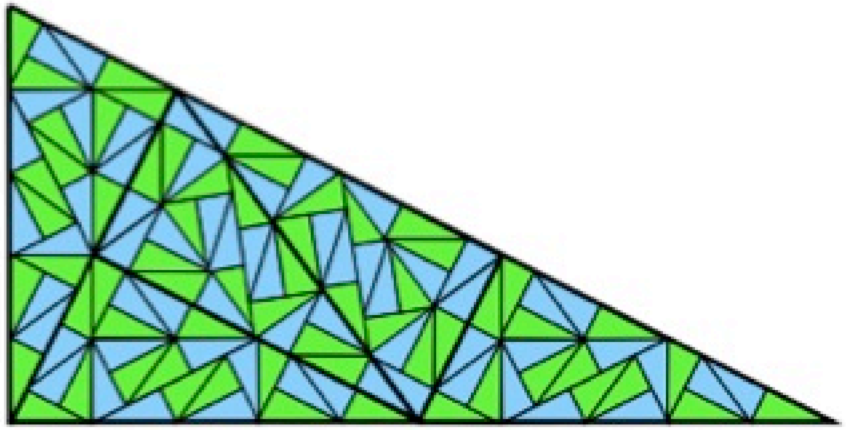
Self-Similar vs. Combinatorial Tiling

- Substitution/Inflate and Subdivide Rule
- No geometric resemblance to itself
- Substitution of non-constant length





Combinatorial Tiling



Self-Similar Tiling

One-Dimensional Symbolic Substitution

- $\mathcal{A}$ is a finite set called an *alphabet* whose elements are *letters*.
- $\mathcal{A}^*$ is the set of all *words* with elements from $\mathcal{A}$.
- A **symbolic substitution** is any map $\sigma : \mathcal{A} \to \mathcal{A}^*$.

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- The block lengths are Fibonacci numbers 1, 2, 3, 5, 8, 13, …
- substitution of non-constant length or combinatorial substitution

Substitution Matrix

- The substitution matrix M is the $n \times n$ matrix with entries given by

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- substitution matrix for one-dimensional Fibonacci substitution:

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Eigenvectors and Eigenvalues

- **Eigenvector** - a non-zero vector v that, when multiplied by square matrix A yields the original vector multiplied by a single number λ

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$$A\vec{v} = \lambda\vec{v}$$

- λ is the **eigenvalue** of A corresponding to v.
- This equation has non-trivial solutions if and only if

$$\det(A - \lambda I) = 0$$

- Solve for λ to find eigenvalues.

Expansion Constant

- Eigenvalues are the roots of the characteristic polynomial

$$\lambda^2 - \lambda - 1 = 0$$

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- Perron Eigenvalue of Fibonacci substitution matrix:

$$\frac{1+\sqrt{5}}{2} = \gamma$$

the golden mean

The Fibonacci Direct Product Substitution

The direct product of the one-dimensional Fibonacci substitution with itself.

$$\{a,b\} \times \{a,b\}$$

where $(a,a) = 1$, $(a,b) = 2$, $(b,a) = 3$, $(b,b) = 4$.

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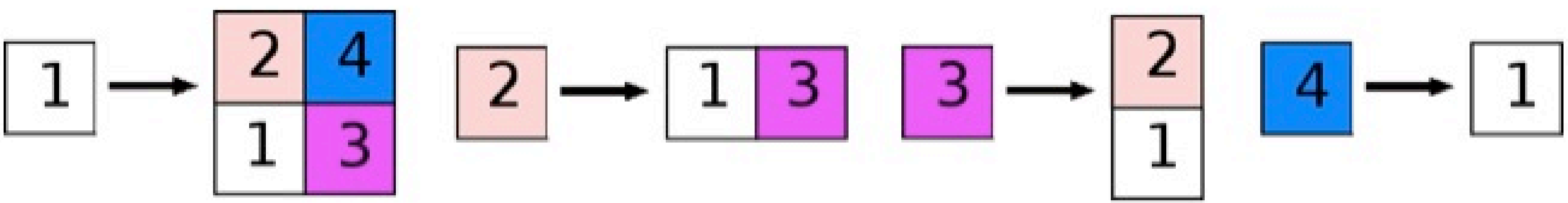

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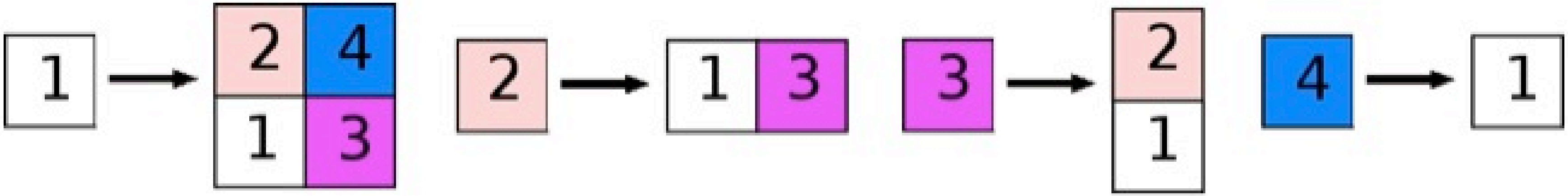
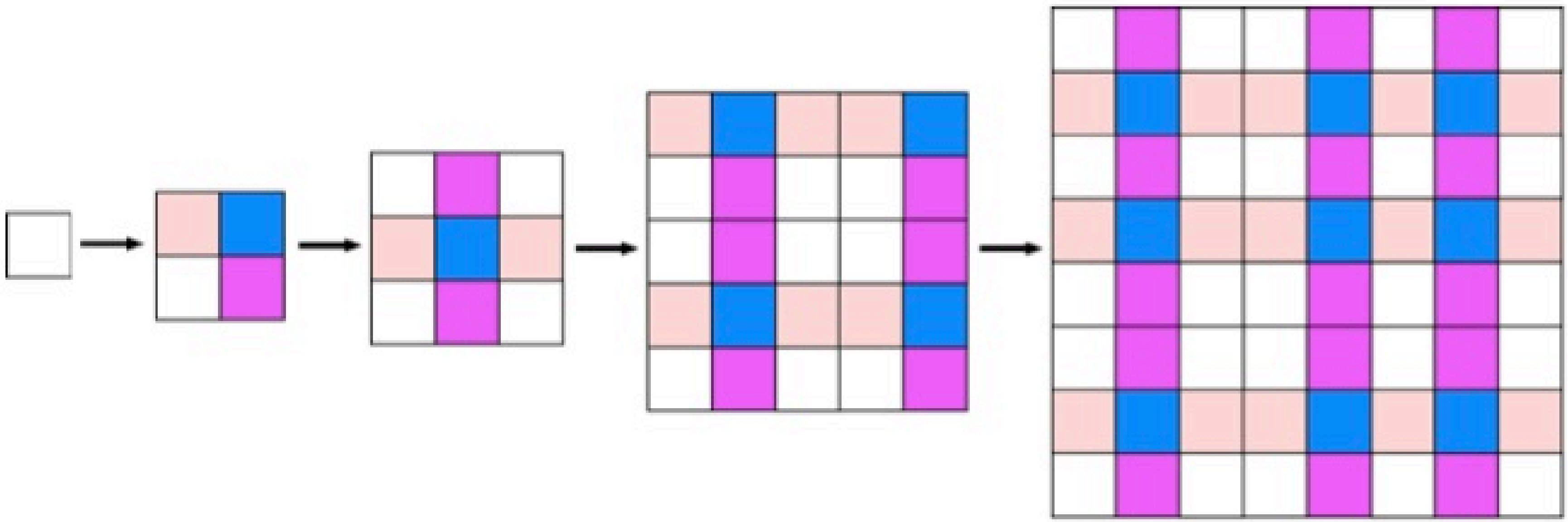


- Not self-similar
- Not an inflate and subdivide rule

Several Iterations of Tile Types



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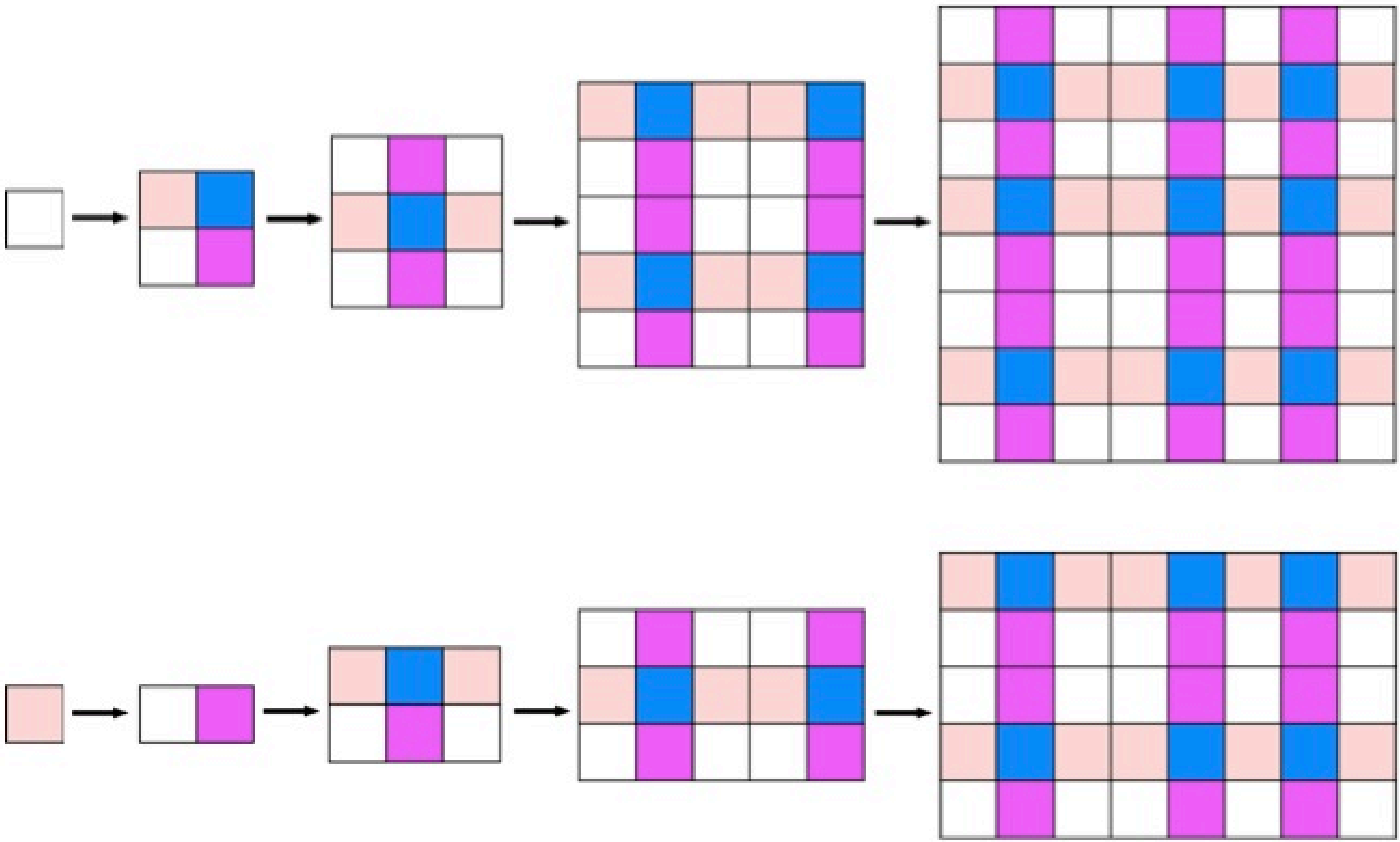
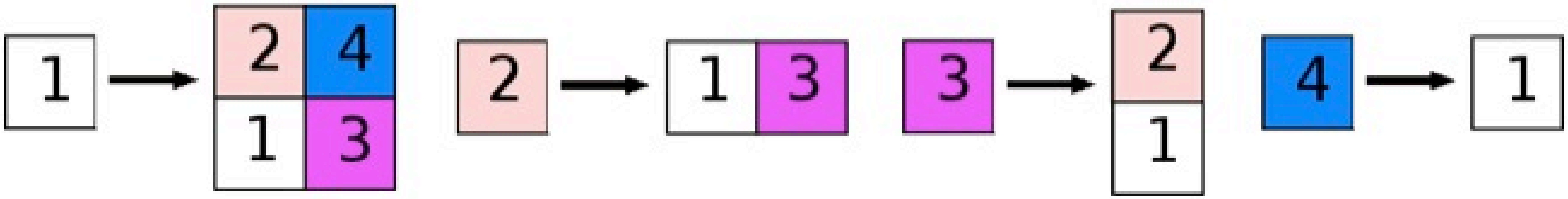


Substitution Matrix



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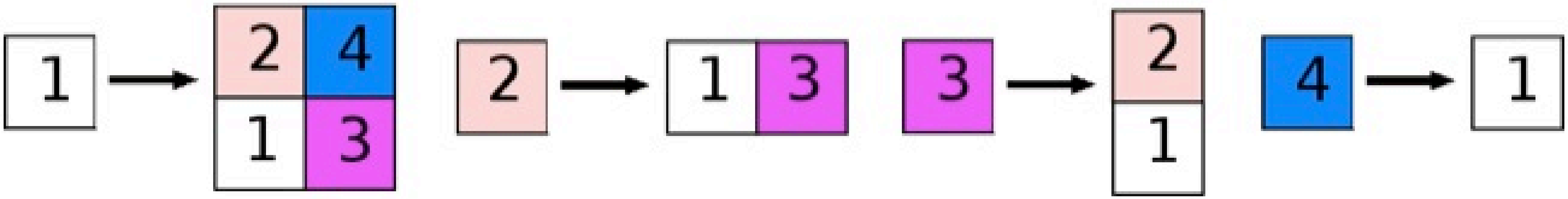

The substitution matrix for the Fibonacci Direct Product is

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Expansion Constant

- Eigenvalues are the roots of the characteristic polynomial

$$\det\left(M - \lambda I\right) = 0$$

$$\lambda^4 - \lambda^3 - 4\lambda^2 - \lambda + 1 = 0$$

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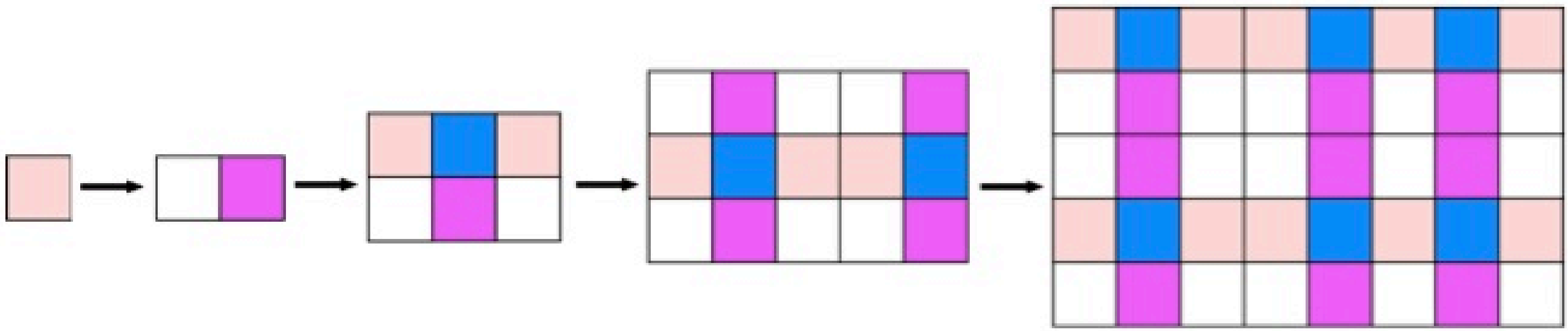
- Perron Eigenvalue of Fibonacci Direct Product substitution matrix:

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = \gamma^2$$

the golden mean squared

Replace and Rescale Method



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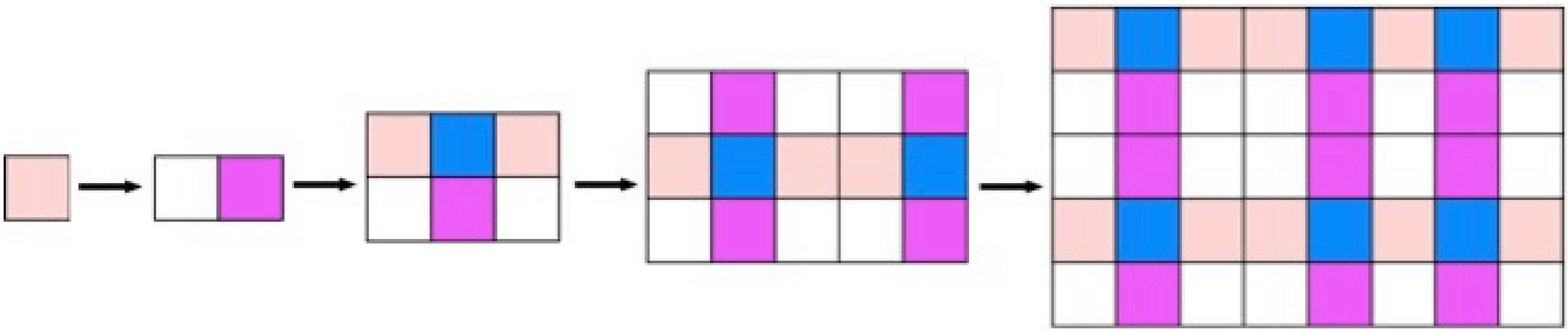
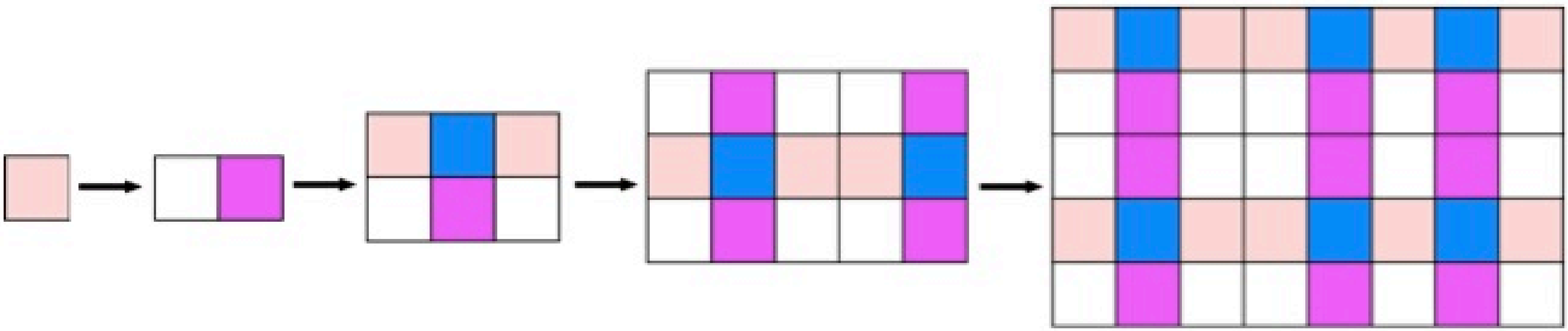


Rescale volumes by the Perron Eigenvalue raised to the n^{th} power: $1/\gamma^{2n}$ where n corresponds to the n^{th}-level of our block.

Replace-and-rescale Method



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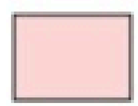
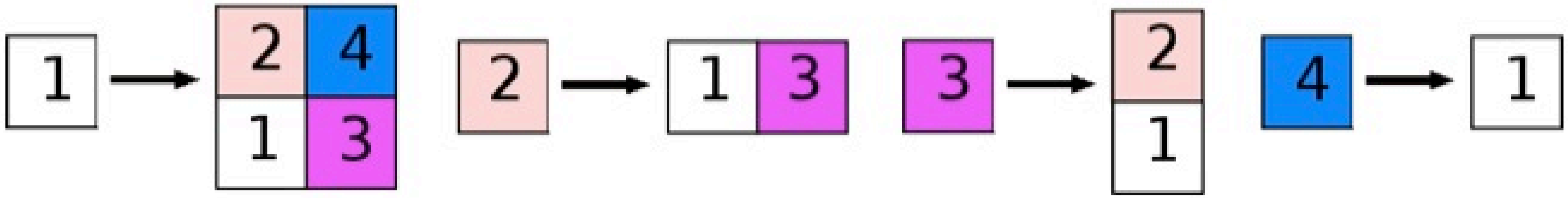


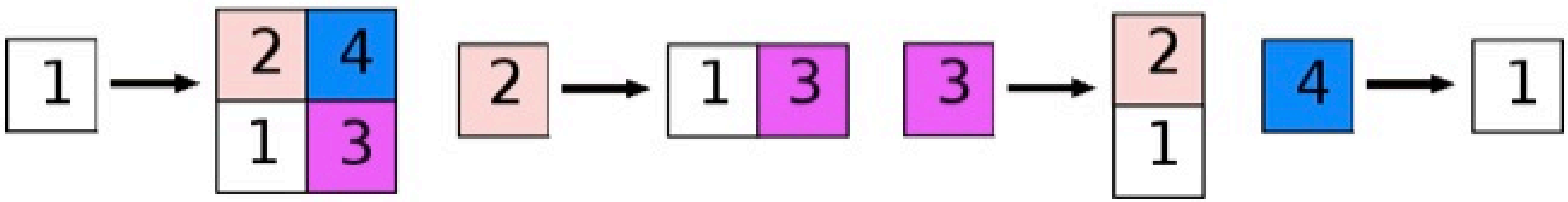
This results in a level-0 tile with different lengths that the original. Repeat for other tile types. The substitution rule is now self-similar.

Self-Similar Fibonacci Direct Product



Combinatorial Substitution Rule

Self-Similar Fibonacci Direct Product



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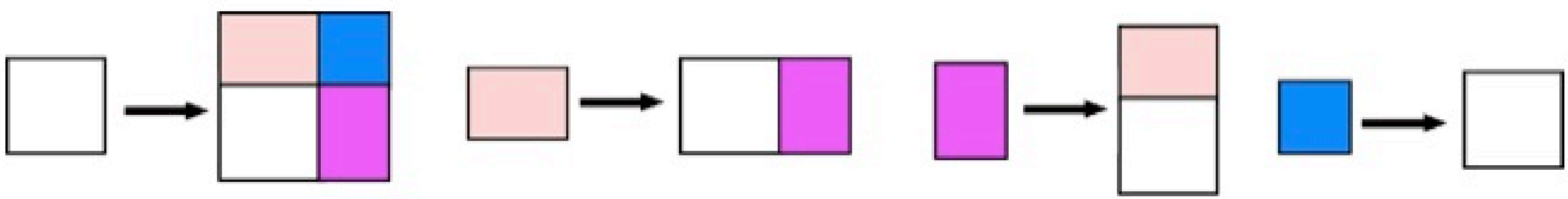
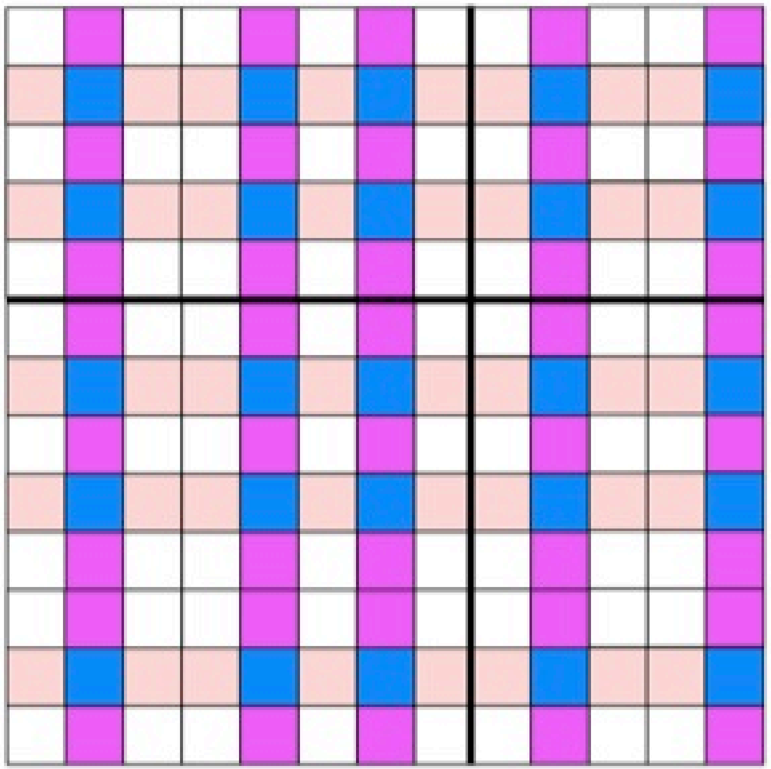


Self-Similar Inflate and Subdivide Rule

Replace-and-rescale Method

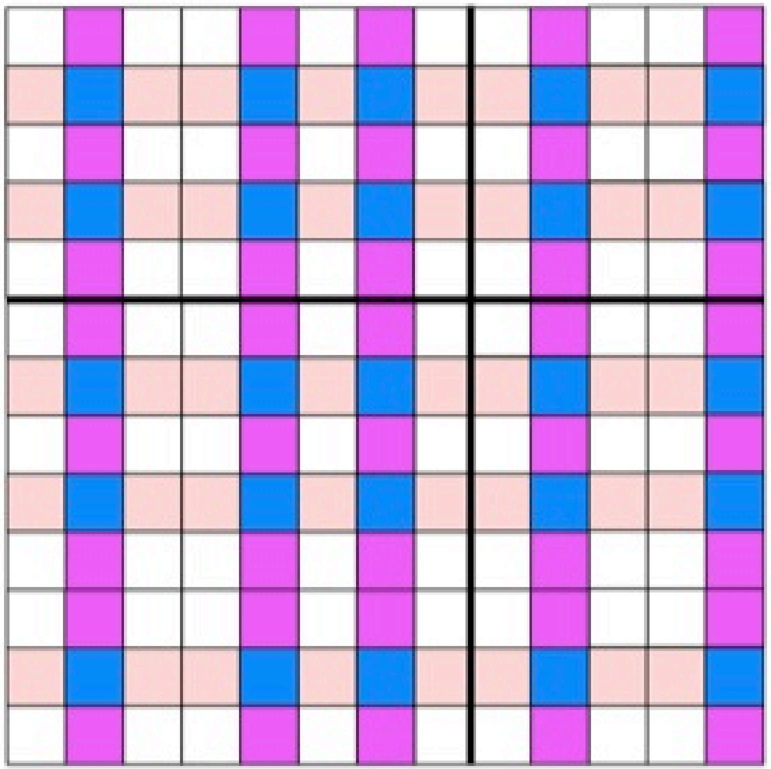
Compare level-5 tiles of the Fibonacci Direct Product (left)



Combinatorial Tiling

Replace-and-rescale Method

Compare level-5 tiles of the Fibonacci Direct Product (left) and the self-similar tiling (right).



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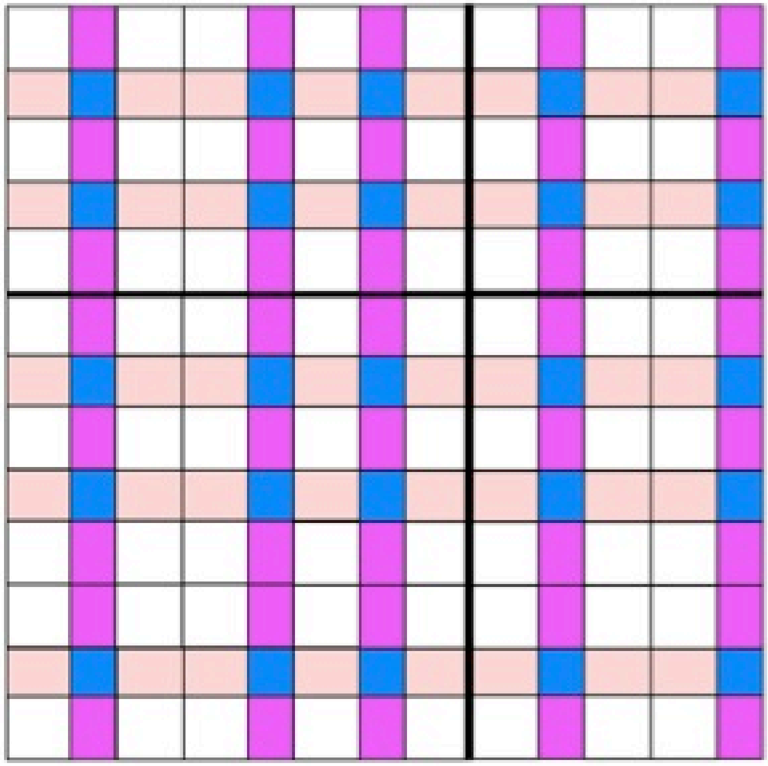
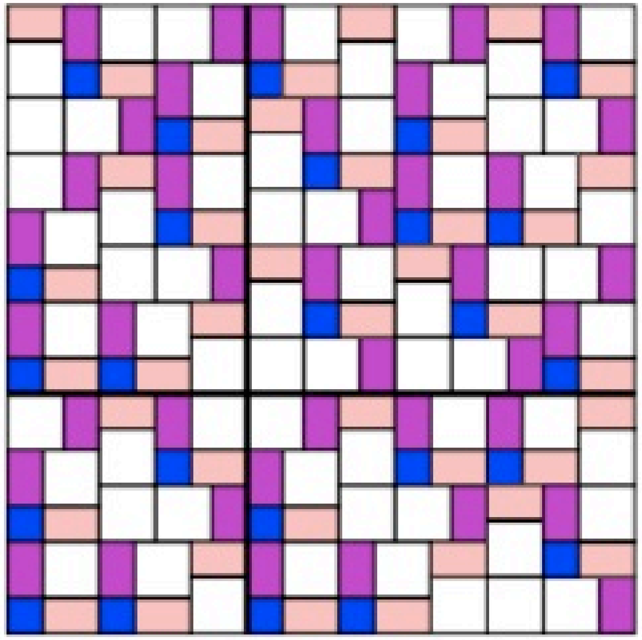


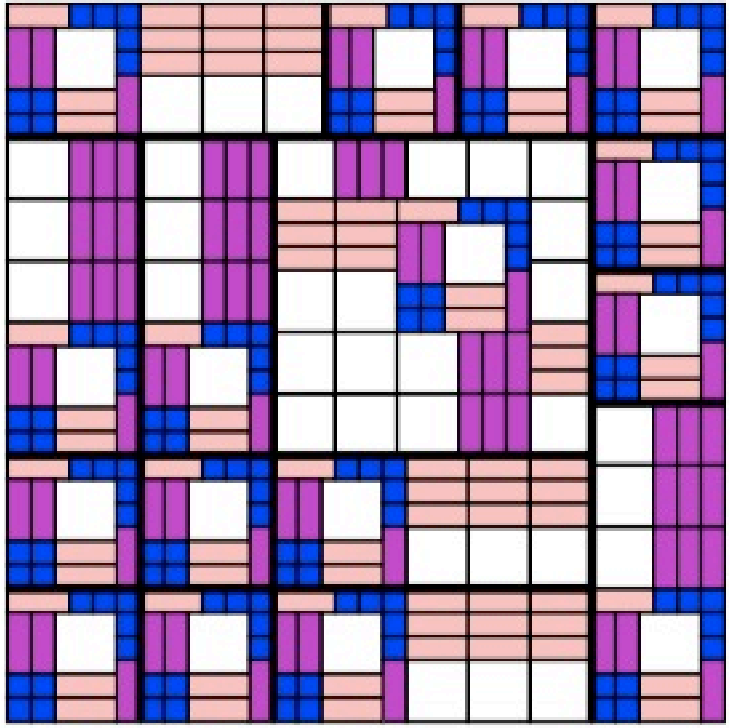
Self-Similar Tiling

Replace-and-rescale Method

- Not known whether replace-and-rescale method always works
- Replace-and-rescale method works in all known examples



Self-Similar Fibonacci DPV



Self-Similar non-Pisot DPV

References

- N.P. Frank, A primer of substitution tilings of the Euclidean plane, Expositiones Mathematicae, 26 (2008) 4, 295-386

Further Readings

- R. Kenyon, B. Solomyak, On the Characterization of Expansion Maps for Self-Affine Tiling, Discrete Comput Geom, 43 (2010), 577-593