# Fixed Point Forecasting: Different Methods of Pinwheel Fractal Generation

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Vassar College

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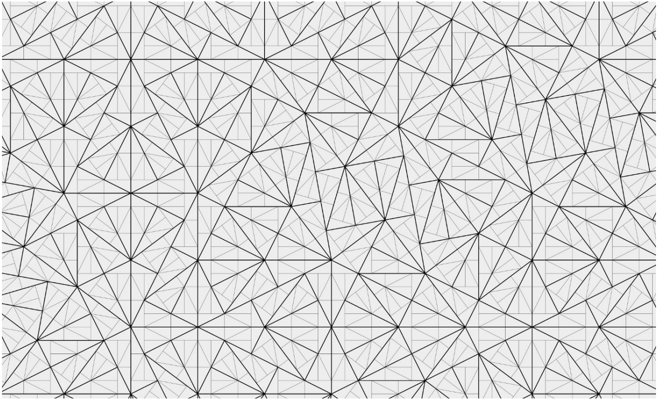
# Pinwheel Tilings



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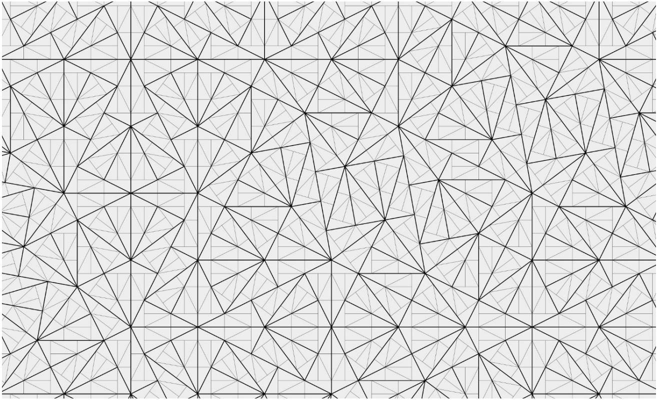

- Non-periodic, self-similar tilings

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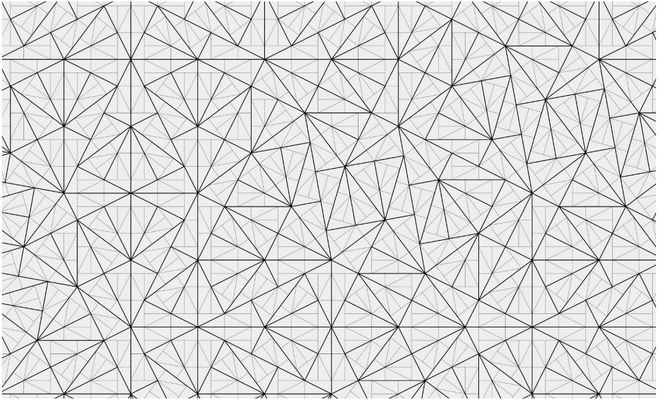

- Non-periodic, self-similar tilings
- Countably infinite distinct tile orientations

# Pinwheel Tiles

- Pinwheel tilings made from right triangle prototiles with side lengths 1, 2 and $\sqrt{5}$

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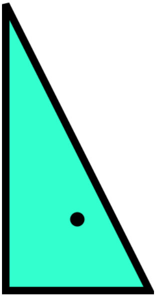
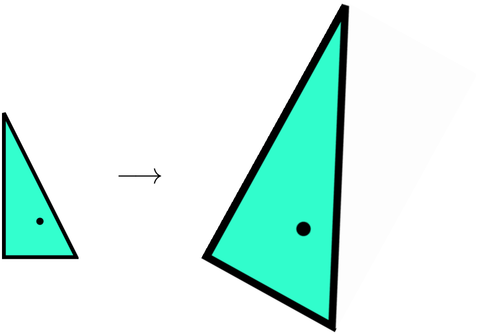
- Pinwheel tilings made from right triangle prototiles with side lengths 1, 2 and $\sqrt{5}$
- Consider a "standard triangle" with vertices at (-.5, -.5), (.5, -.5), and (-.5, 1.5):



# Inflation and Subdivision



- Given "standard triangle," multiply by $M_P = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$.

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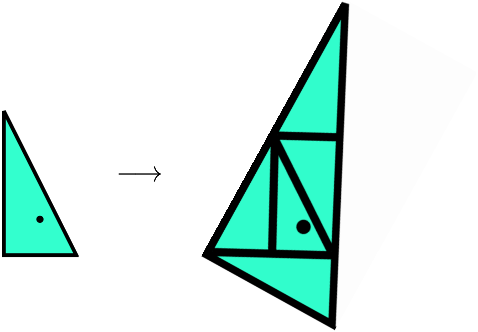
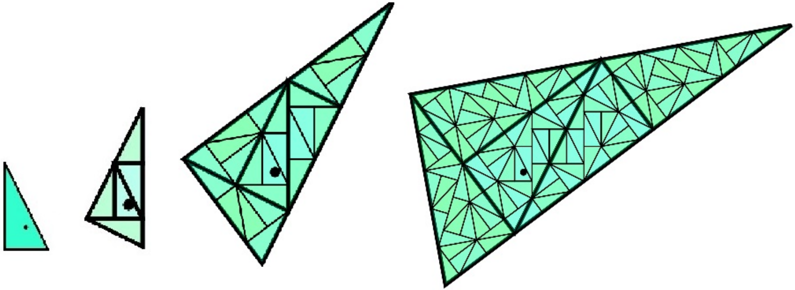


- Given "standard triangle," multiply by $M_P = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$.
- Subdivide inflated tile

# Substitution Rule

Iterations of inflation and subdivision produce a pinwheel tiling:



### Contraction Mapping

For a metric space $(M, d)$, a function $f : M \to M$ such that $\exists k \in \mathbb{R}$ where $k < 1$ and $\forall x, y \in M$,

$$d(f(x), f(y)) \leq kd(x, y)$$

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Example: Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = M_P^{-1}(x, y)$, where $M_P^{-1} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix}$.

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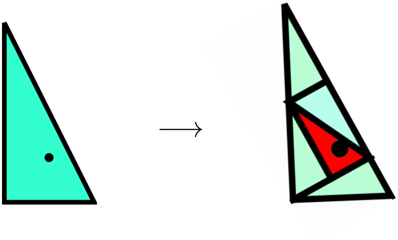

### Iterated Function System

A finite set of contraction mappings on a complete metric space. That is,

$$\{f_i : M \to M | i = 1, 2, ..., N, N \in \mathbb{N}\}$$

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## Theorem

*For any iterated function system on $\mathbb{R}^n$, there exists a unique fixed set $S$. That is, there exists $S \subseteq \mathbb{R}^n$ such that*

$$S = \bigcup_{i=1}^{N} f_i(S).$$

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$$f_1(x, y) = M_P^{-1} * R_y(x, y) + (-0.4, -0.2),$$
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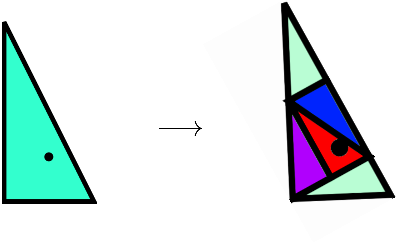
$$f_3(x, y) = R_\pi * M_P^{-1}(x, y).$$



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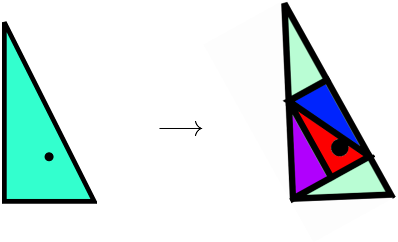
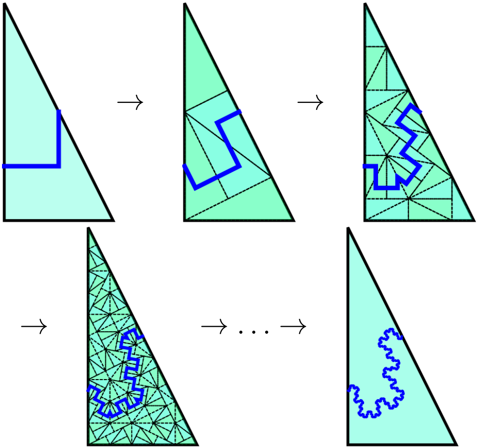


This has a fixed set $A = \bigcup_{i=1}^{3} f_i(A)$. We call $A$ the aorta.

# Stages of the Aorta Generation



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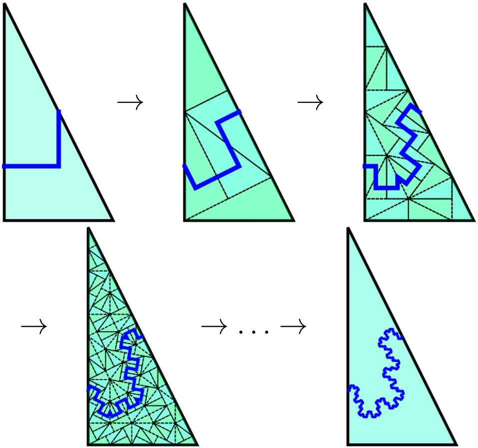
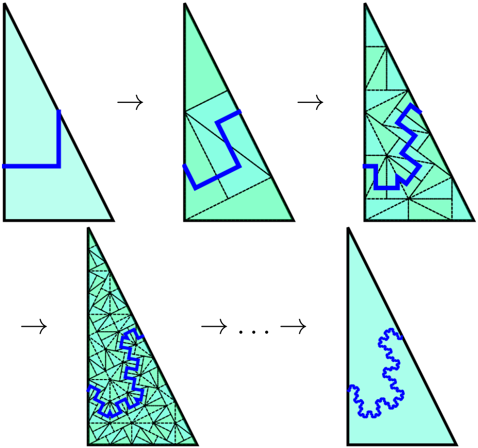


- Three control points: $(0,0)$, $(-0.5,0)$, $(0,0.5)$

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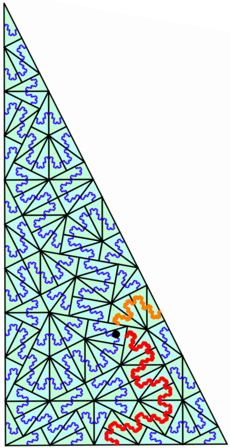
- These points are invariant under our iterated function system.

# Continuation



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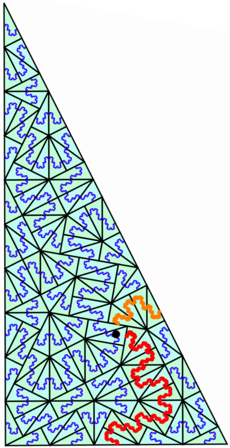

- Discontinuities only happen in two orientations

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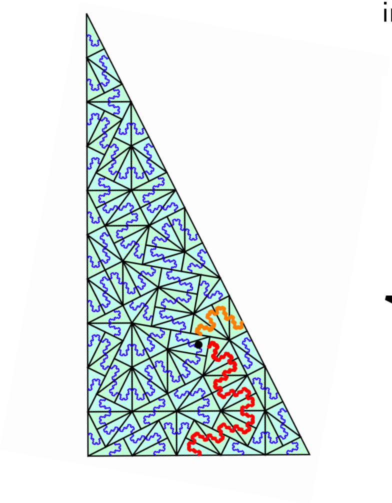

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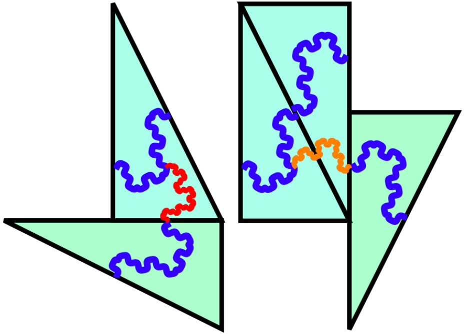

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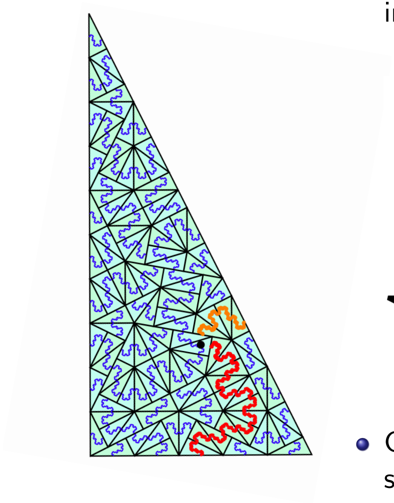

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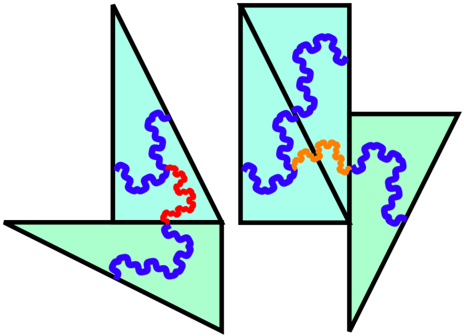

- Copy/paste part of aorta at side control points

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- New tiling of the plane with a finite number of tiles
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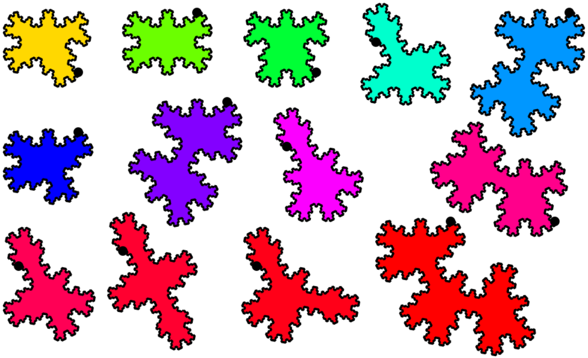

# Fractile Substitution Rule

- Pinwheel substitution rule induces a well-defined substitution on the fractiles

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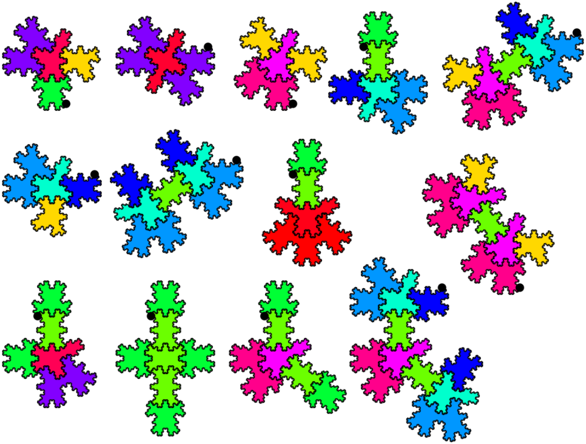

- Primary result of paper by Drs. Frank and Whittaker

# New Method

- Identify points of interest beforehand
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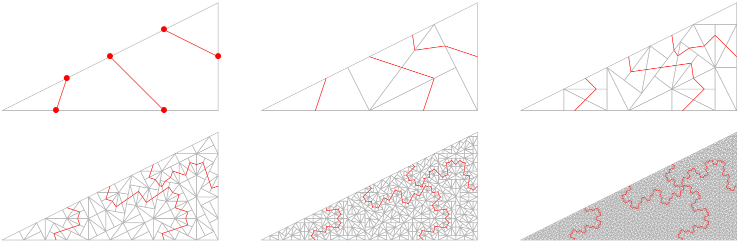
Method 1



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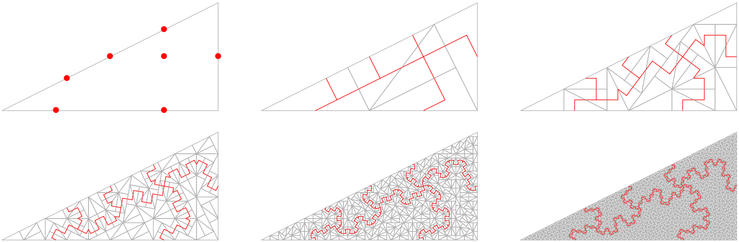
Method 2



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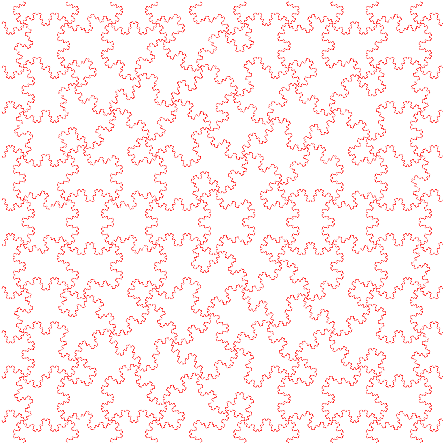
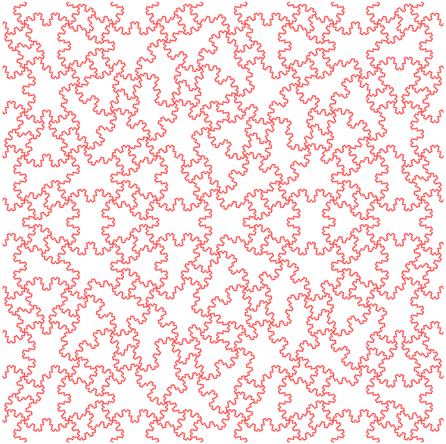


# New Method

Method 2



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- New method somewhat ad-hoc
- Identify relationship between methods
- Use new method approach on other tilings

# References

Natalie Frank and Michael Whittaker, 2011
A Fractal Version of the Pinwheel Tiling
*The Mathematical Intelligencer* 33(2), 7 – 17.