More Explicit Formulas for Euler and Bernoulli Numbers

Francesca Romano

Siena College Mathematics

22 September 2012
The Problem:

- There are explicit formulas for Euler and Bernoulli numbers that result from the classical Faà di Bruno formula.
The Problem:

- There are explicit formulas for Euler and Bernoulli numbers that result from the classical Faà di Bruno formula.
- Can we establish more explicit formulas by coming from the multivariable point of view?
Generating Functions

- A function $f$ is the generating function of $(a_n)_{n=0}^\infty$ if

$$f(x) = \sum_{n=0}^\infty a_n x^n.$$
A function $f$ is the **generating function** of $(a_n)_{n=0}^{\infty}$ if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$ 

"A generating function is a clothesline on which we hang up a sequence of numbers for display." - Herbert Wilf, *Generatingfunctionology*
A function $f$ is the **generating function** of $(a_n)_{n=0}^\infty$ if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$ 

"A generating function is a clothesline on which we hang up a sequence of numbers for display." -Herbert Wilf, *Generatingfunctionology*

A function $f$ is the **exponential generating function** of $(a_n)_{n=0}^\infty$ if

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n.$$
The classical Euler numbers $E_n$ have exponential generating function $h(x) = \text{sech}(x)$, i.e. the $n$th Taylor coefficient of $h$ about 0 is $\frac{E_n}{n!}$. 

1, 0, -1, 0, 5, 0, -61, 0, 1385, 0, -50521, 0, 2702765, 0, -199360981,...
Euler Numbers

- The classical Euler numbers $E_n$ have exponential generating function $h(x) = \text{sech}(x)$, i.e. the $n$th Taylor coefficient of $h$ about 0 is $\frac{E_n}{n!}$.
- $1, 0, -1, 0, 5, 0, -61, 0, 1385, 0, -50521, 0, 2702765, 0, -199360981,...$
Euler Numbers

- The classical Euler numbers $E_n$ have exponential generating function $h(x) = \text{sech}(x)$, i.e. the $n$th Taylor coefficient of $h$ about 0 is $\frac{E_n}{n!}$.
- $1, 0, -1, 0, 5, 0, -61, 0, 1385, 0, -50521, 0, 2702765, 0, -199360981,...$
- Similarly the multivariable Euler numbers $E_\alpha$ (cf. DiNardo and Oliva) have exponential generating function $h_E(x_1, ..., x_\nu) = \text{sech}(x_1 + ... + x_\nu)$. This sequence can similarly be defined using Taylor series expansions as $E_\alpha = \alpha! T_\alpha(h_E; 0)$ where $\alpha = (\alpha_1, ..., \alpha_\nu) \in \mathbb{N}_0^\nu$. 
Euler Numbers

- The classical Euler numbers $E_n$ have exponential generating function $h(x) = \text{sech}(x)$, i.e. the $n$th Taylor coefficient of $h$ about 0 is $\frac{E_n}{n!}$.
- $1, 0, -1, 0, 5, 0, -61, 0, 1385, 0, -50521, 0, 2702765, 0, -199360981, ...$
- Similarly the multivariable Euler numbers $E_\alpha$ (cf. DiNardo and Oliva) have exponential generating function $h_E(x_1, ..., x_\nu) = \text{sech}(x_1 + ... + x_\nu)$. This sequence can similarly be defined using Taylor series expansions as $E_\alpha = \alpha! T_\alpha(h_E; 0)$ where $\alpha = (\alpha_1, ..., \alpha_\nu) \in \mathbb{N}_0^\nu$.

\[
\begin{array}{ccccccc}
1 & 0 & -1 & 0 & 5 & 0 \\
0 & -1 & 0 & 5 & 0 & -61 \\
-1 & 0 & 5 & 0 & -61 & 0 \\
0 & 5 & 0 & -61 & 0 & 1385 \\
5 & 0 & -61 & 0 & 1385 & 0 \\
0 & -61 & 0 & 1385 & 0 & -50521
\end{array}
\]
The classical Bernoulli numbers $B_n$ have exponential generating function $h(x) = \frac{x}{e^x - 1}$, i.e. the $n$th Taylor coefficient of $h$ about 0 is $\frac{B_n}{n!}$.
The classical Bernoulli numbers $B_n$ have exponential generating function $h(x) = \frac{x}{e^x - 1}$, i.e. the $n$th Taylor coefficient of $h$ about 0 is $\frac{B_n}{n!}$.

1, -1/2, 1/6, 0, -1/30, 0, 1/42, 0, -1/30, ...
The classical Bernoulli numbers $B_n$ have exponential generating function $h(x) = \frac{x}{e^x - 1}$, i.e. the $n$th Taylor coefficient of $h$ about 0 is $\frac{B_n}{n!}$.

1, -1/2, 1/6, 0, -1/30, 0, 1/42, 0, -1/30, ...

Similarly the multivariable Bernoulli numbers $B_\alpha$ (cf. DiNardo and Oliva) have exponential generating function

$$h_B(x_1, \ldots, x_\nu) = \frac{x_1 + \ldots + x_\nu}{e^{x_1 + \ldots + x_\nu} - 1}.$$  

This sequence can similarly be defined using Taylor series expansions as $B_\alpha$ to be $\alpha! T_\alpha(h_B; 0)$ where $\alpha = (\alpha_1, \ldots, \alpha_\nu) \in \mathbb{N}_0^\nu$. 
The classical Bernoulli numbers $B_n$ have exponential generating function $h(x) = \frac{x}{e^x - 1}$, i.e. the $n$th Taylor coefficient of $h$ about 0 is $\frac{B_n}{n!}$.

1, -1/2, 1/6, 0, -1/30, 0, 1/42, 0, -1/30, ...

Similarly the multivariable Bernoulli numbers $B_\alpha$ (cf. DiNardo and Oliva) have exponential generating function

$$h_B(x_1, \ldots, x_\nu) = \frac{x_1 + \ldots + x_\nu}{e^{x_1 + \ldots + x_\nu} - 1}.$$

This sequence can similarly be defined using Taylor series expansions as $B_\alpha$ to be $\alpha! \, T_\alpha(h_B; 0)$ where $\alpha = (\alpha_1, \ldots, \alpha_\nu) \in \mathbb{N}_0^\nu$.

There is an analogous square pattern...
Suppose $f$ and $g$ are functions that are both differentiable everywhere. Every calculus student knows that composite function is also differentiable and

$$
\frac{d(f \circ g)}{dx}(x) = \frac{df}{dx}(g(x)) \cdot \frac{dg}{dx}(x).
$$
Suppose $f$ and $g$ are functions that are both differentiable everywhere. Every calculus student knows that composite function is also differentiable and
\[
\frac{d}{dx} (f \circ g)(x) = \frac{df}{dx}(g(x)) \cdot \frac{dg}{dx}(x).
\]

What about $(f \circ g)''$?
Suppose $f$ and $g$ are functions that are both differentiable everywhere. Every calculus student knows that composite function is also differentiable and

$$
\frac{d(f \circ g)}{dx}(x) = \frac{df}{dx}(g(x)) \cdot \frac{dg}{dx}(x).
$$

What about $(f \circ g)^{''}$?

We have

$$
\frac{d^2(f \circ g)}{dx^2}(x) = \frac{df}{dx}(g(x)) \cdot \frac{d^2g}{dx^2}(x) + \frac{d^2f}{dx^2}(g(x)) \cdot (\frac{dg}{dx}(x))^2.
$$
The Faà di Bruno formula is what you get if you continue the above. It expresses "the chain rule for higher derivatives".
The Faà di Bruno formula is what you get if you continue the above. It expresses "the chain rule for higher derivatives".

The classical single variable Faà di Bruno formula is as follows:

\[
(f \circ g)^{(n)} = \sum_{\pi \in P_n} \frac{n}{!}(n^{!})! (\frac{d^m f}{dx^m} \circ g) \prod_{i=1}^{n} [\frac{d^i g}{dx^i}]^{\pi_i}.
\]
The Faà di Bruno formula is what you get if you continue the above. It expresses "the chain rule for higher derivatives".

The classical single variable Faà di Bruno formula is as follows:

$$(f \circ g)^{(n)} = \sum_{\pi \in P_n} \frac{n!}{\lambda(\pi)!} \left( \frac{d^m f}{dx^m} \circ g \right) \prod_{i=1}^{n} \left[ \frac{d^i g}{dx^i} \right]^{\pi_i}.$$ 

The multivariable Faà di Bruno formula is as follows:

$$(f \circ g)^{|\alpha|} = \sum_{\pi \in P(\alpha)} \frac{\alpha!}{\lambda(\pi)!} \left( \frac{\partial^m f}{\partial y^m} \circ g \right) \prod_{k=1}^{s} \prod_{i=1}^{\mu} \left[ \frac{\partial |P_k|}{\partial x|P_k| y_i} \right]^{m_{ki}}.$$
If $E_n$ is the $n$ Euler number, then for all $m \in \mathbb{N}$:

$$E_n = \sum_{\pi \in P_n} (-1)^m \binom{m}{\lambda(\pi)} \binom{n}{\pi} = \sum_{\pi \in C_n} (-1)^m \binom{n}{\pi}.$$
If \( E_n \) is the \( n \) Euler number, then for all \( m \in \mathbb{N} \):

\[
E_n = \sum_{\pi \in \mathcal{P}_n \text{ all even parts}} (-1)^m \binom{m}{\lambda(\pi)} \binom{n}{\pi} = \sum_{\pi \in \mathcal{C}_n \text{ all even parts}} (-1)^m \binom{n}{\pi}.
\]

(Bannon and R. 2012) If \( E_\alpha \) is the \( \alpha \) multivariable Euler number, then for all \( m \in \mathbb{N} \):

\[
E_{|\alpha|} = \sum_{\pi \in \mathcal{P}(\alpha, \text{even})} (-1)^m \binom{m}{\lambda(\pi)} \binom{\alpha}{\pi} = \sum_{\pi \in \mathcal{S}(\alpha, \text{even})} (-1)^m \binom{\alpha}{\pi}.
\]
If $B_n$ is the $n$ Bernoulli number, then for all $m \in \mathbb{N}$:

$$B_n = \sum_{\pi \in P_n} \frac{(-1)^m}{1 + m \lambda(\pi)} \binom{m}{\pi} \binom{n}{\pi} = \sum_{\pi \in C_n} \frac{(-1)^m}{1 + m \lambda(\pi)} \binom{n}{\pi}.$$
If $B_n$ is the $n$ Bernoulli number, then for all $m \in \mathbb{N}$:

$$B_n = \sum_{\pi \in P_n} \frac{(-1)^m}{1 + m} \binom{m}{\lambda(\pi)} \binom{n}{\pi} = \sum_{\pi \in C_n} \frac{(-1)^m}{1 + m} \binom{n}{\pi}.$$ 

(Bannon and R. 2012) If $B_\alpha$ is the $\alpha$ multivariable Bernoulli number, then for all $m \in \mathbb{N}$:

$$B_{|\alpha|} = \sum_{\pi \in \rho(\alpha)} \frac{(-1)^m}{1 + m} \binom{m}{\lambda(\pi)} \binom{\alpha}{\pi} = \sum_{\pi \in s^+(\alpha)} \frac{(-1)^m}{1 + m} \binom{\alpha}{\pi}.$$
This work resulted in a paper that is currently under review. If you would like a copy of this paper please see me after the talk.