# The Checkerboard Challenge

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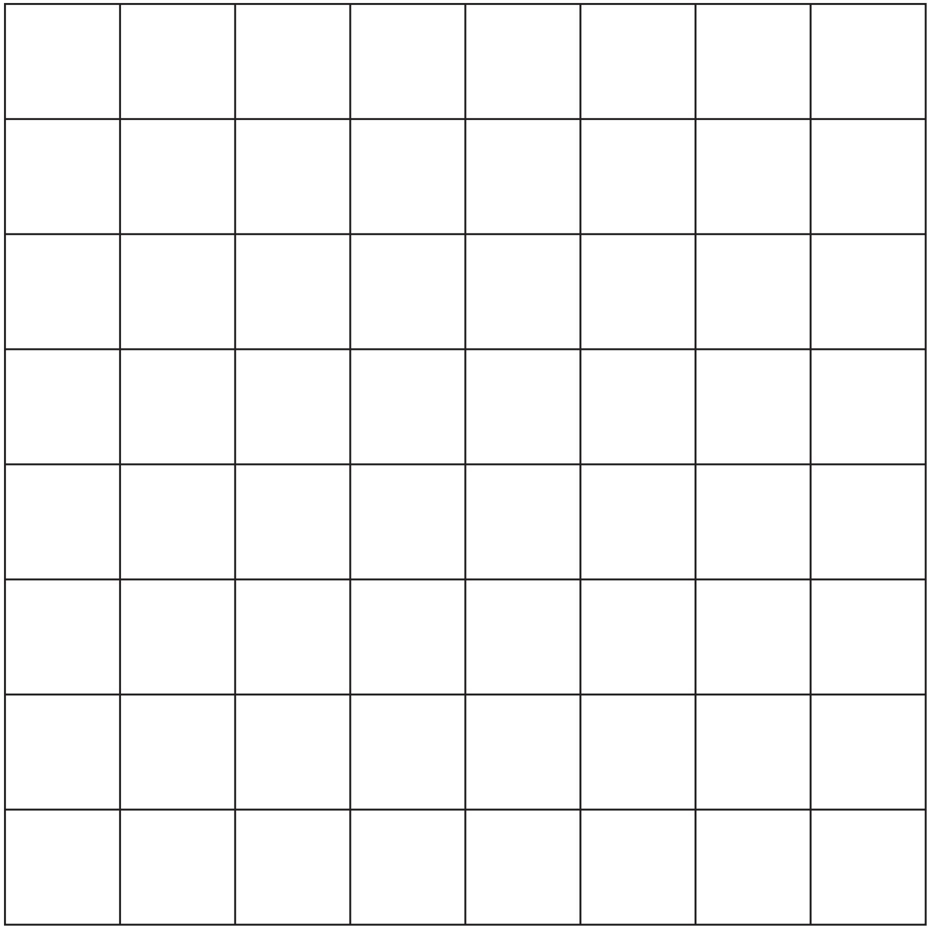
HRUMC 2013

# The Checkerboard



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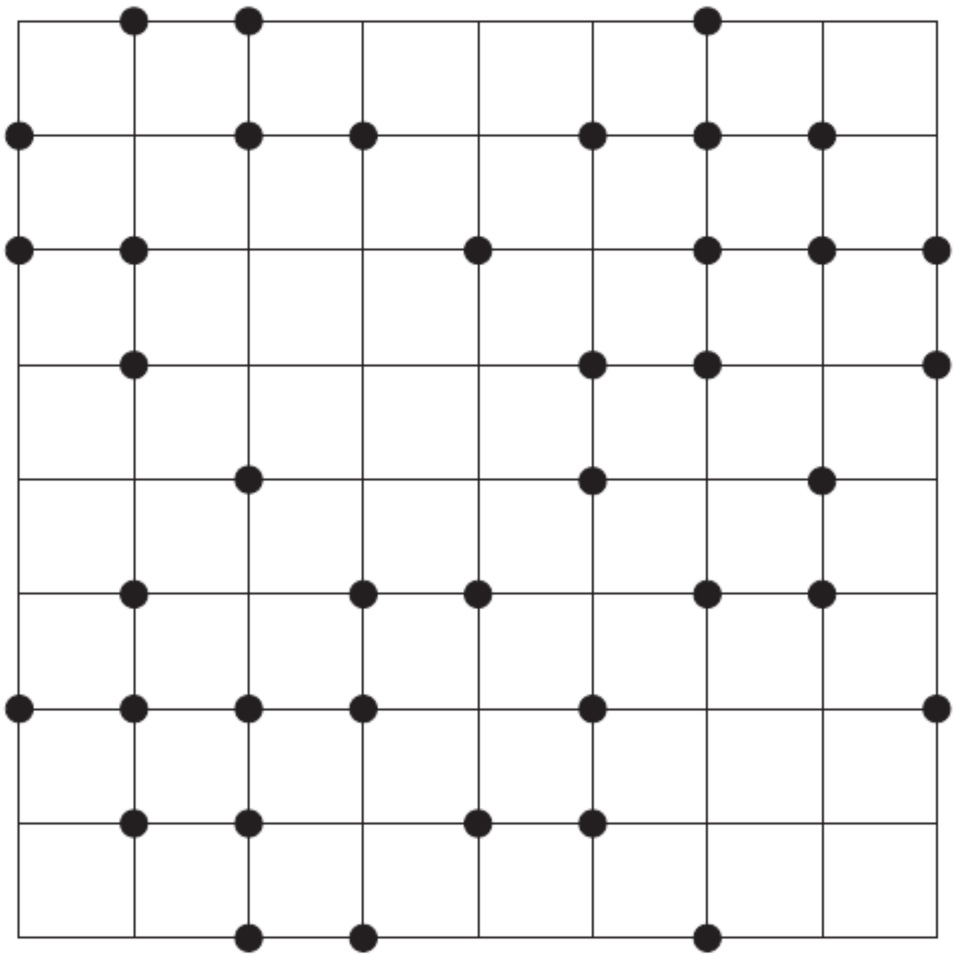

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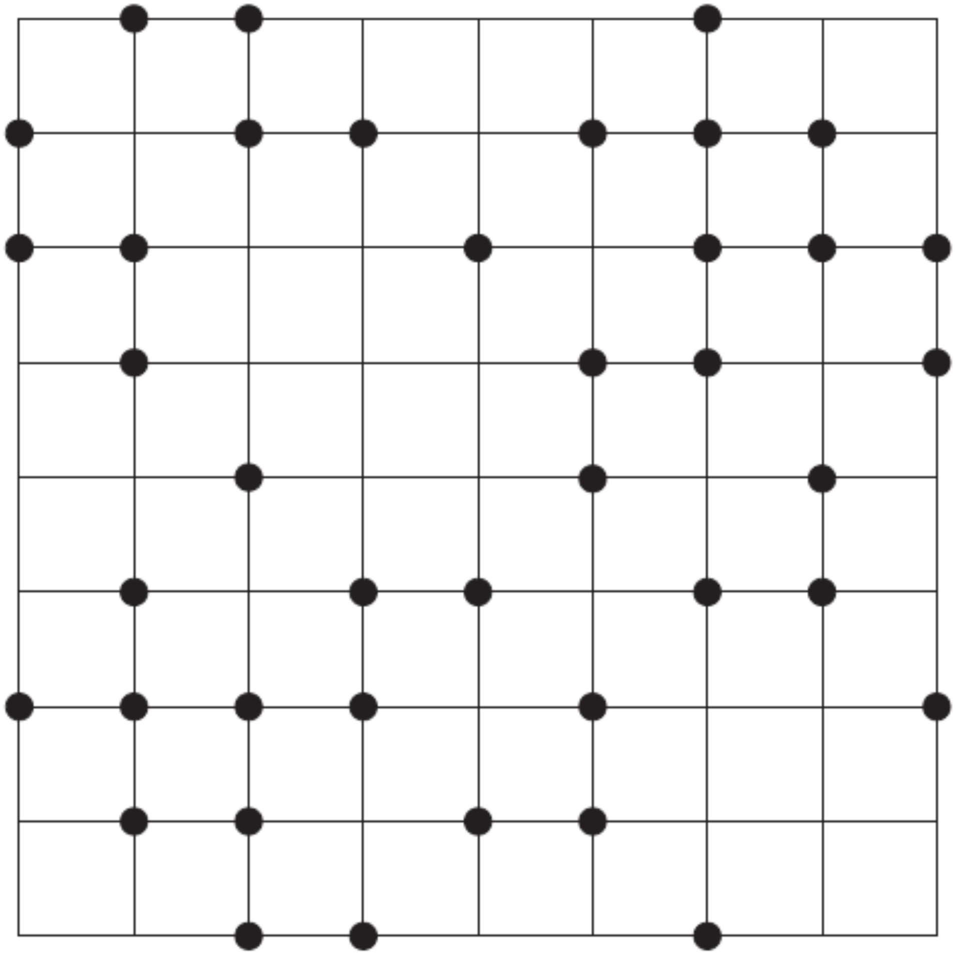

- The main diagonal of the grid contains no coins

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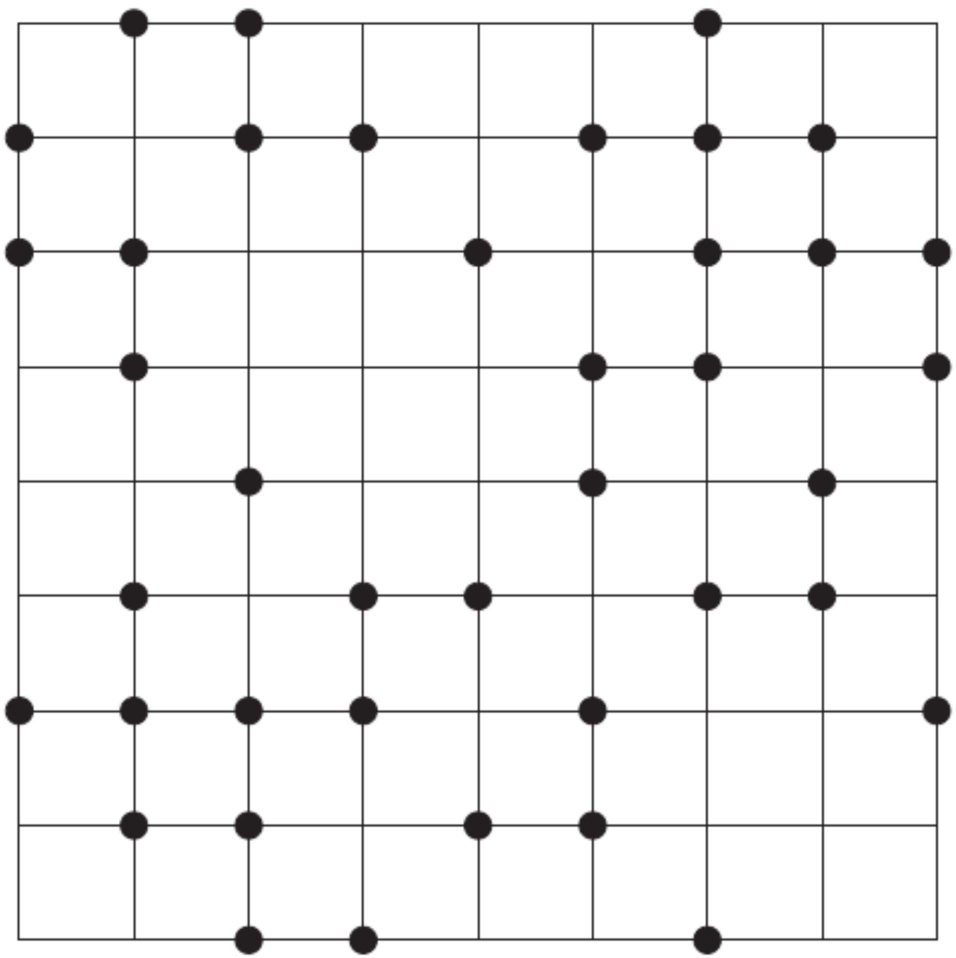
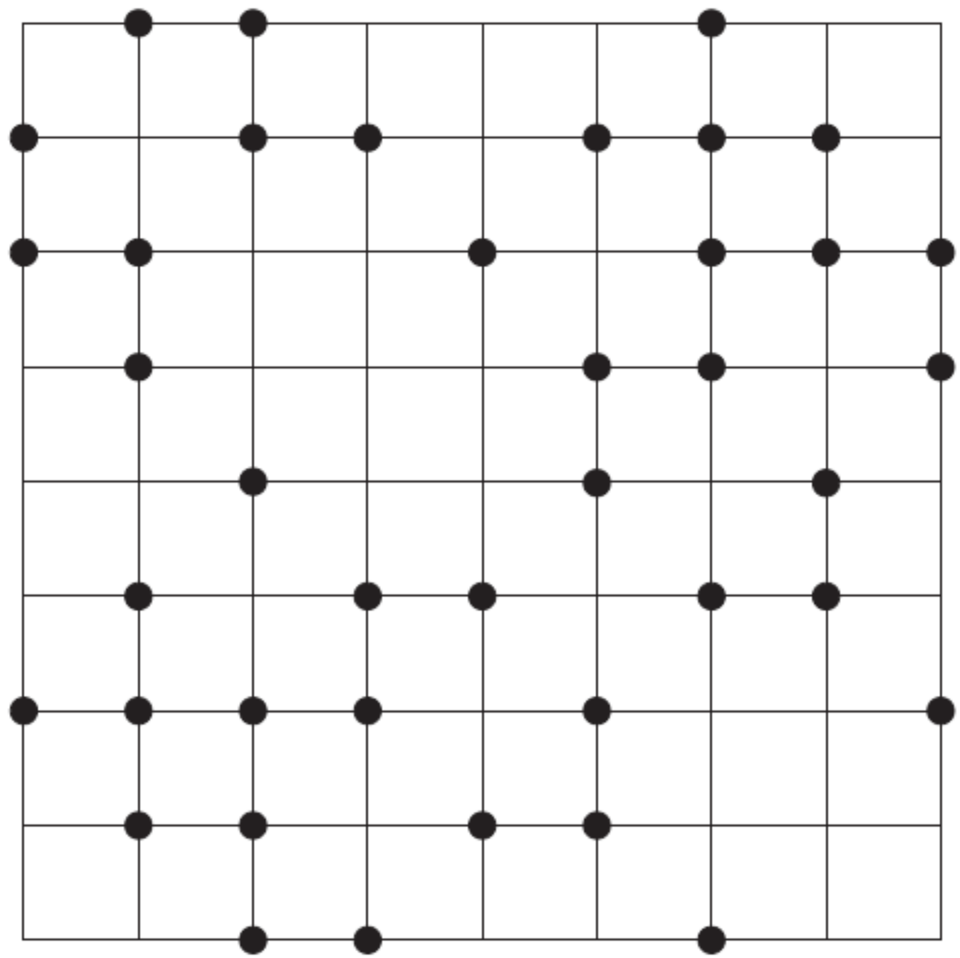


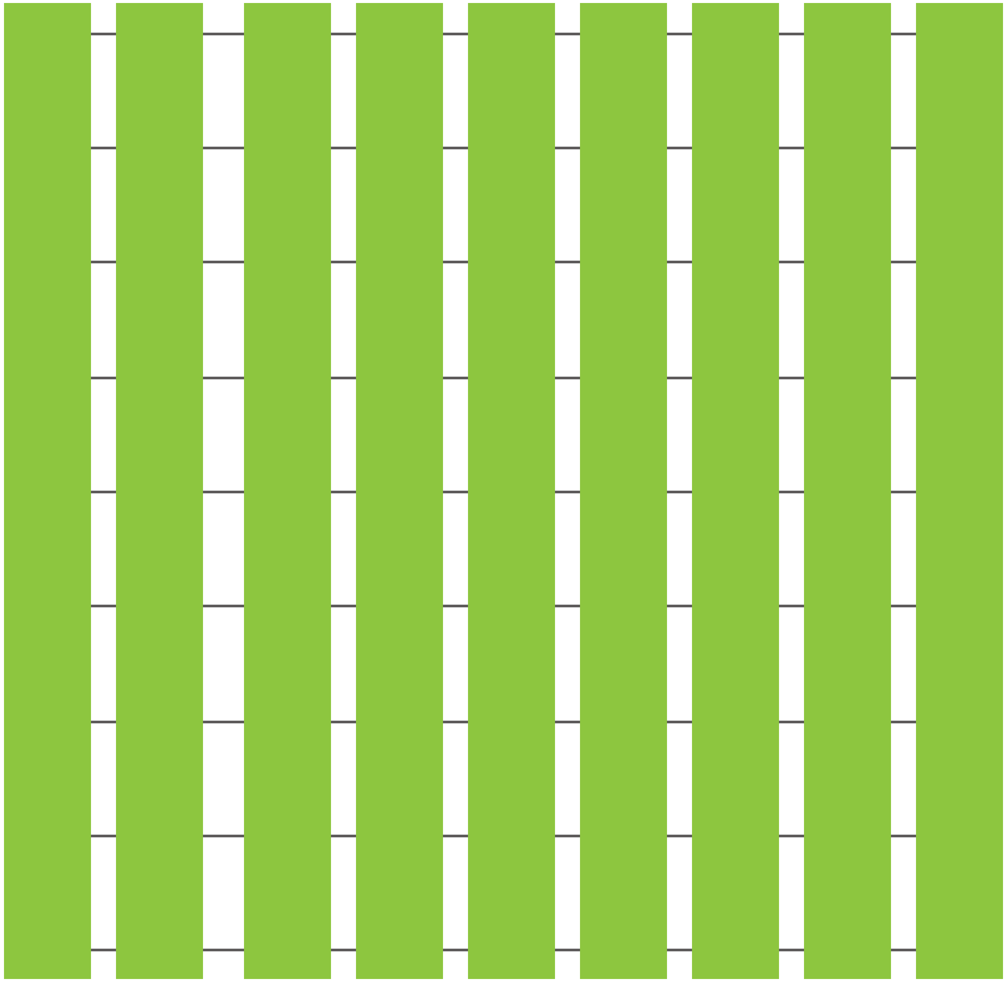
- The main diagonal of the grid contains no coins
- The arrangement of the coins is diagonally symmetric

# The Challenge



Cover $n$ columns, $n < 9$ such that an even number of coins remains visible in each row.

# The Challenge



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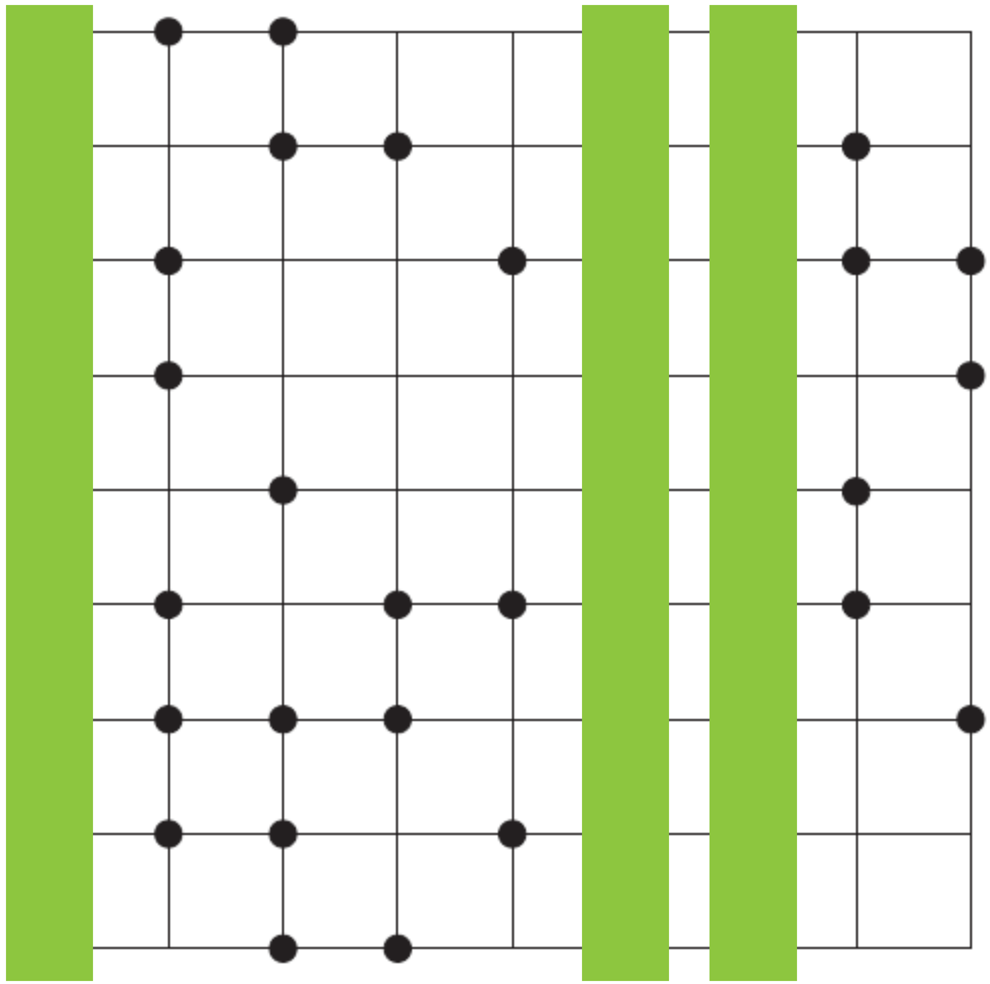
# The Solution?



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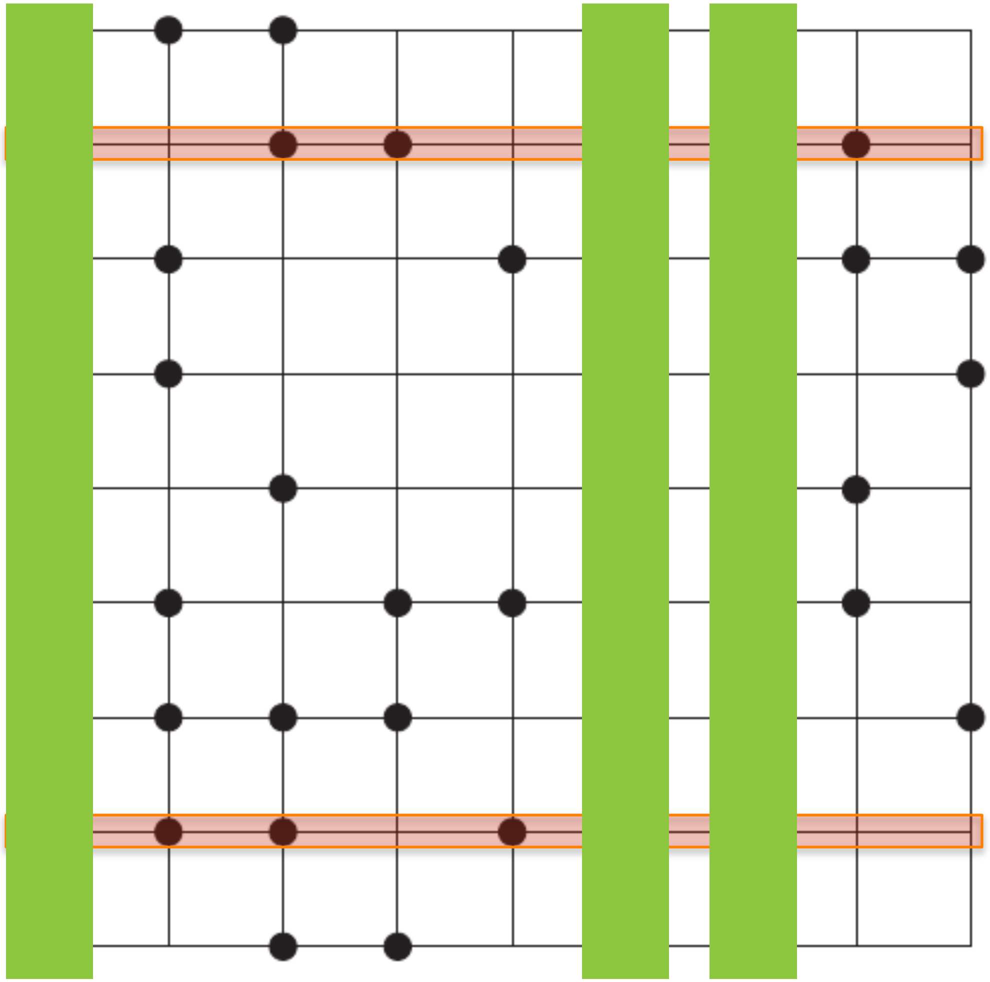

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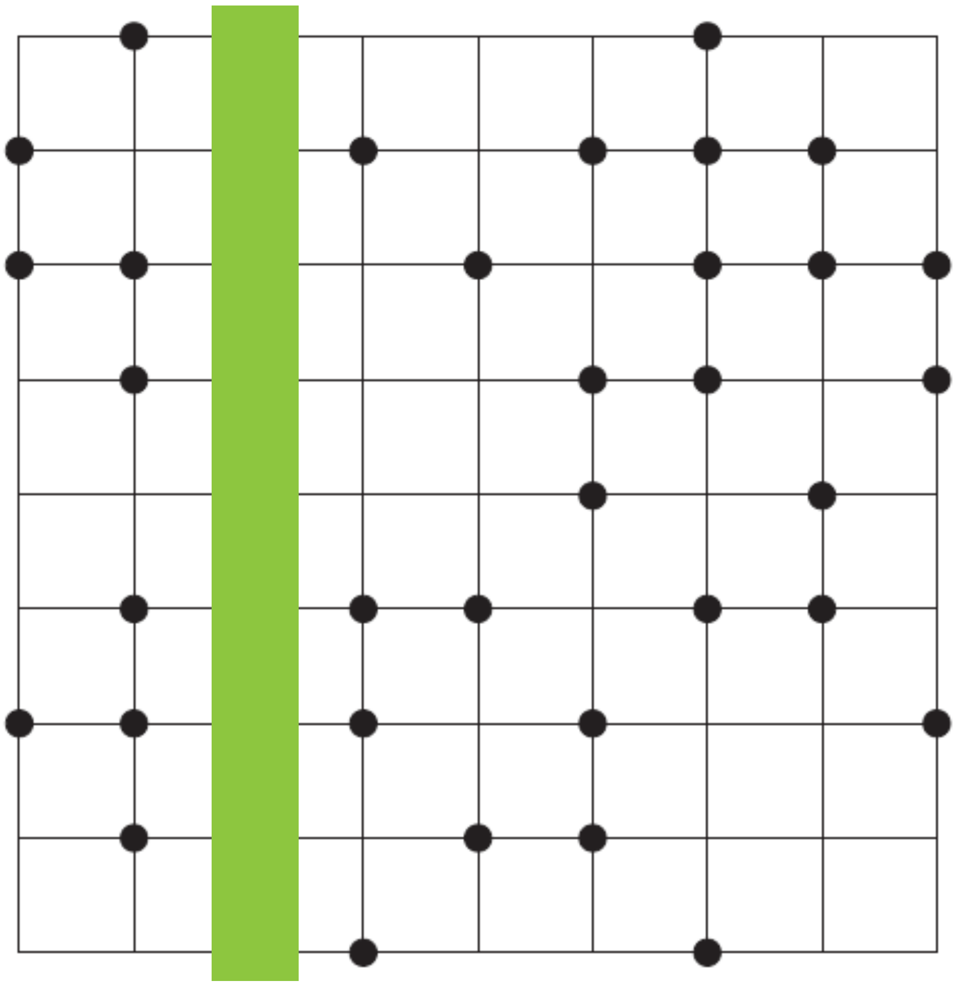

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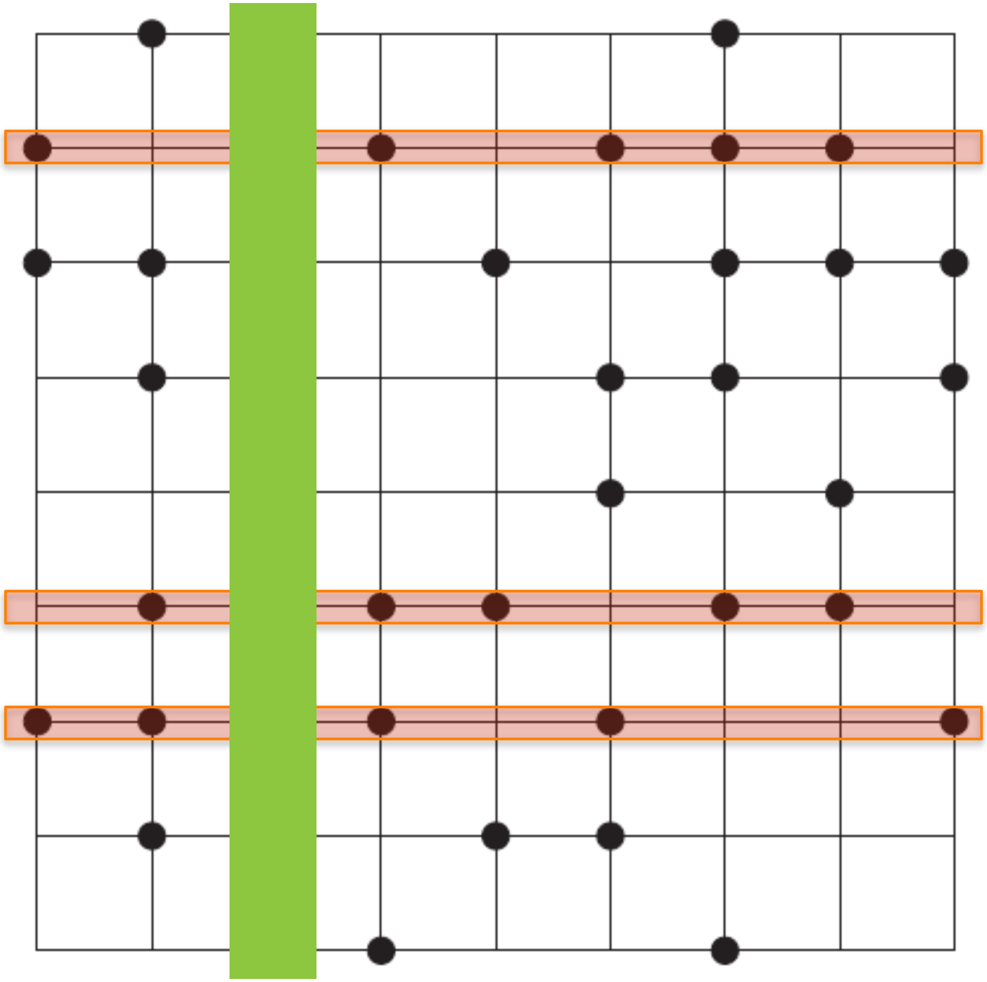

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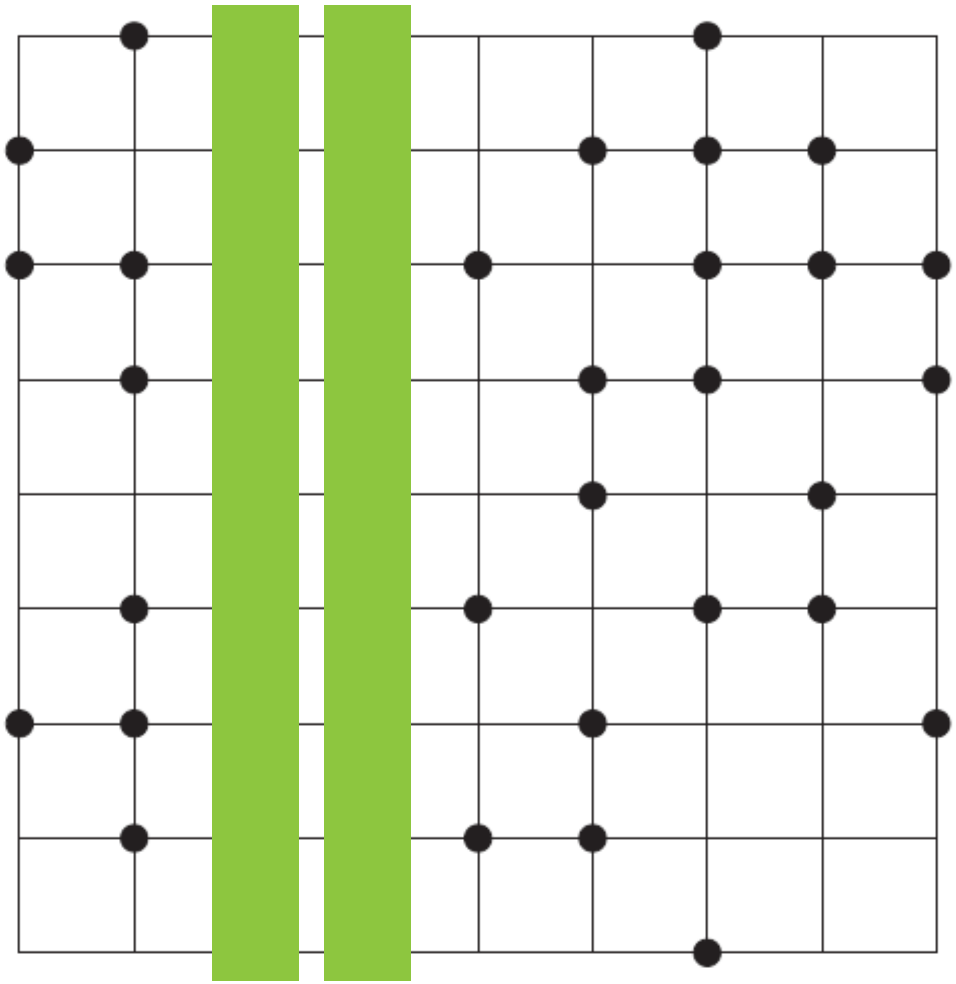

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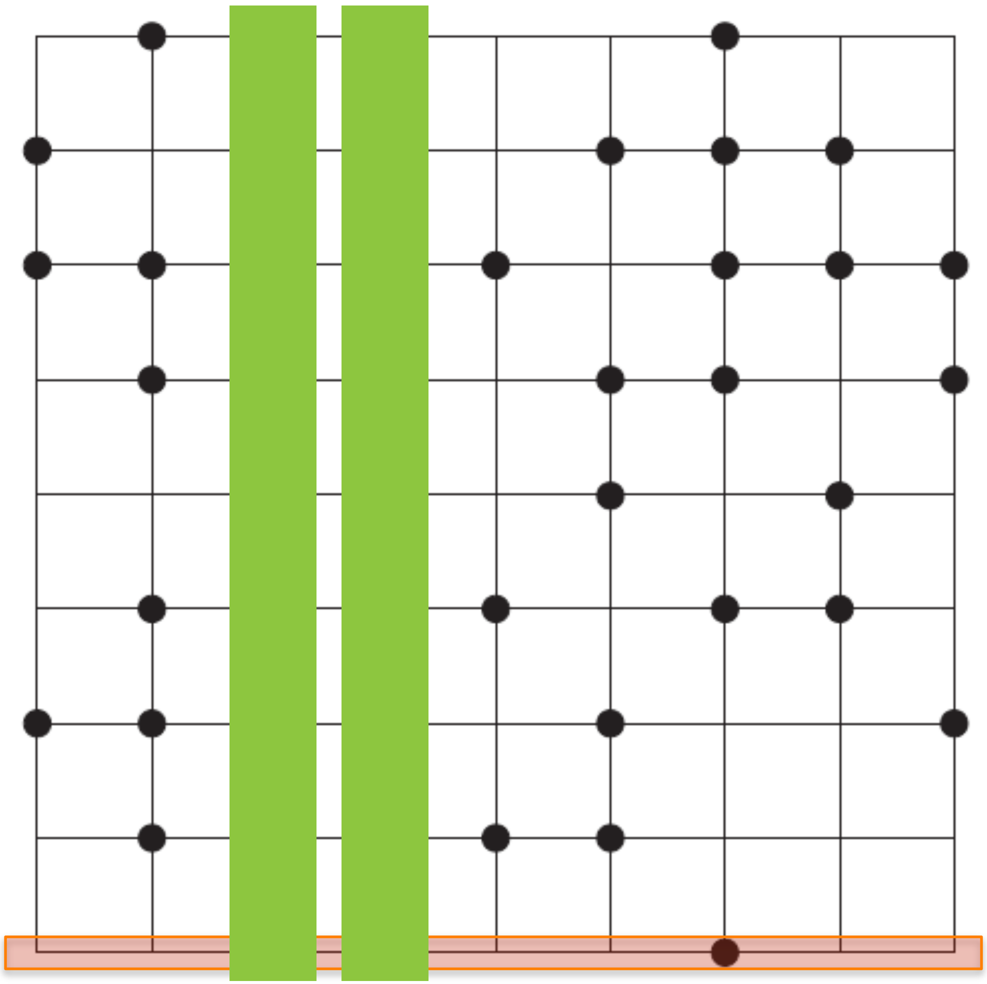

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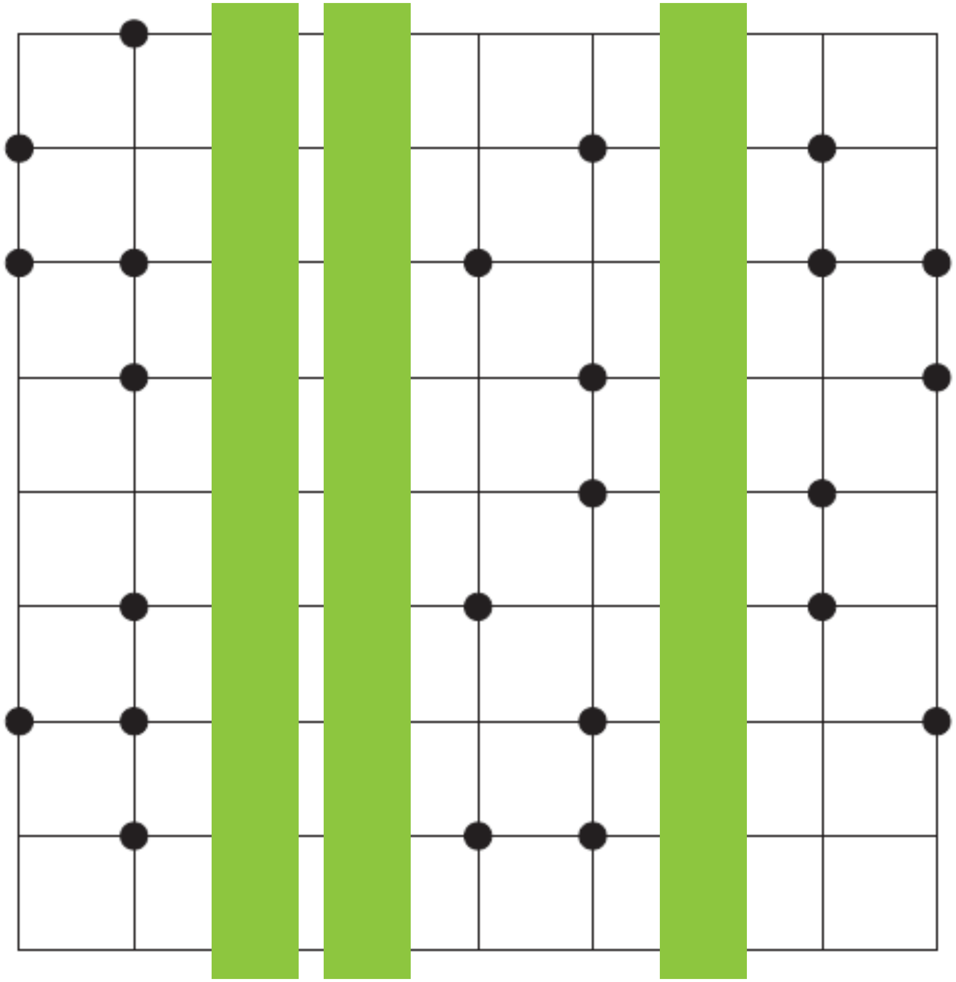

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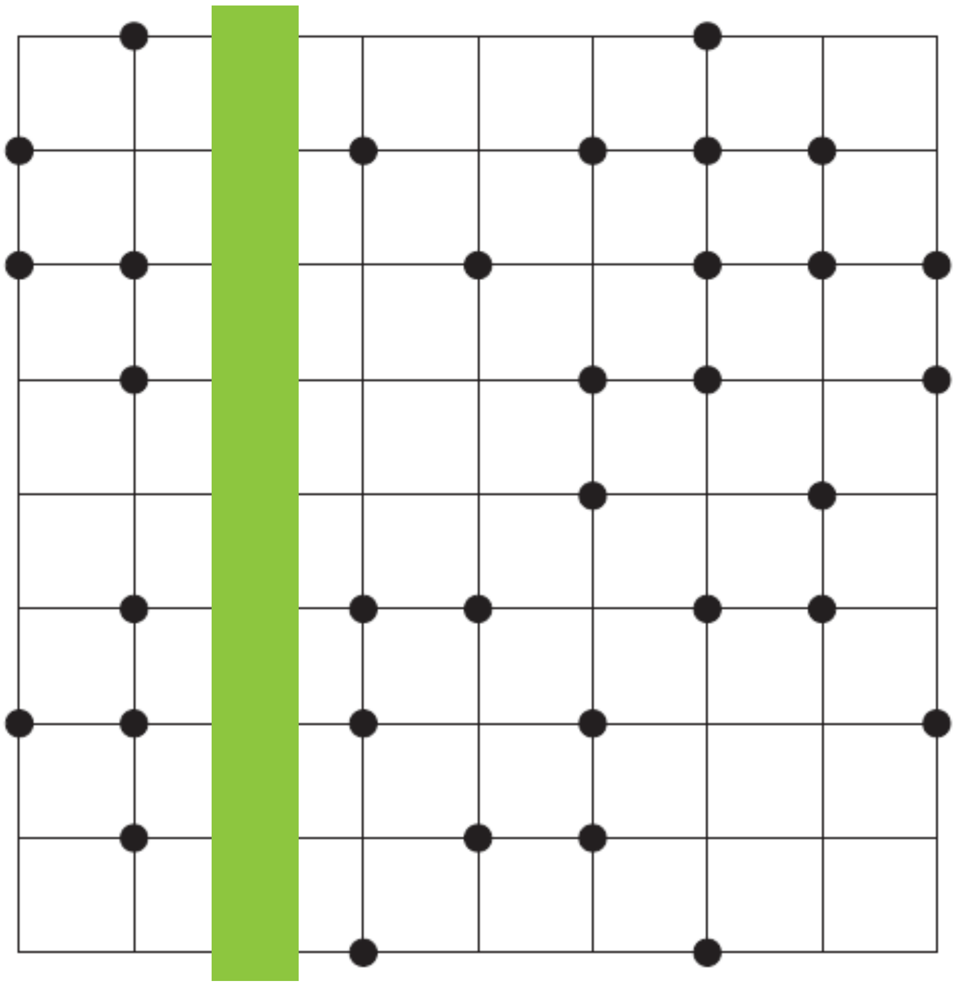

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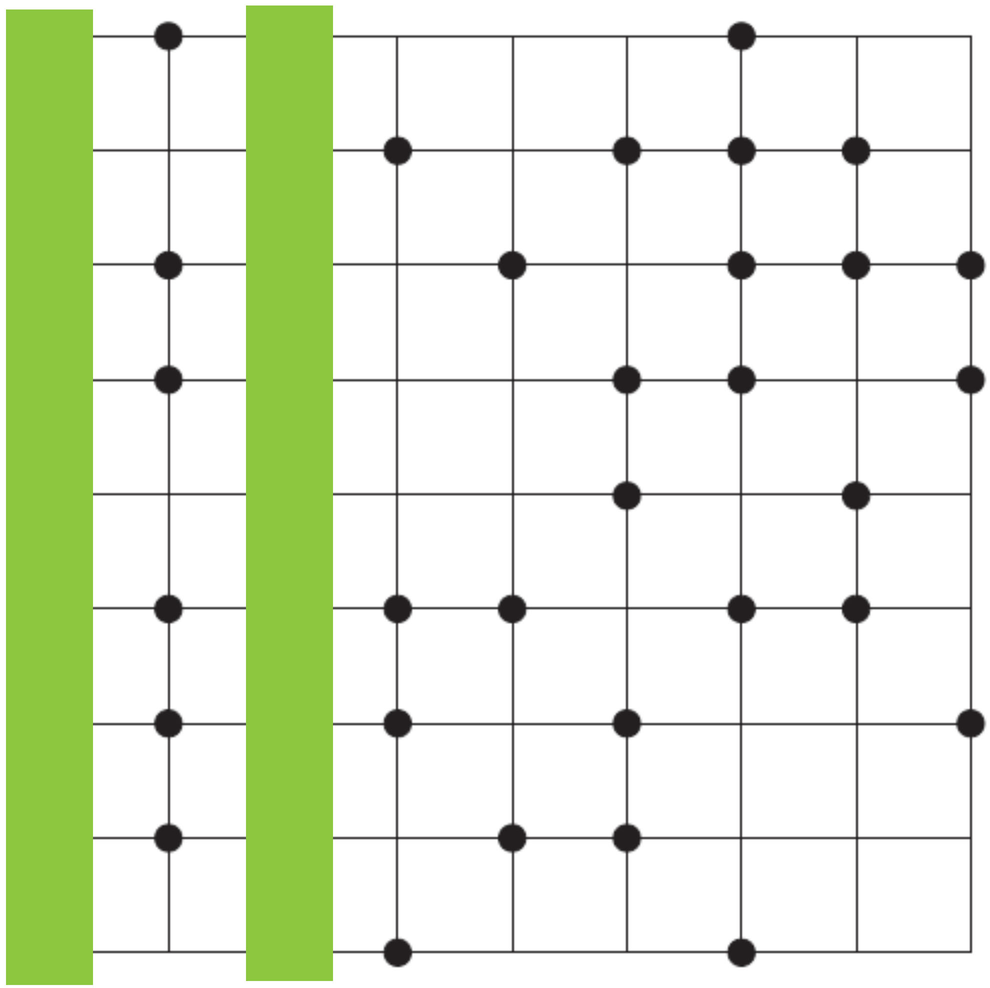
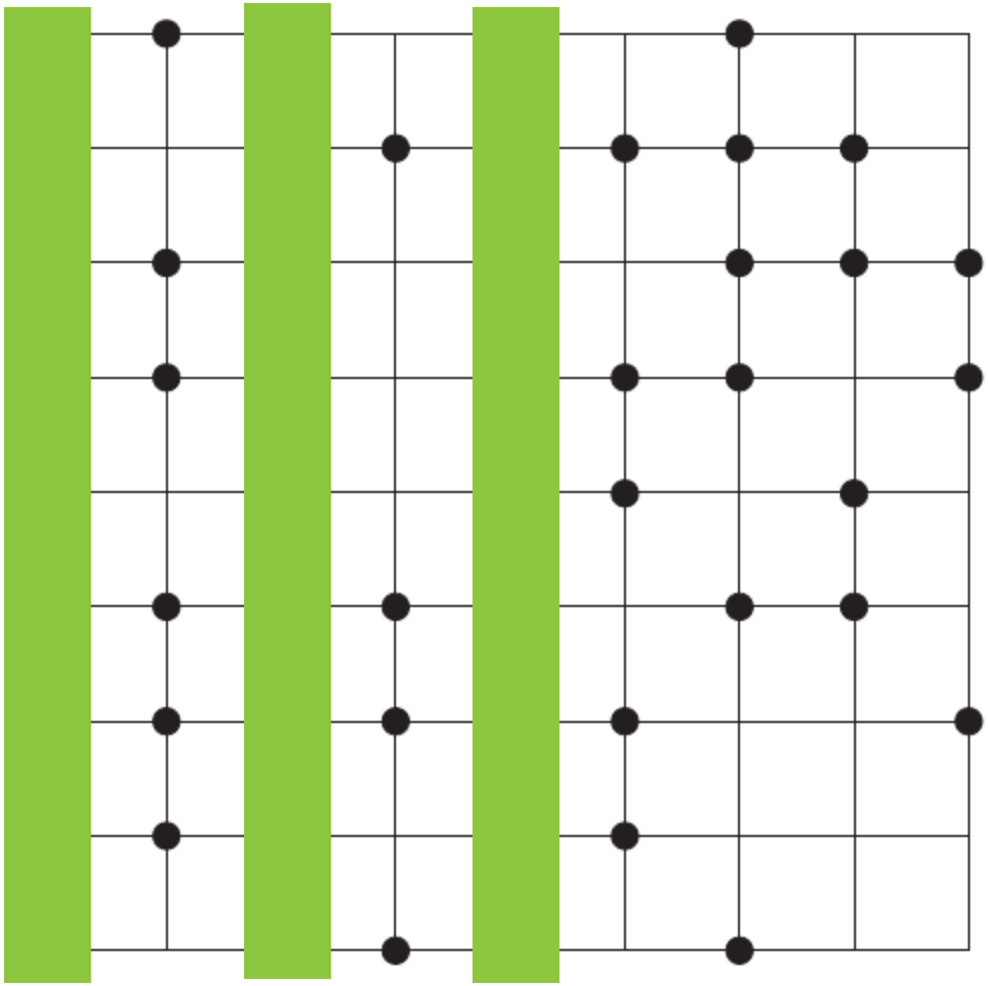
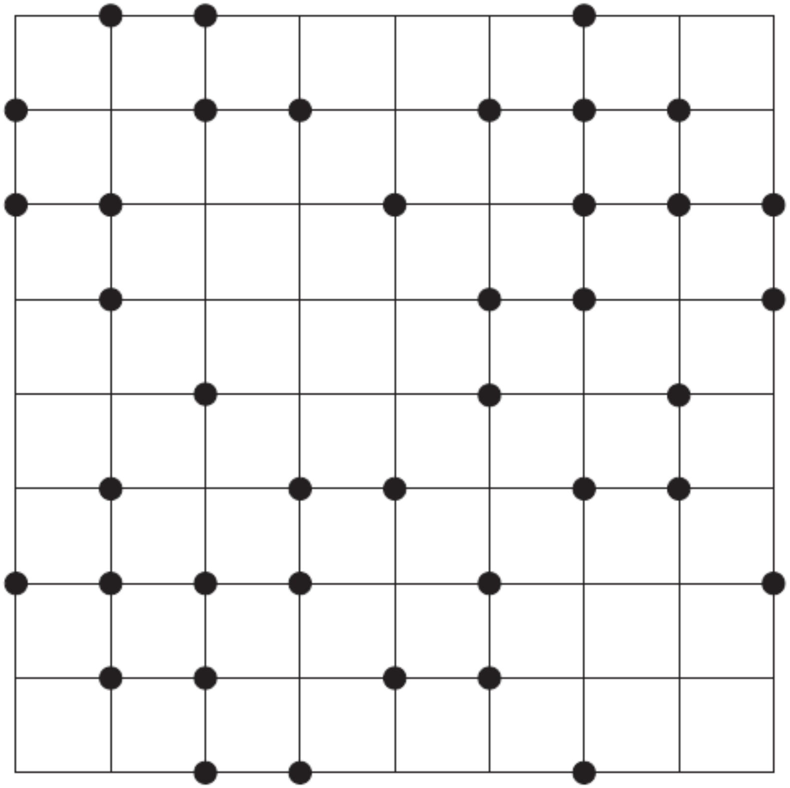


# The Solution!!!

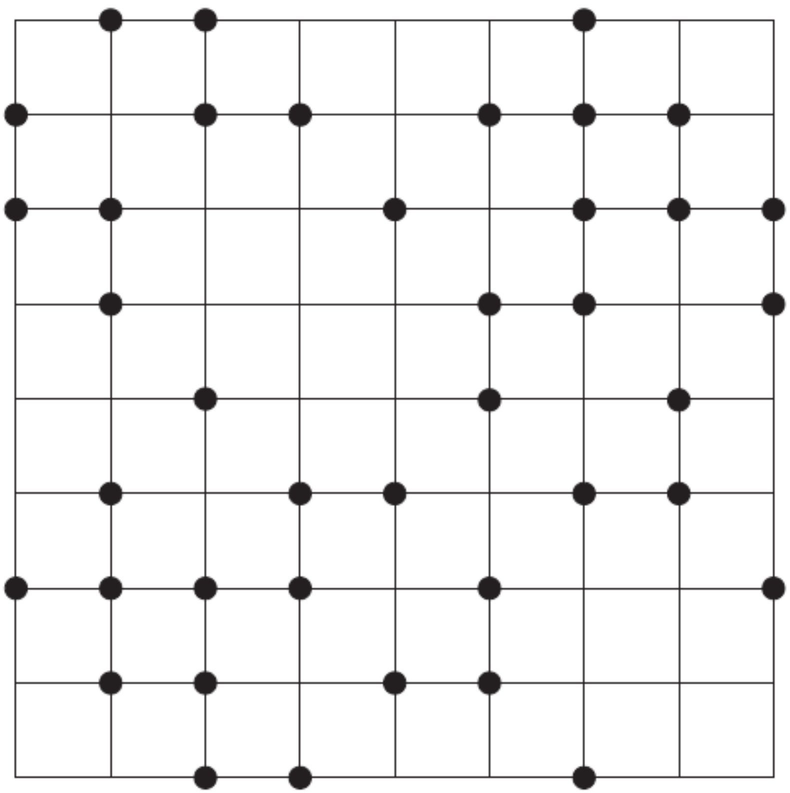


# Existence:
# Does a Solution Exist for Every 'Checkerboard'?

# Checkerboards and Matrices



# Checkerboards and Matrices



$$\equiv \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

# Checkerboards and Matrices

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

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$A_j$ - $j^{\text{th}}$ column of the matrix $A$

# Checkerboards and Matrices

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$A_j$ - $j^{\text{th}}$ column of the matrix $A$

$$A_2 + A_4 + A_6 + A_7 + A_8 + A_9 = \begin{bmatrix} 2 \\ 4 \\ 4 \\ 4 \\ 2 \\ 4 \\ 4 \\ 2 \\ 2 \end{bmatrix}$$

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$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

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$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$A_j$ - $j^{\text{th}}$ column of the matrix $A$

solution matrix

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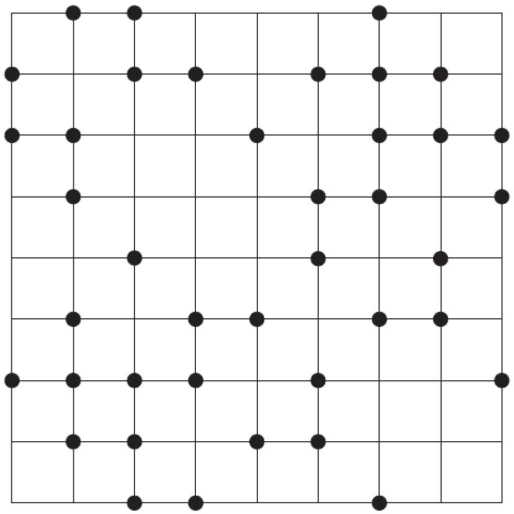
# Checkerboards and Matrices



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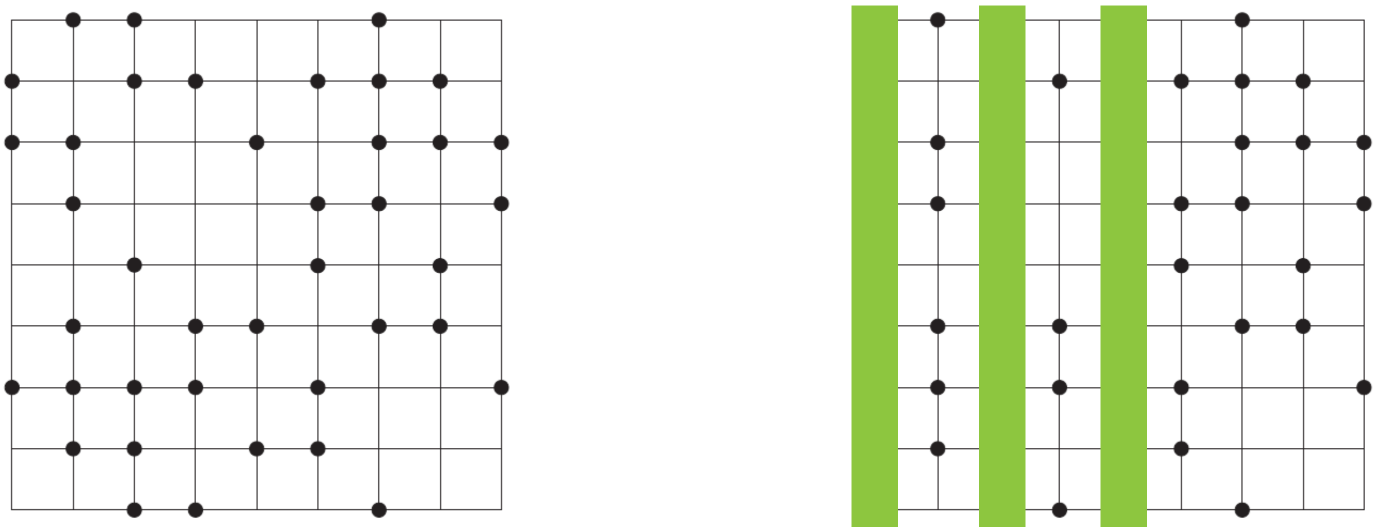

$$A \,.\, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 4 \\ 4 \\ 2 \\ 4 \\ 4 \\ 2 \\ 2 \end{bmatrix}$$

# Working over $[\mathbb{Z}]_2$ field

$[\mathbb{Z}]_2$ is the smallest finite field consisting of two elements 0 and 1.

By modular arithmetic, for all integers $z$

$z \equiv 0 \pmod 2$, if $z$ is even

$z \equiv 1 \pmod 2$, if $z$ is odd

# Working over $[\mathbb{Z}]_2$ field

$$A \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 4 \\ 4 \\ 2 \\ 4 \\ 4 \\ 2 \\ 2 \end{bmatrix}$$

# Working over $[\mathbb{Z}]_2$ field

$$A \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 4 \\ 4 \\ 2 \\ 4 \\ 4 \\ 2 \\ 2 \end{bmatrix} \equiv A \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$A\boldsymbol{X} = \mathbf{0}$$

# Working over $[\mathbb{Z}]_2$ field

$$\boldsymbol{AX} = \mathbf{0}$$

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

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# Working over $[\mathbb{Z}]_2$ field

$$\boldsymbol{AX} = \mathbf{0}$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = 0,\ x_2 = 1,\ x_3 = 0,\ x_4 = 1,\ x_5 = 0,\ x_6 = 1,\ x_7 = 1,\ x_8 = 1,\ x_9 = 1$$

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# Existence Theorem

**Definition:** Let $m$ be an odd number. Over the field $[\mathbb{Z}]_2$, a **checkerboard matrix** is an $m \times m$ symmetric matrix with diagonal elements equal to 0.

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**Terminology:** An **elementary product** of an $m \times m$ matrix is a product of $m$ elements of the matrix such that the each element in the product is located on a unique row $i$ and a unique column $j$, where $0 \leq i, j \leq m$. The set of ordered pairs $(i, j)$ is the corresponding **transversal**.

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**Theorem:** A checkerboard matrix has a non-trivial nullspace.

# PROOF

Let $A$ be an $m \times m$ checkerboard matrix.

**We want to show: det(*A*) = 0 over** $[\mathbb{Z}]_2$ **.**

# Proof

Let $A$ be an $m \times m$ checkerboard matrix.

**We want to show: det($A$) = 0 over $[\mathbb{Z}]_2$ .**

Det(A) is an alternating sum of elementary products of $A$.

Since we are working over $[\mathbb{Z}]_2$ , +1 = -1.

Det(A) is just an ordinary sum of the elementary products of A.

An $m \times m$ matrix has $m!$ elementary products.

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Case I: If a transversal contains an ordered pair ($i$, $i$), i.e it represents a diagonal element, the corresponding elementary product is equal to 0.

Case II: If a transversal represents no diagonal element, it will have a 'mirror image' obtained by reflection across the diagonal.

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Since each elementary product consists of an odd number of elements, each transversal is distinct from its mirror image.

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Since each elementary product consists of an odd number of elements, each transversal is distinct from its mirror image.

Since $A$ is symmetric, the elementary products corresponding to the the transversal and its mirror image are equal. Since we are working over $[\mathbb{Z}]_2$, their sum is equal to 0.

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Since each elementary product consists of an odd number of elements, each transversal is distinct from its mirror image.

Since $A$ is symmetric, the elementary products corresponding to the the transversal and its mirror image are equal. Since we are working over $[\mathbb{Z}]_2$, their sum is equal to 0.

So, the sum of the elementary products is 0.

# PROOF

Let $A$ be an $m \times m$ checkerboard matrix.

**We want to show: det($A$) = 0 over $[\mathbb{Z}]_2$ .**

Det(A) is an alternating sum of elementary products of $A$.

Since we are working over $[\mathbb{Z}]_2$ , +1 = -1.

Det(A) is just an ordinary sum of the elementary products of A.

An $m \times m$ matrix has $m$! elementary products.

**The sum of these $m$! elementary products is 0.**

QED

# Reference

L. Zulli, The Incredibly Knotty Checkerboard Challenge, *Mathematics Magazine* 71(1998), 378-385

# Acknowledgements

Professor Richard Bedient

Hamilton College Mathematics Department

Williams College

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & & & 0 \\ 1 & & & 0 \\ 0 & & & 1 \\ 0 & & & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} & & 1 & 1 \\ & & 1 & 1 \\ & & 0 & 1 \\ & & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} & 1 & 1 & & 1 \\ & 0 & 1 & & 1 \\ & 1 & 0 & & 1 \\ & 1 & 1 & & 1 \\ & 1 & 1 & & 0 \end{bmatrix}$$