CAYLEY DIGRAPHS AND THE ART OF BELL RINGING

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What is a Cayley digraph?

A Cayley digraph, or *directed graph*, is a graph which is a visual representation of a group.

Formally, we say that *for any group G and a set of generators S of G, we define Cay(S:G) as follows:*

1) *The elements of G are the vertices of Cay(S:G) and*
2) *For x and y in G, there is an arc from x to y if and only if xs=y for some s in S.*

Properties of Cayley Digraphs

There are 4 properties of Cayley digraphs:

1) The digraph is connected; that is, we can ge from any vertex *g* to any vertex *h* by traveling along consecutive arcs, starting at *g* and ending at *h*.

Reasoning: *Every equation gx=h has a solution in a group.*

Properties of Cayley Digraphs

2) At most one arc goes from a vertex *g* to a vertex *h.*

Reasoning: The solution of gx=h is unique.

Properties of Cayley Digraphs

3) Each vertex *g* has exactly one arc of each type starting at *g*, and one of each type ending at *g*.

Reasoning: For g in G and each generator b we can compute gb, and $(gb^{-1})b=g$.

Properties of Cayley Digraphs

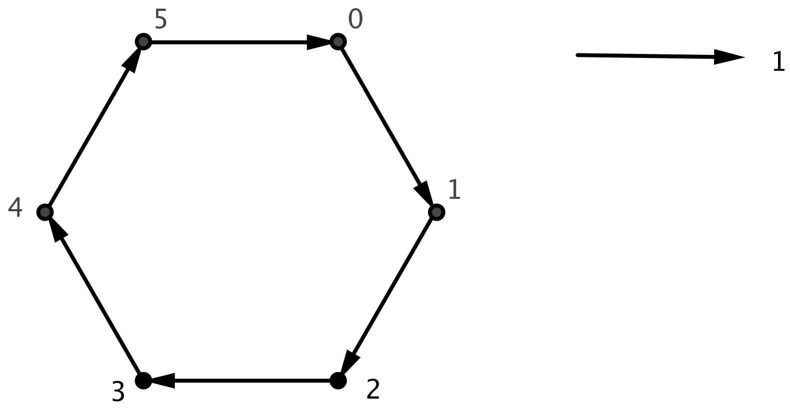
4) If two different sequences of arc types starting from vertex *g* lead to the same vertex *h*, then those same sequences of arc types starting from any vertex *u* will lead to the same vertex *v*.

Reasoning: If $qg=h$ and $rg=h$, then $uq=ug^{-1}h=ur$.

Examples of Cayley Digraphs

The cyclic group Z_8={0,1,2,3,4,5,6,7} with generating set S={1}

Cay({1}:Z_8)



Examples of Cayley Digraphs

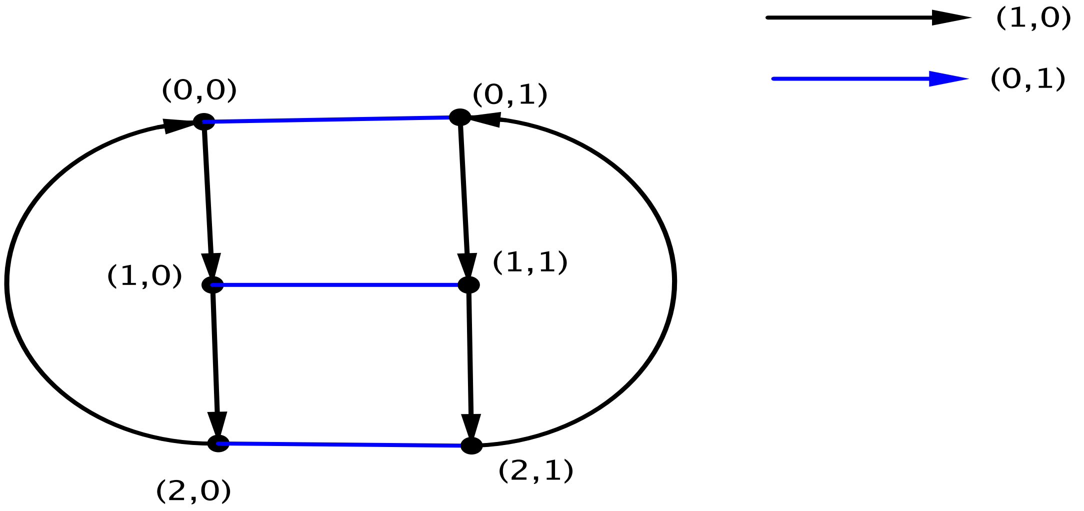
The Cartesian product of two groups *G* and *H* is given by

$G \times H=\{(x,y): x \text{ in } G, y \text{ in } H\}$.

For example, $Z_3 \times Z_2=\{(0,0),(1,0),(2,0),(1,1),(2,1),(0,1)\}$

Examples of Cayley Digraphs

Cay({(1,0),(0,1)}:Z_3 x Z_2}



Examples of Cayley Digraphs

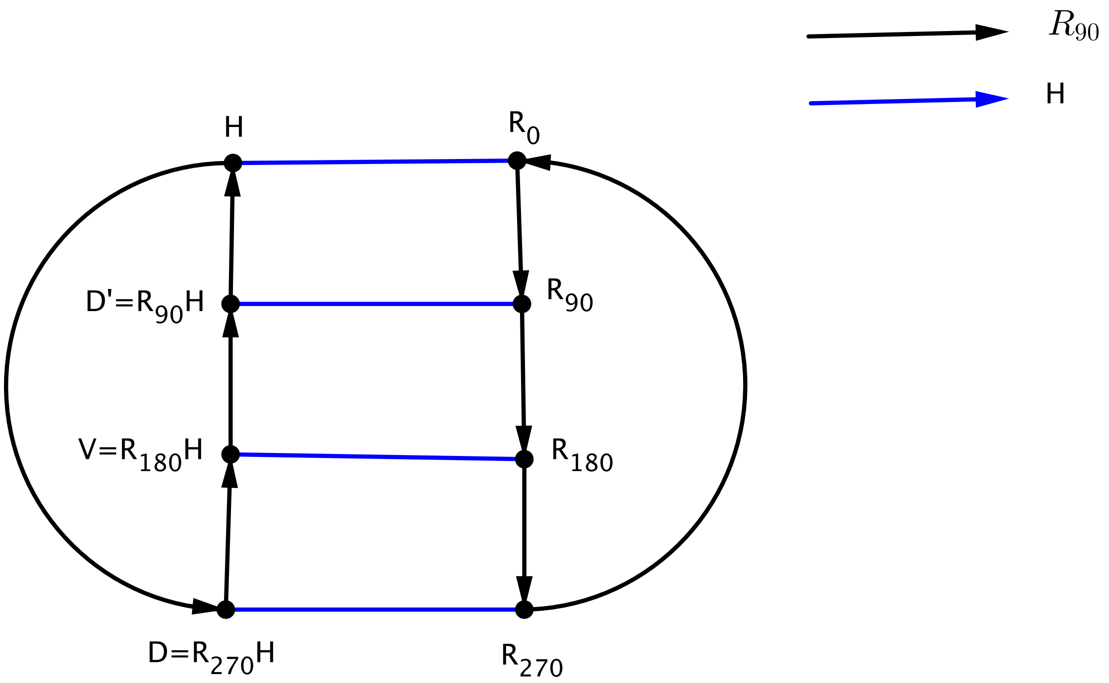
The dihedral group D_n is the group whose elements represent the symmetries of a regular *n*-sided polygon.

$D_4=\{R_0, R_{90}, R_{180}, R_{270}, H, V, D, D'\}$ would represent the symmetries of a square.

Here, R_0 would be a rotation of 0 degrees, R_n a rotation of n degrees, H a horizontal reflection, V a vertical reflection, and D and D' are the diagonal reflections.

Examples of Cayley Digraphs

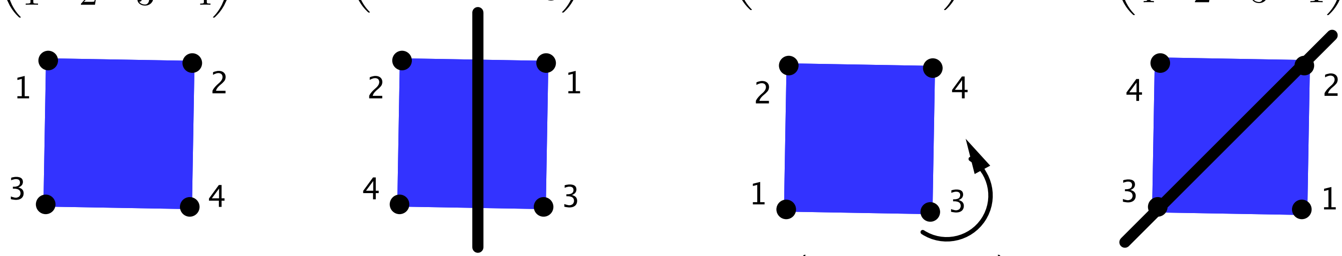
Cay({R_{90}, H}: D_4)



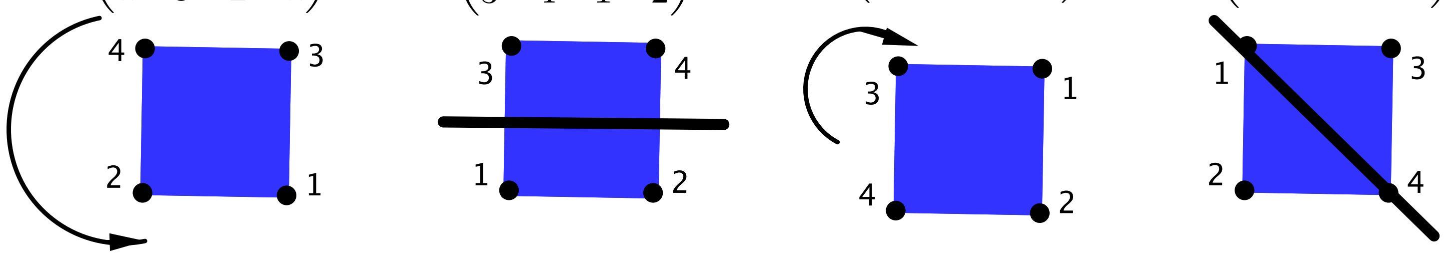
More on the Dihedral Group D_4

D_4 can also be represented as depicted below:

$$R_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad V = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \quad R_{90} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}$$



$$R_{180} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \quad R_{270} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \quad D' = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$



This will become more important to us later on.

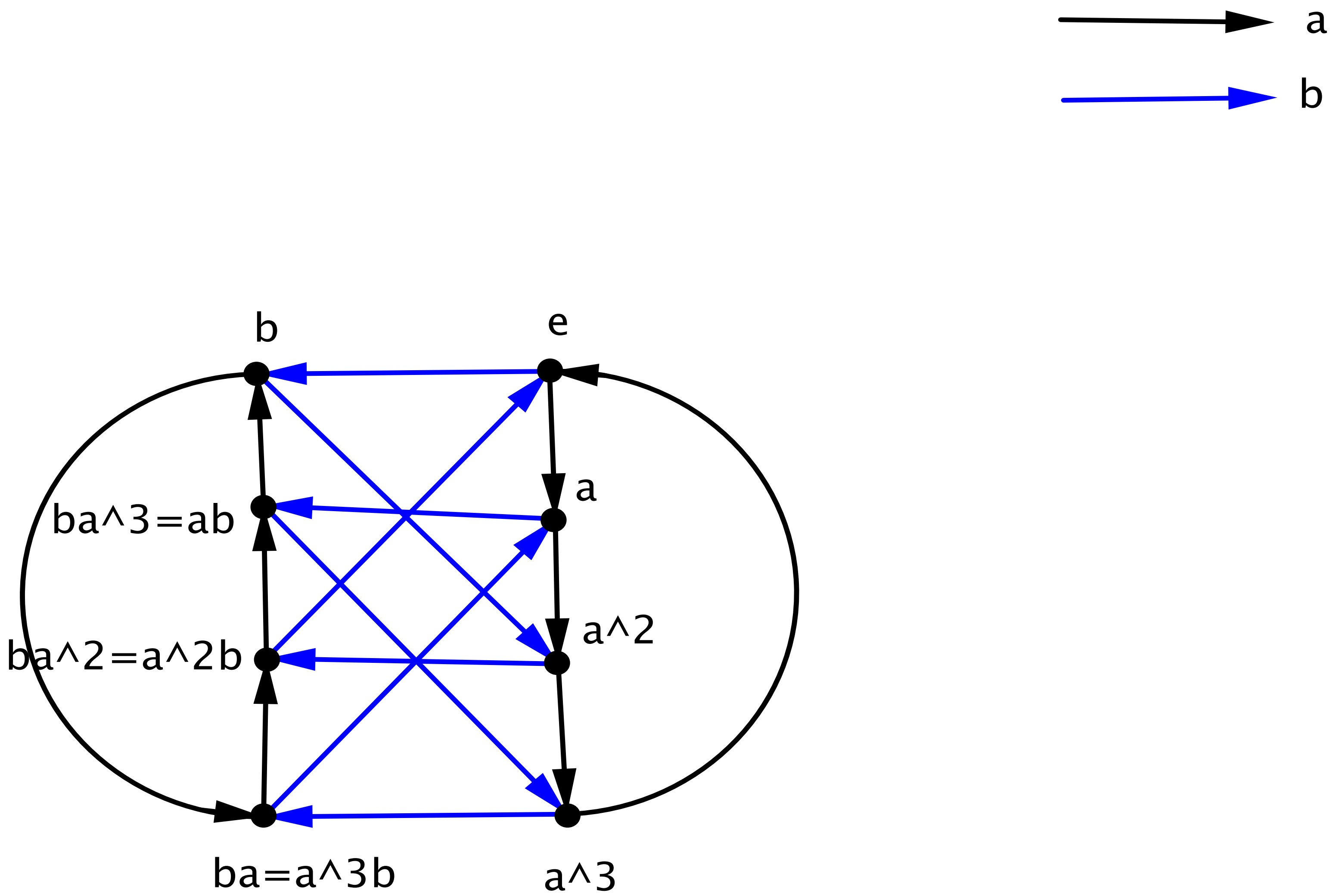
Examples of Cayley Digraphs

The quaternion group Q_4 is an example of a nonabelian group of order 8.

$Q_4=\{e, a, a^2, a^3, b, ba, ba^2, ba^3\}$

$Q_4=\{a,b: a^4=e, b^2=a^2, ba=a^3b\}$

Example of Cayley Digraphs



Hamiltonian Circuits and Paths

Hamiltonian Circuit is a sequence of arcs such that each vertex in a graph is visited exactly once before returning to the starting vertex.

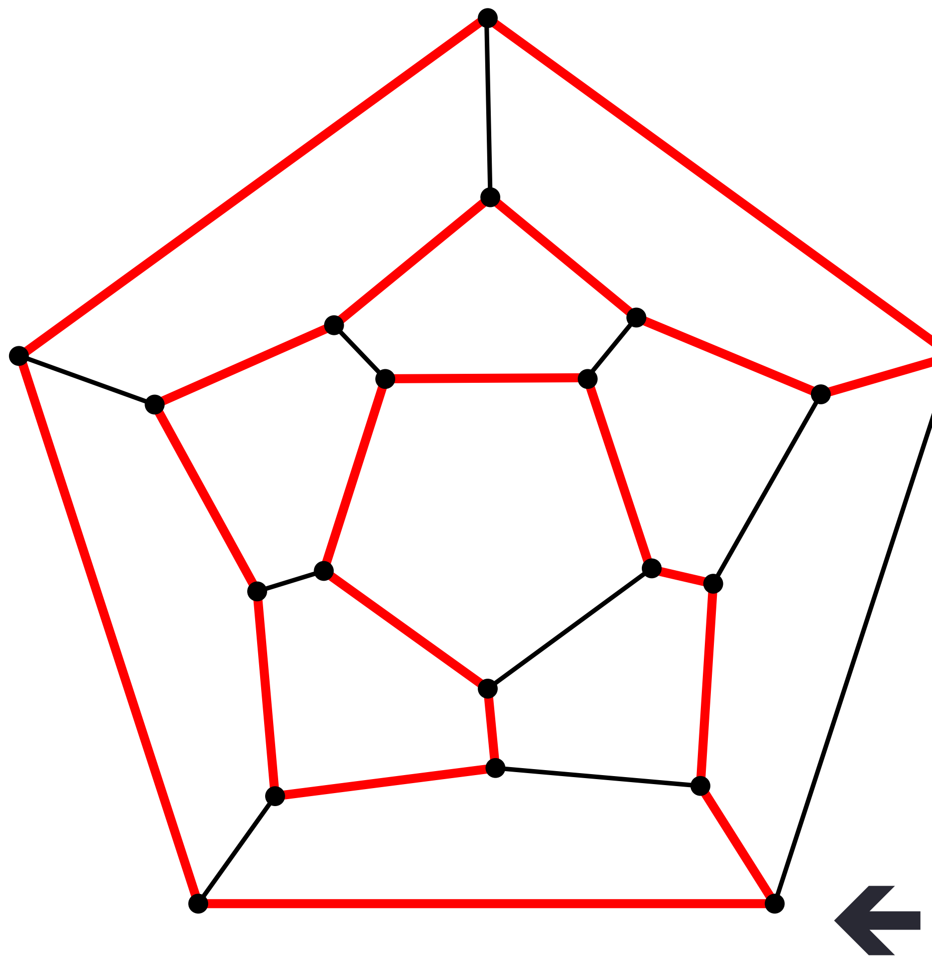
Hamiltonian Path is a sequence of arcs such that each vertex in a graph is visited exactly once.

Around the World

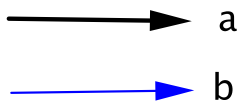
In 1859, Sir William Hamilton created a puzzle called *Around the World*. He illustrated the problem as a dodecahedron on a map, where each vertex was labeled with a famous city. He then asked if it was possible to visit each city exactly once, beginning and ending at the same city.

This is when the notion of Hamiltonian circuits and paths were first introduced. The natural questions which then arose were: When does a Cayley digraph contain a Hamiltonian circuit? A Hamiltonian path? Both? Neither?

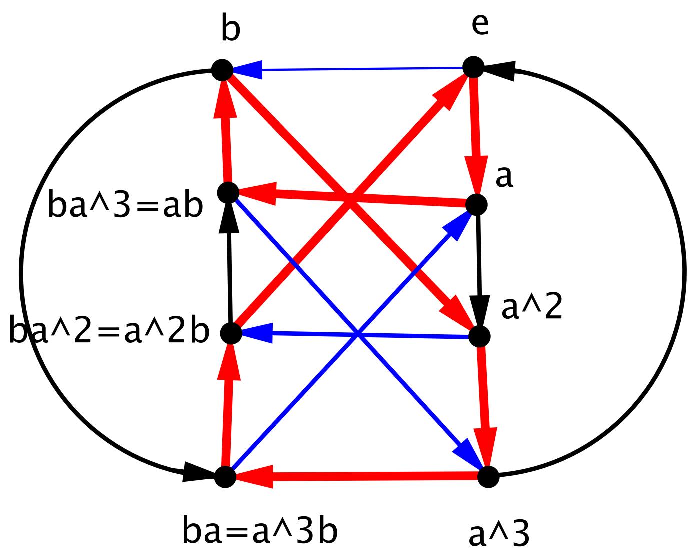
Around the World



Hamiltonian Circuit in a Cayley Digraph

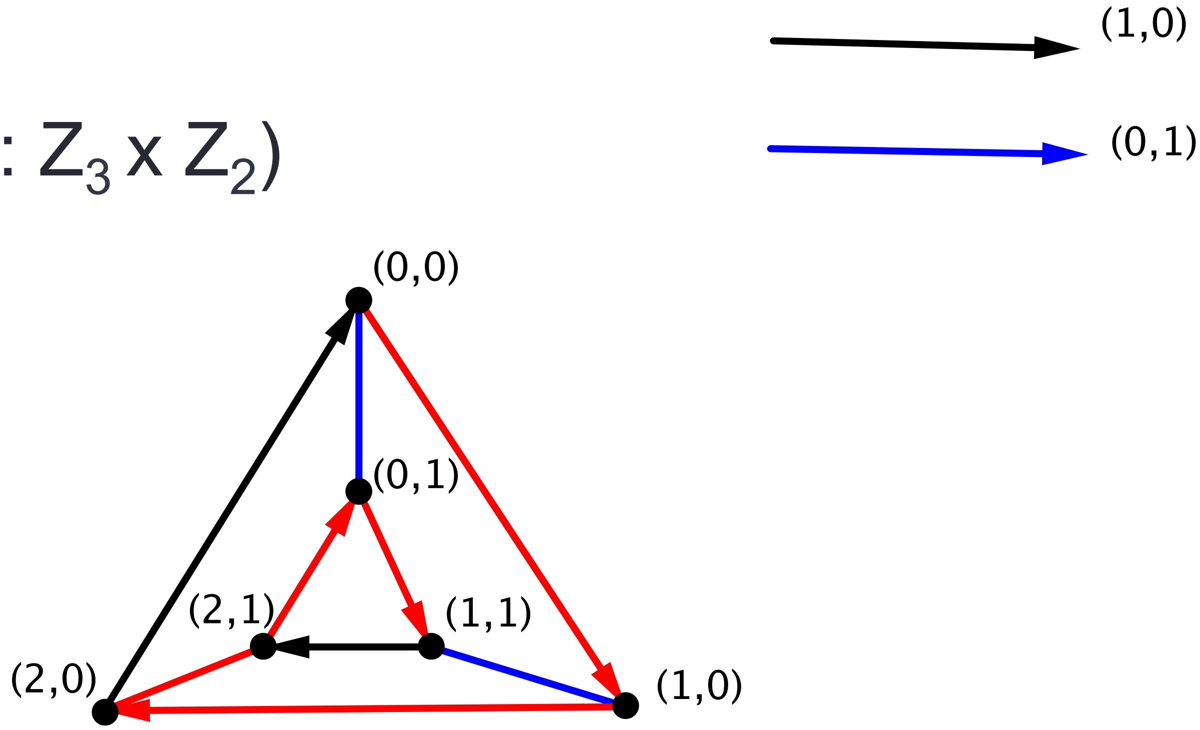


Cay({a,b}: Q_4)



Hamiltonian Path in a Cayley Digraph

Cay({(1,0),(0,1)}: Z_3 x Z_2)



Relevant Theorems

Theorem 1) Cay({(1,0),(0,1)}: Z_m x Z_n) does not have a Hamiltonian circuit when m and n are relatively prime and greater than 1.

Theorem 2) Cay({(1,0),)(0,1)}: Z_m x Z_n) has a Hamiltonian circuit when n divides m.

Theorem 3) Let G be a finite abelian group, and let S be any nonempty generating set for G. Then Cay(S:G) has a Hamiltonian path.

Bell Ringing

Bell Ringing, or campanology, was invented in England in the 1600's. It involves a concept called change ringing, which is when every permutation on n bells is rung without repetition.

It was about 200 years later before mathematicians realized that there was a relationship between bell ringing and group theory.

Bell Ringing

The official rules of bell ringing are as follows:

1) The bells are first rung in order from highest pitch to lowest pitch, also known as *rounds.*
2) Every bell is rung exactly once in each permutation.
3) Each bell moves at most one position from one row to the next.
4) No permutation is repeated until returning to *rounds* after all possible permutations have been rung.

It is also incredible to find out that no ringer is allowed sheet music; each ringer must memorize the order and placement of their bell throughout the entire piece.

Bell Ringing-Definitions

Some definitions that we need to help us understand the art of bell ringing are:

Extent- an extent on n bells is the order the permutations on n bells need to be rung so that no permutation is rung twice, and the bells move at most one spot in the permutations.

Treble- the bell with the highest pitch; it will be the smallest bell rung and the bell with the lowest number.

Tenor- the bell with the lowest pitch; it will be the largest bell rung and the bell with the highest number.

Bell Ringing-Definitions

Transposition- a cycle in a permutation with only two elements. It switches the order of the two elements.

Lead- a lead is an order of permutations of bells in which the treble starts in the first position, moves to the end, and then moves back to the first position again.

Bell Ringing

There are many different pieces in bell ringing. The one we will examine more closely is the piece called Plain Bob Minimus, which consists of ringing the permutations on 4 bells. This means that the number of permutations being rung is 4!=24.

The permutations that comprise Plain Bob Minimus are 1234 (rounds), 2143, 2413, 4231, 4321, 3412, 3142, 1324, 1342, 3124, 3214, 2341, 2431, 4213, 4123, 1432, 1423, 4132, 4312, 3421, 3241, 2314, 2134, 1234.

Bell Ringing

The permutations that comprise Plain Bob Minimus are the elements of the group S_4, the symmetric group on 4 numbers. This makes sense because the symmetric group is comprised of all of the permutations on 4 numbers.

The feature that defines Plain Bob Minimus is the order in which the permutations are rung. On the next slide we will see the order in which the 24 permutations are rung.

Plain Bob Minimus

```
1234    1342    1423
X X     X X     X X
2143    3124    4132
 X       X       X
2413    3214    4312
X X     X X     X X
4231    2341    3421
 X       X       X
4321    2431    3241
X X     X X     X X
3412    4213    2314
 X       X       X
3142    4123    2134
X X     X X     X X
1324    1432    1243
  X       X       X
                1234
```

The x's represent the transpositions that take place to get to the next permutation in the sequence.

As you can see, there are 3 types of transpositions taking place here; the simultaneous switch of the first two and last two numbers,
the middle two numbers,
and the last two numbers.

Graphing Plain Bob Minimus

We mentioned that the permutations that comprise Plain Bob Minimus are the elements of S_4. After 200 years of bell ringing, mathematicians modeled Plain Bob Minimus using a Cayley digraph. They then realized that the question of ringing the full extent on 4 bells is equivalent to finding a Hamiltonian circuit within the Cayley digraph.

The first step needed was to determine the set of generators for Plain Bob Minimus.

Graphing Plain Bob Minimus

Mathematicians knew the order in which the bells were rung, so they had a good idea of what the generating set may be. Let’s revisit the order of Plain Bob Minimus and label the transpositions we see.

Graphing Plain Bob Minimus

1234
X X a
2143
X b
2413
X X a
4231
X b
4321
X X a
3412
X b
3142
X X a
1324

Consider the first 8 permutations rung; the pattern we have for the first 8 is abababa.

From this, we realize that we must have at least 2 generators (a and b) to generate the first 8 elements of S_4

Graphing Plain Bob Minimus

The last permutation in the first column of permutations is 1324. If we look at the order of permutations, the next permutation would be 1342.

To get from 1324→1342 we need to do a transposition of the last two numbers.

Since this is necessary in order to get all the permutations, this will be a new generator, called c, that we will add to the set of generators.

Graphing Plain Bob Minimus

```
1234
 X  X    a
2143
   X     b
2413
 X  X    a
4231
   X     b
4321
 X  X    a
3412
   X     b
3142
 X  X    a
1324
      X  c
```

So now we see that the pattern of generators is abababac, or $(ab)^3ac$.

We will call this product w.

Graphing Plain Bob Minimus

Now, to get the next 8 permutations that comprise Plain Bob Minimus, we multiply each permutation in the first column by the element w on the left.

When multiplied out, w equals the permutation 1342.

So take the first permutation, 1234 and multiply it on the left by 1342:

$$\begin{pmatrix} 1234 \\ 1342 \end{pmatrix}\begin{pmatrix} 1234 \\ 1234 \end{pmatrix} = \begin{pmatrix} 1234 \\ 1342 \end{pmatrix}$$

If this is done to each of the permutations in the first column, we get the next column of 8 permutations:

Graphing Plain Bob Minimus

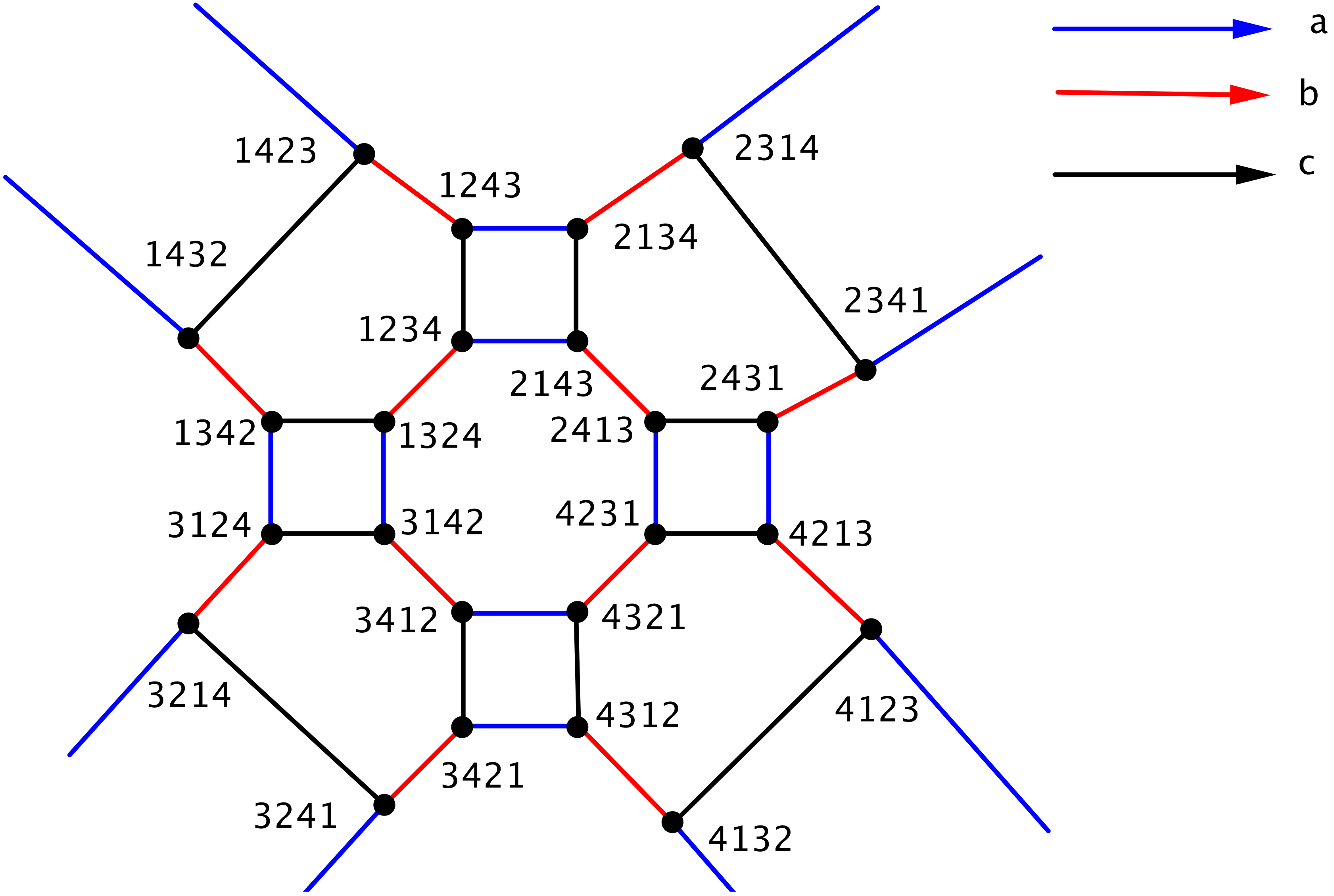
1234		1342	
X X	a	X X	a
2143		3124	
X	b	X	b
2413		3214	
X X	a	X X	a
4231		2341	
X	b	X	b
4321		2431	
X X	a	X X	a
3412		4213	
X	b	X	b
3142		4123	
X X	a	X X	a
1324		1432	
X	c	X	c

Graphing Plain Bob Minimus

1234		1342		1423	
X X	a	X X	a	X X	a
2143		3124		4123	
X	b	X	b	X	b
2413		3214		4312	
X X	a	X X	a	X X	a
4231		2341		3421	
X	b	X	b	X	b
4321		2431		3241	
X X	a	X X	a	X X	a
3412		4213		2314	
X	b	X	b	X	b
3142		4123		2134	
X X	a	X X	a	X X	a
1324		1432		1243	
X	c	X	c	X	c
				1234	

Finally, if we multiply the first column of 8 permutations on the left by w^2 we get the last 8 permutations, giving us all 24 elements of S_4.

This tells us that there are only the 3 generators as we originally thought, and now we can create the digraph Cay(S:S_4) where $S=\{a,b,c\}$.



a
b
c
1423
1243
2314
2134
1432
2341
1234
2143
2431
1342
1324
2413
3124
3142
4231
4213
3412
4321
3214
4312
4123
3421
3241
4132

Plain Bob Minimus

It turns out that those 3 columns of 8 permutations are each *leads* of Plain Bob Minimus; they have the treble (1) going from the front to the back to the front again.

Also, the first lead should look familiar to us.

This set of 8 elements,

{1234, 2143, 2413, 4231, 4321, 3412, 3142, 1324}

is the dihedral group D_4.

Finally, each lead is a coset of the group S_4, so when the bell ringers ring Plain Bob Minimus, they are really ringing the cosets of the group S_4.

Campanalogia Improved:

OR, THE

ART *of* RINGING

MADE EASY,

By Plain and Methodical Rules and Directions, whereby the Ingenious Practitioner may, with a little Practice and Care, attain to the Knowledge of Ringing all Manner of *Double*, *Tripple*, and *Quadruple Changes*.

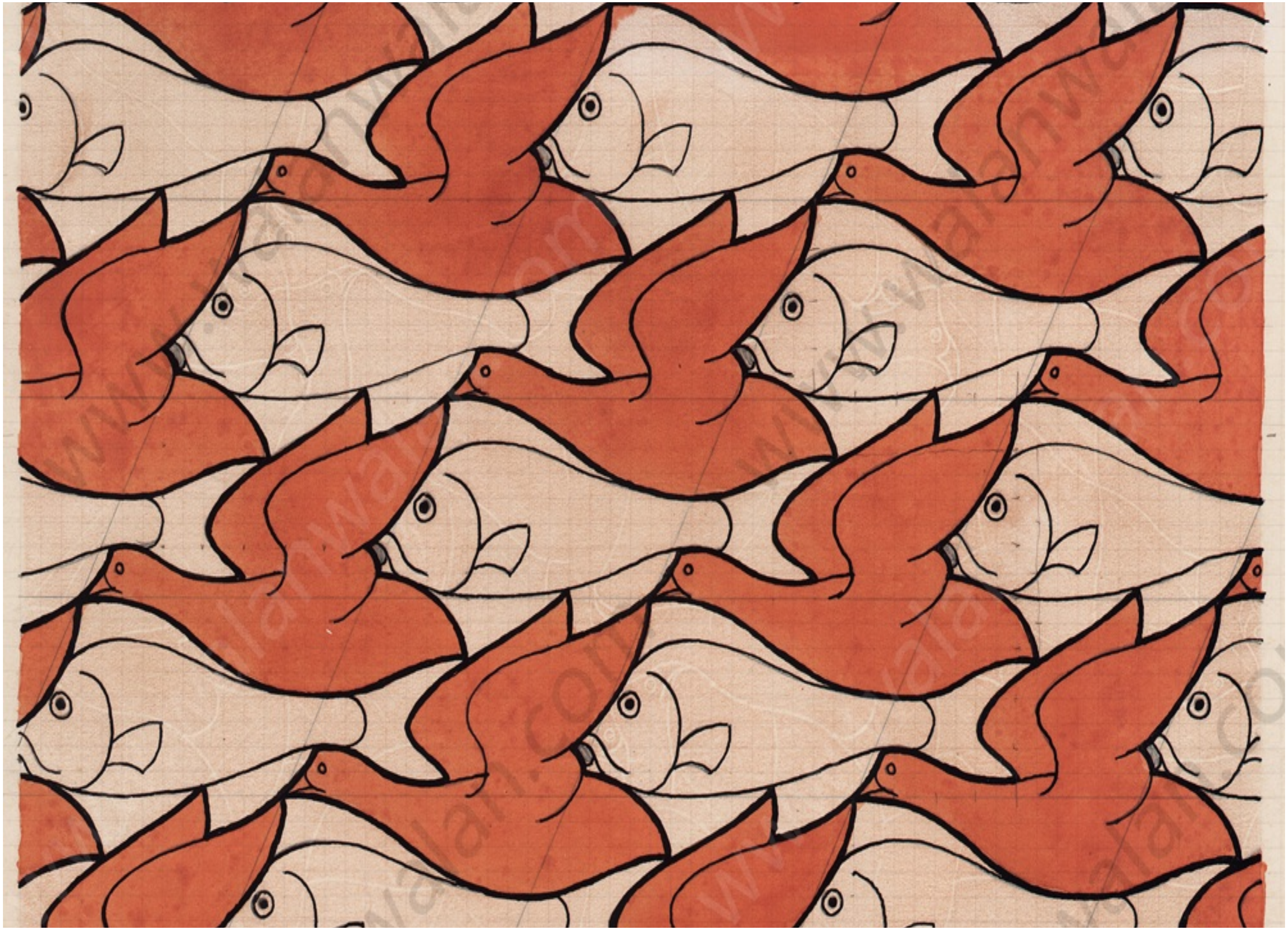
With Variety of *New Peals* upon Five, Six, Seven, Eight, and Nine Bells. As alſo the Method of calling *Bobs* for any *Peal* of *Tripples* from 168 to 2520 (being the *Half Peal*:) Alſo for any *Peal* of *Quadruples*, or *Cators* from 324 to 1140.

Never before Publiſhed.

The THIRD EDITION, Corrected.

Escher Patterns

Another application that uses Cayley digraphs are Escher patterns. An algorithm was written in 1981 which used Hamiltonian paths and Cayley digraphs to recreate the Escher-type repeating patterns.



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