# Fractals in Finance: Price Jumps and a Peculiar Smile

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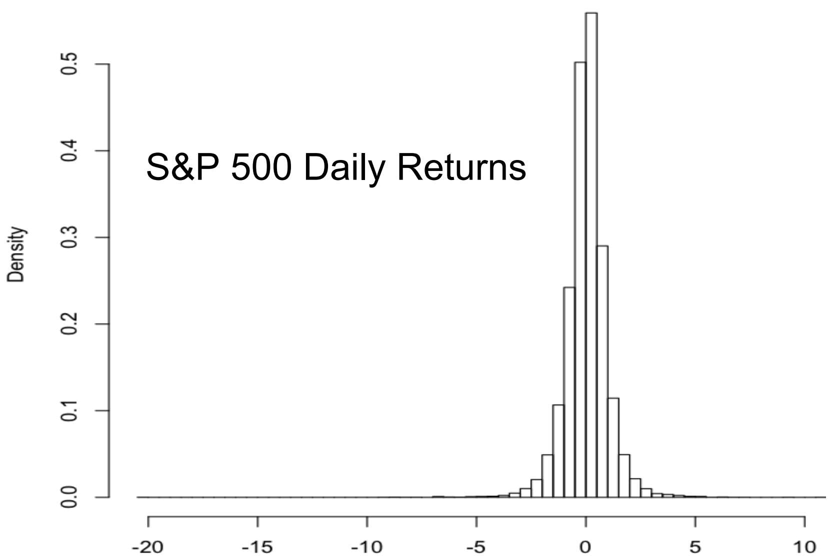
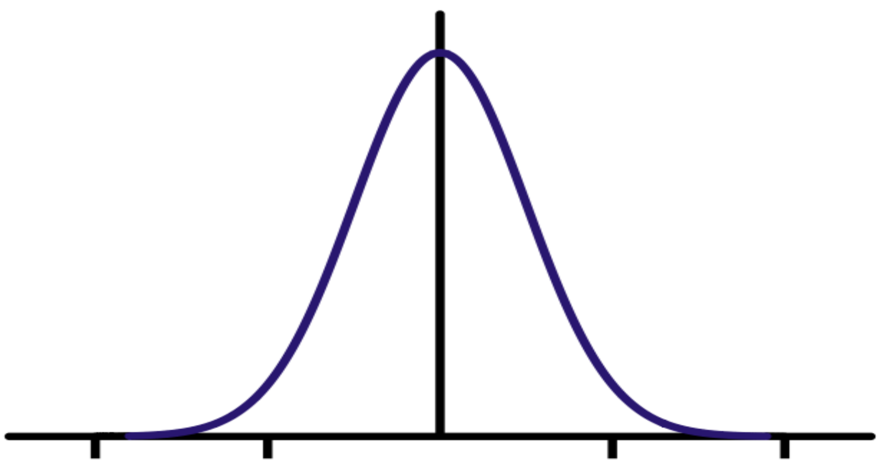
Slides: http://tinyurl.com/SeilerProb13

# The Normal Approach

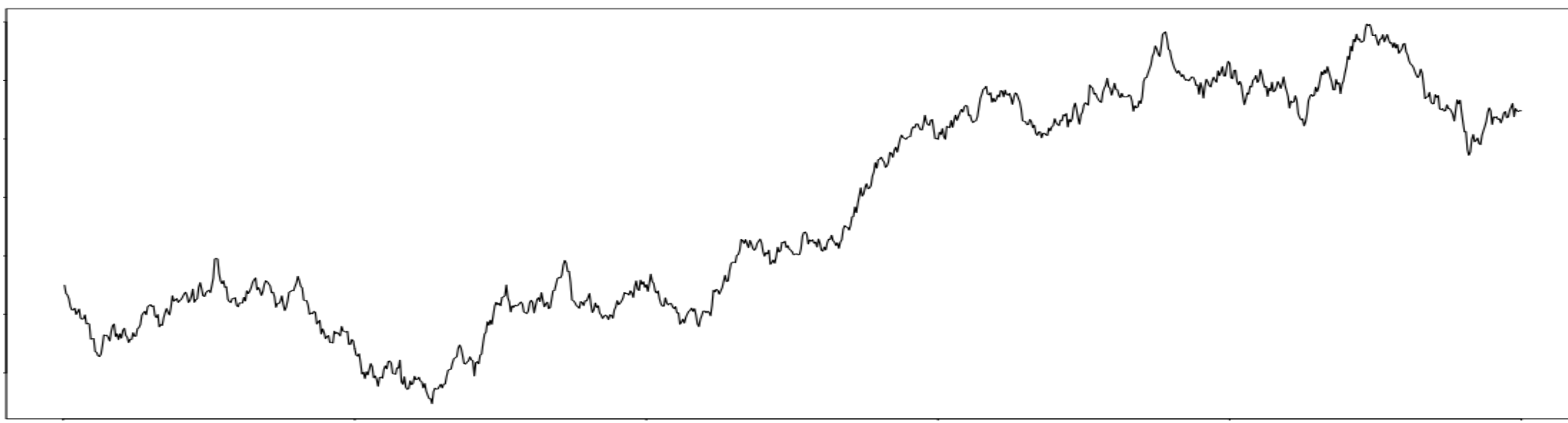
Normal Distribution

# Brownian Motion

1. $W_0 = 0$.
2. $W_t$ is almost surely continuous.
3. $W_t$ has idependent increments.
4. $W_t - W_s \sim N(0, t-s)$

$$\exp\left[\mathrm{i}\mu \mathrm{t} - \frac{1}{2}\sigma^2 \mathrm{t}^2\right]$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(\mathrm{x}-\mu)^2}{2\sigma^2}\right]$$

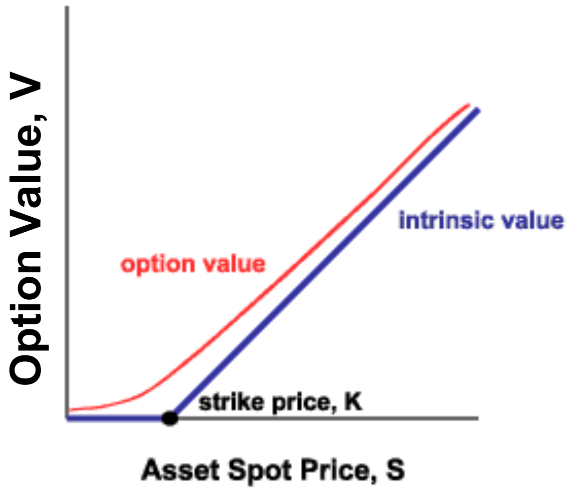


Brownian Simulation of a Stock Price

# Black-Scholes

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

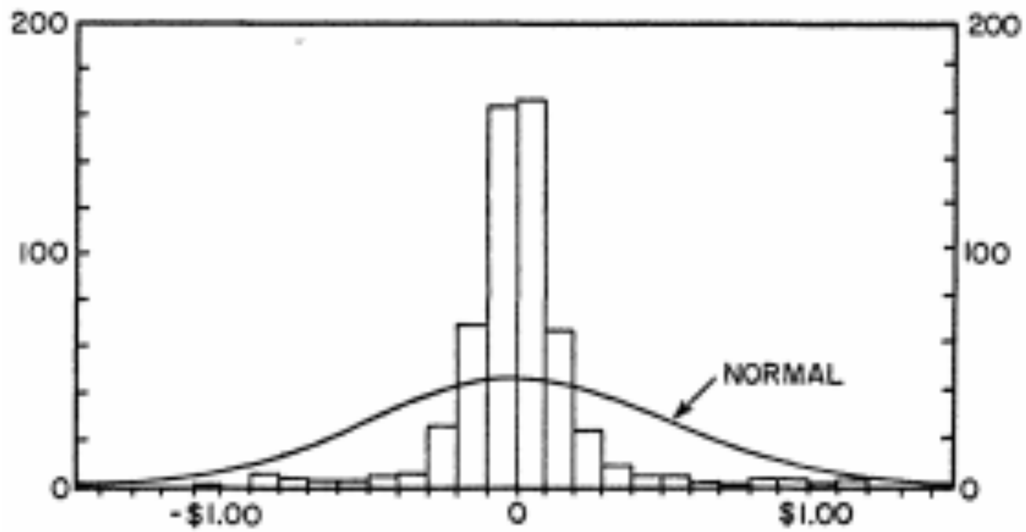
$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$



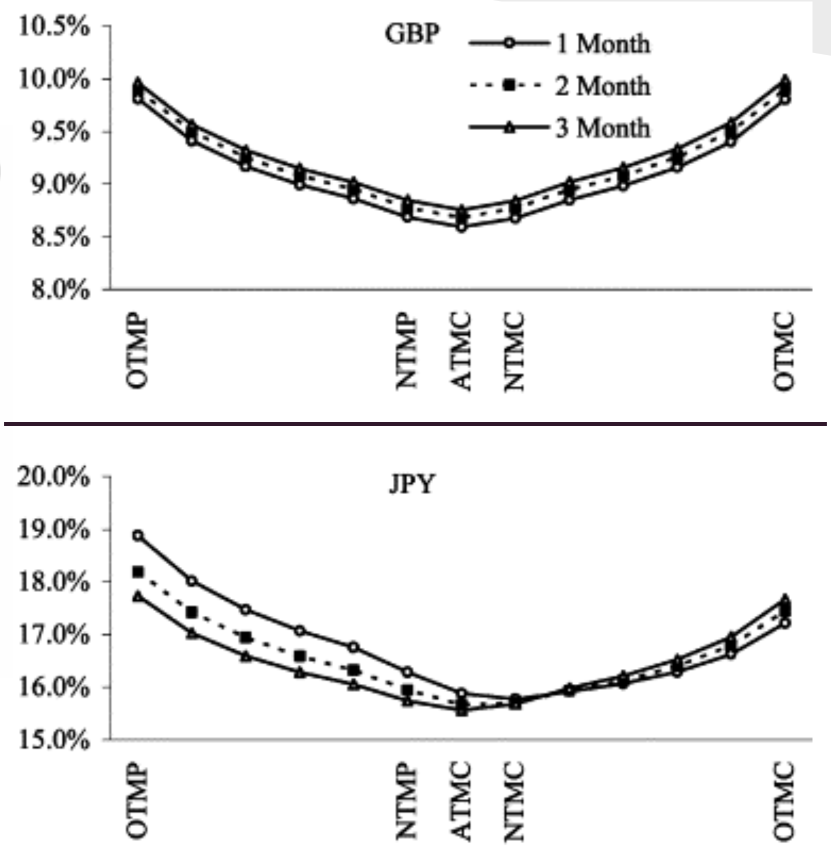

http://en.wikipedia.org/wiki/File:Option_value.gif

$$\begin{aligned} V_{call}(S,t) &= N(d_1)S - N(d_2)Ke^{-r\tau} \\ d_1 &= \frac{\ln(\frac{S}{K}) + \tau\left(r + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}} \\ d_2 &= d_1 - \sigma\tau \end{aligned}$$

# Bad Fits and Smiles



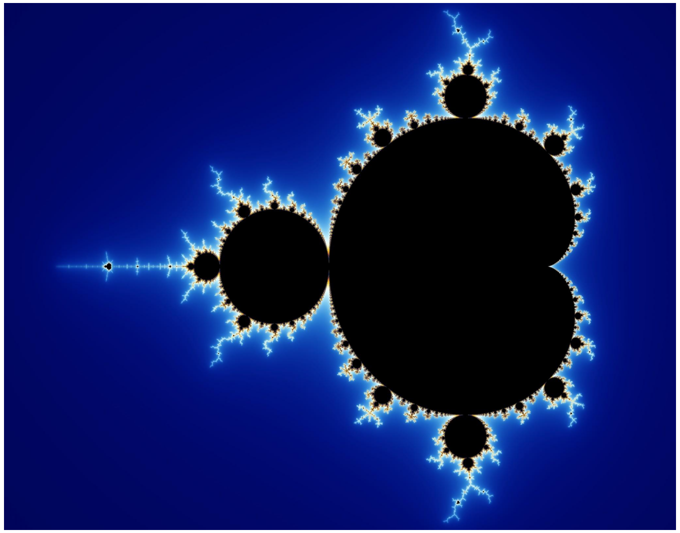

Monthly wool price returns [5]




Quoted Foreign Exchange Volatility Smiles

http://www.sciencedirect.com/science/article/pii/S0261560602000736

# Mandelbrot and Stability



Mandlebrot (top)  and the Mandlebrot Set (bottom)
http://commons.wikimedia.org/wiki/File:
Mandel_zoom_00_mandelbrot_set.jpg

$$\phi(t) = \exp\left[\mathrm{it}\mu - |\mathrm{ct}|^{\alpha}(1 - \mathrm{i}\beta\mathrm{sgn}(\mathrm{t})\Phi)\right]$$

$$\Phi(t) = \begin{cases} \tan\left(\frac{\pi\alpha}{2}\right) & \text{for } \alpha \neq 1 \\ -\frac{2}{\pi}\log|t| & \text{for } \alpha = 1 \end{cases}$$

$$\alpha \in (0,2],\ \beta \in [-1,1],\ \mathrm{c} \in (0,\infty),\ \mu \in \mathbb{R}$$

$$\Psi(t) = \int_{-\infty}^{\infty}\left(e^{itx} - 1 - it\tau_{\alpha}(x)\right)W(dx)$$

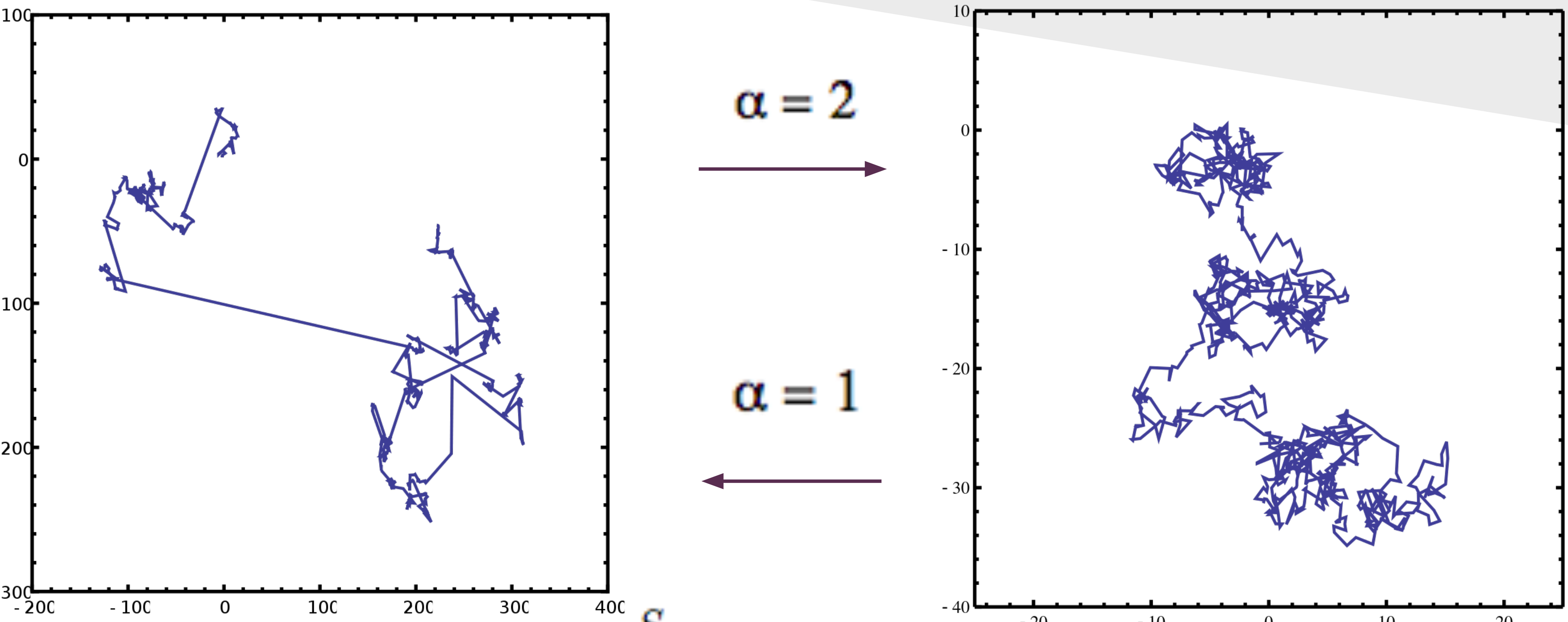
$$W(x) = \begin{cases} Cq|x|^{-1-\alpha} & \text{for } x < 0 \\ Cp|x|^{-1-\alpha} & \text{for } x > 0 \end{cases}$$

$$\tau_{\alpha}(x) = \begin{cases} x & \text{for } \alpha > 1 \\ \sin(x) & \text{for } \alpha = 1 \\ 0 & \text{for } \alpha < 1 \end{cases}$$

$$C, p, q > 0$$

$$p + q = 1$$

# Lévy Flights and Damped Lévy-Stable Processes




$$\ln \frac{S_{t+\Delta t}}{S_t} = \bar{\mu}\Delta t + \sigma\phi$$

$$\bar{\mu} = r - D - \Psi_{TL}(-i\tilde{\sigma})$$

with $\sigma = \Delta t^{1/\alpha}\tilde{\sigma} > 0, \lambda = \tilde{\lambda}\Delta t^{1/\alpha} > 0, \alpha > 1$, and $\phi \sim DL_\alpha$ with $\mu = 0$.

$$\Psi_{TL}(t) = xc^\alpha[p(\lambda - it)^\alpha + q(\lambda + it)^\alpha - \lambda^\alpha - i\alpha\lambda^{\alpha-1}(q-p)t]$$

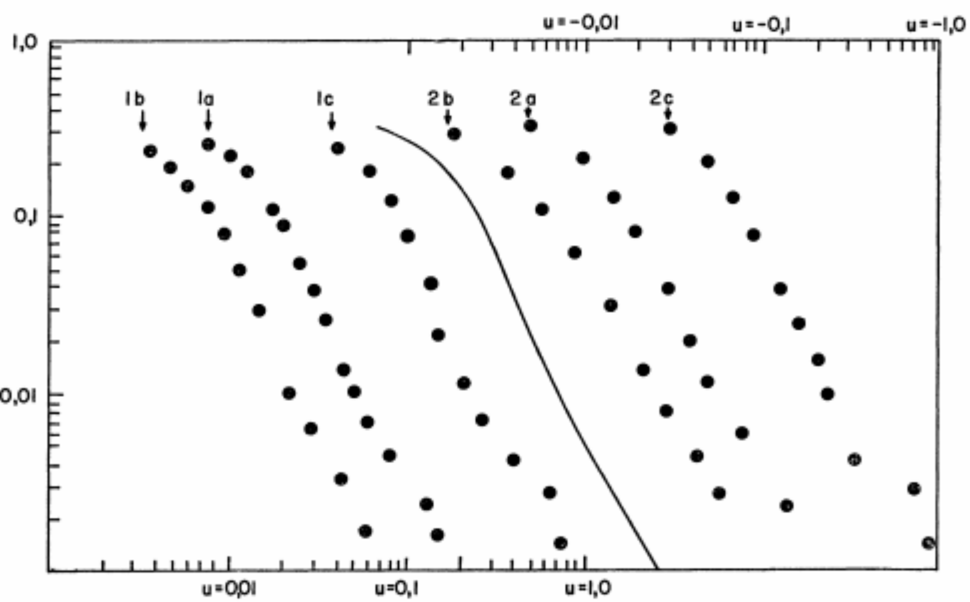
# Generalized Black-Scholes

$$-\frac{\partial \hat{V}}{\partial t} = \left\{ c^{\alpha} \left[ p \left( \tilde{\lambda} + i\zeta\tilde{\sigma} \right)^{\alpha} + q \left( \tilde{\lambda} - i\zeta\tilde{\sigma} \right)^{\alpha} - \tilde{\lambda}^{\alpha} + \alpha\tilde{\lambda}^{\alpha-1}\beta i\zeta\tilde{\sigma} \right] - i\bar{\mu}x - r \right\} \hat{V}$$

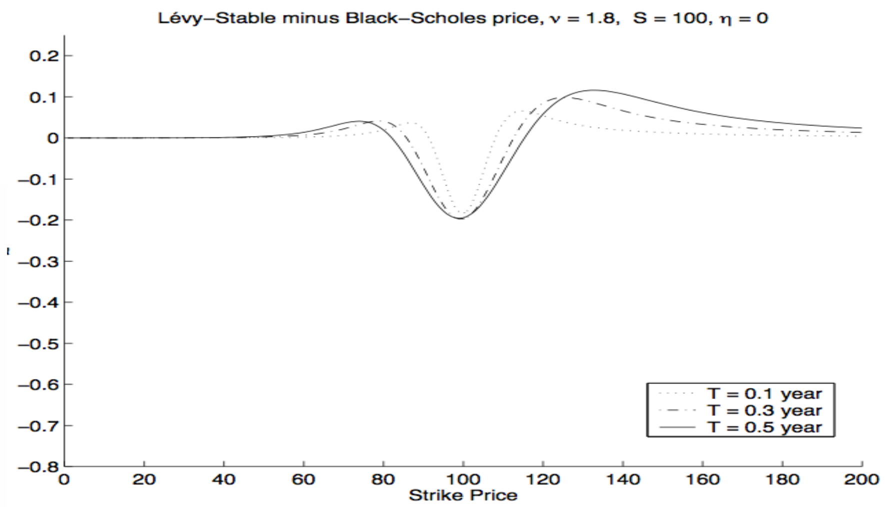
$$-\frac{\partial \hat{V}}{\partial t} = \left[ -\frac{1}{2}\tilde{\sigma}^2\zeta^2 + i\zeta\left(\frac{1}{2}\tilde{\sigma}^2 + D\right) - r(1 + i\zeta) \right] \hat{V}$$

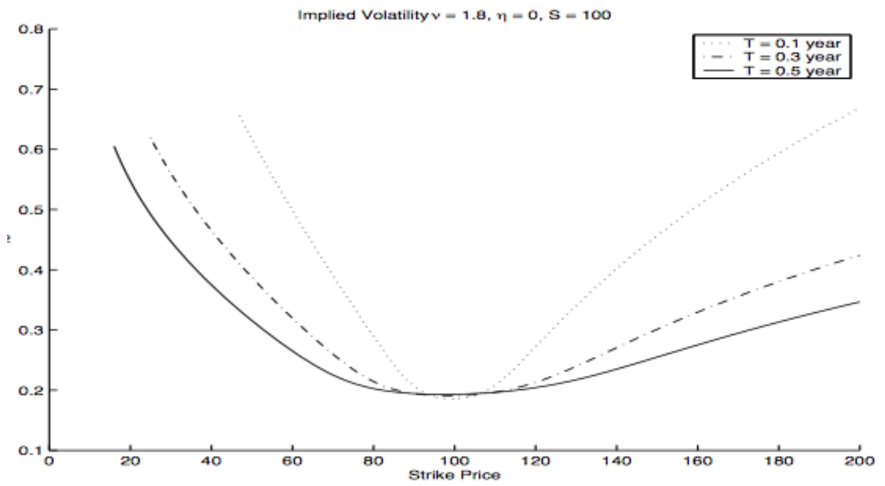
$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

# Improved Fits and a Natural Smile




Log-log plots of cotton price tails (points) vs prediction of stable distribution with alpha=1.7 (line) [5]




Improved option pricing and natural vol smile [1]

# References


[1] Alvaro Cartea and Sam Howison. Distinguishing limits of levy-stable processes, and applications to option pricing. Working paper, Oxford Financial Research Centre, 2003.

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[3] Eugene F. Fama. Mandelbrot and the stable paretian hypothesis. *The Journal of Business*, Vol. 36, No. 4, 1963.

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[5] Benoit Mandelbrot. The variation of certain speculative prices. *The Journal of Business*, Vol. 36, No. 4, 1963.

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