More Explicit Formulas for Euler and Bernoulli Numbers

Francesca Romano

Siena College Mathematics

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- There are explicit formulas for Euler and Bernoulli numbers that result from the classical Faà di Bruno formula.
- Can we establish more explicit formulas by coming from the multivariable point of view?

Generating Functions

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- "A generating function is a clothesline on which we hang up a sequence of numbers for display." -Herbert Wilf, *Generatingfunctionology*
- A function f is the **exponential generating function** of $(a_n)_{n=0}^{\infty}$ if

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n.$$

Euler Numbers

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- Similarly the multivariable Euler numbers $E_{\boldsymbol{\alpha}}$ (cf. DiNardo and Oliva) have exponential generating function $h_E(x_1, ..., x_\nu) = \text{sech}(x_1 + ... + x_\nu)$. This sequence can similarly be defined using Taylor series expansions as $E_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}! T_{\boldsymbol{\alpha}}(h_E; \mathbf{0})$ where $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_\nu) \in \mathbb{N}_0^\nu$.

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1	0	−1	0	5	0
0	−1	0	5	0	−61
−1	0	5	0	−61	0
0	5	0	−61	0	1385
5	0	−61	0	1385	0
0	−61	0	1385	0	−50521

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- Similarly the multivariable Bernoulli numbers $B_{\boldsymbol{\alpha}}$ (cf. DiNardo and Oliva) have exponential generating function

$$h_B(x_1, \ldots, x_\nu) = \frac{x_1 + \ldots + x_\nu}{e^{x_1 + \ldots + x_\nu} - 1}.$$

This sequence can similarly be defined using Taylor series expansions as $B_{\boldsymbol{\alpha}}$ to be $\boldsymbol{\alpha}! T_{\boldsymbol{\alpha}}(h_B; \mathbf{0})$ where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_\nu) \in \mathbb{N}_0^\nu$.

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- There is an analogous square pattern...

The Chain Rule

- Suppose f and g are functions that are both differentiable everywhere. Every calculus student knows that composite function is also differentiable and

$$\frac{d(f \circ g)}{dx}(x) = \frac{df}{dx}(g(x)) \cdot \frac{dg}{dx}(x).$$

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- What about $(f \circ g)''$?
- We have

$$\frac{d^2(f \circ g)}{dx^2}(x) = \frac{df}{dx}(g(x)) \cdot \frac{d^2 g}{dx^2}(x) + \frac{d^2 f}{dx^2}(g(x)) \cdot (\frac{dg}{dx}(x))^2.$$

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$$(f \circ g)^{(n)} = \sum_{\pi \in P_n} \frac{\binom{n}{\pi}}{\lambda(\pi)!} (\frac{d^m f}{dx^m} \circ g) \prod_{i=1}^{n} [\frac{d^i g}{dx^i}]^{\pi_i}.$$

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- The multivariable Faà di Bruno formula is as follows:

$$(f \circ g)^{|\boldsymbol{\alpha}|} = \sum_{\pi \in p(\boldsymbol{\alpha})} \frac{\binom{\boldsymbol{\alpha}}{\pi}}{\lambda(\pi)!} \left(\frac{\partial^{|m|} f}{\partial y^m} \circ g\right) \prod_{k=1}^{s} \prod_{i=1}^{\mu} \left[\frac{\partial^{|P_k|}}{\partial x^{|P_k|}} y_i\right]^{m_{ki}}.$$

Formulas for Multivariable Euler Numbers

- If E_n is the n Euler number, then for all $m \in \mathbb{N}$:

$$E_n = \sum_{\substack{\pi \in P_n \\ \text{all even parts}}} (-1)^m \binom{m}{\lambda(\pi)} \binom{n}{\pi} = \sum_{\substack{\pi \in C_n \\ \text{all even parts}}} (-1)^m \binom{n}{\pi}.$$

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- (Bannon and R. 2012) If $E_{\boldsymbol{\alpha}}$ is the $\boldsymbol{\alpha}$ multivariable Euler number, then for all $m \in \mathbb{N}$:

$$E_{|\boldsymbol{\alpha}|} = \sum_{\pi \in p(\boldsymbol{\alpha}, \text{even})} (-1)^m \binom{m}{\lambda(\pi)} \binom{\boldsymbol{\alpha}}{\pi} = \sum_{\pi \in s^+(\boldsymbol{\alpha}, \text{even})} (-1)^m \binom{\boldsymbol{\alpha}}{\pi}.$$

- If B_n is the n Bernoulli number, then for all $m \in \mathbb{N}$:

$$B_n = \sum_{\pi \in P_n} \frac{(-1)^m}{1+m} \binom{m}{\lambda(\pi)} \binom{n}{\pi} = \sum_{\pi \in C_n} \frac{(-1)^m}{1+m} \binom{n}{\pi}.$$

Formulas for Multivariable Bernoulli Numbers

- If B_n is the n Bernoulli number, then for all $m \in \mathbb{N}$:

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Conclusion

- This work resulted in a paper that is currently under review. If you would like a copy of this paper please see me after the talk.