Integrals and Interesting Series Involving the Central Binomial Coefficient

MA 411 Senior Seminar

Robyn Taylor

Faculty Advisor: Professor Rob Poodiack

Abstract

- We compute the sum of series involving the central binomial coefficient and deemed "interesting" by D. H. Lehmer. We do this using a different method involving integrals. This produces results that match Lehmer's and leads to the discovery of patterns leading to conjectures on the sums of related series.

Overview

- Introduction of Concepts
- Lehmer’s Work
- Alternate Proof
- Other Series
- Results

Basic Concepts

- Series
- Taylor, Maclaurin, and Binomial Series
- Binomial Coefficient
- Central Binomial Coefficient
- Beta Function

Series

- Basic Series

$$S = \sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots$$

- Taylor Series

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

- Maclaurin Series

$$f(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \cdots + \frac{f^{(n)}(0)}{n!}(x)^n + \cdots$$

Binomial Coefficient

- Binomial Series

$$(x+y)^n = x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + y^n$$

$$\binom{n}{k} = {}_nC_k = C(n,k) = \frac{n!}{k!\,(n-k)!}$$

- n is for the row
- k is column in row

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Central Binomial Coefficient

- Similar to Binomial Coefficient
- Center column of Pascal's Triangle

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} = \frac{2^n(2n-1)!!}{n!}$$

Beta Function

- Convert factorials to an integral

$$B(m,n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!} = \int_0^1 t^{m-1}(1-t)^{n-1}dt, \qquad \text{For } m,n \in \mathbb{Z}^+.$$

Lehmer’s Work

- Article in Aug-Sep ’85 American Mathematical Monthly
- “Interesting”- a series with a simple explicit formula for its *n*th term and its sum can be expressed in terms of known constants

Lehmer's Work

I. $\sum_{n=0}^{\infty} a_n \binom{2n}{n}$ and II. $\sum_{n=0}^{\infty} \frac{a_n}{\binom{2n}{n}}$

- Type II is more mysterious and less well understood
- Lehmer used combinatorics and counting methods

Alternate Proof

- Differential Equations
- Series
- Recursion Equation
- Integrals
- Functions
- Results

Differential Equations

$$y = (\arcsin x)^2$$

$$y' = \frac{2(\arcsin x)}{\sqrt{1 - x^2}}$$

$$y'' = \left(\frac{2(\arcsin x)}{\sqrt{1 - x^2}}\right)\left(\frac{1}{1 - x^2}\right) + \frac{2}{1 - x^2}$$

Differential Equations

- Alter y' by clearing fractions, multiplying both sides by $(1 - x^2)$, differentiating and regrouping the equation gives

$$y''(1 - x^2) - y'x = 2$$

Series

$$y(x) = (\arcsin x)^2 = \sum_{n=0}^{\infty} a_n x^n$$

$$y''(1 - x^2) - y'x = 2$$

$$-a_1 x + 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} [(n+1)(n+2)a_{n+2} - n^2 a_n]x^n = 2$$

- Coefficients $a_1 = 0, a_2 = 0, a_3 = 0$

Recursion Equation

$$a_{n+2} = \frac{n^2}{(n+1)(n+2)} a_n$$

$$= \frac{n^2(n-2)^2(n-4)^2 \ldots (2)^2}{(n+2)(n+1)(n)(n-1)(n-2)(n-3) \ldots (4)(3)}$$

$$a_{2n} = \frac{[2^{n-1}(n)!]^2}{\frac{1}{2} n^2 (2n)!}$$

Recursion Equation

$$a_{2n} = \frac{[2^{n-1}(n)!]^2}{\frac{1}{2}n^2(2n)!}$$

- Recall $\binom{2n}{n} = \frac{(2n)!}{n!(2n-n)!} = \frac{(2n)!}{(n!)^2}$

$$a_{2n} = \frac{2^{2n-1}}{n^2\binom{2n}{n}}$$

Recursion Equation

$$(\arcsin(x))^2 = \frac{1}{2}\sum_{n=1}^{\infty}\frac{2^{2n}}{n^2\binom{2n}{n}}x^{2n}$$

- Matches Lehmer's equation (13)
- Take derivative with respect to x

$$\frac{2x(\arcsin(x))}{\sqrt{1-x^2}} = \sum_{n=1}^{\infty}\frac{2^{2n}x^{2n}}{n\binom{2n}{n}}$$

- Matches Lehmer's equation (9)

Ratio Test

- We perform the ratio test on the series for $(\arcsin(x))^2$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2x)^{2n+2}[(n+1)!]^2(n^2(2n)!)}{(n+1)^2(2n+2)!\,(2x)^{2n}(n!)^2}\right|$$

$$= 4|x|^2\frac{(n+1)^2(n^2)}{(2n+1)(2n+2)(n+1)^2} = 4|x|^2\frac{(n^2)}{(2n+1)(2n+2)}$$

Ratio Test

- For $4|x|^2 \frac{(n^2)}{(2n+1)(2n+2)}$
- Let $n \to \infty$, and $|x|^2 < 1$
- Radius of convergence is 1 for both $(\arcsin(x))^2$ and $\frac{2x(\arcsin(x))}{\sqrt{1-x^2}}$
- But this is a bound!

Raabe's Test

$$\rho = \lim_{n\to\infty}\left[n\left(\frac{u_n}{u_{n+1}} - 1\right)\right]$$

- If $\rho > 1$, the series converges
- If $\rho < 1$, the series diverges
- If $\rho = 1$, the series may converge or diverge

Raabe's Test

$$\rho = \lim_{n\to\infty}\left[n\left(\frac{u_n}{u_{n+1}} - 1\right)\right]$$

$$\rho = \lim_{n\to\infty}\left[n\left(\frac{(1+n)(1+2n)}{2n^2} - 1\right)\right]$$

$$\rho = \lim_{n\to\infty}\left[n\left(\frac{1+3n}{2n^2}\right)\right]$$

$$\rho = \lim_{n\to\infty}\left[\frac{n+3n^2}{2n^2}\right] = \frac{3}{2}$$

$$\frac{3}{2} > 1$$

- According to Raabe's test our series converges

Integrals

$$B(m,n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!} = \int_0^1 t^{m-1}(1-t)^{n-1}dt, \qquad \text{For } m,n \in \mathbb{Z}^+.$$

$$\frac{1}{\binom{2n}{n}} = \frac{(n!)^2}{(2n)!} = \frac{n!\,n!}{(2n)!} = \frac{n(n-1)!\,n!}{(2n)!}$$

$$nB(n,n+1) = \frac{n(n-1)!\,n!}{(2n)!} = n\int_0^1 t^{n-1}(1-t)^n dt$$

Integrals

$$\frac{1}{\binom{2n}{n}} = n\int_0^1 t^{n-1}(1-t)^n dt$$

$$\sum_{n=0}^{\infty} \frac{x^n}{\binom{2n}{n}} = \int_0^1 \sum_{n=0}^{\infty} n\, x^n t^{n-1}(1-t)^n dt$$

$$= \int_0^1 x(1-t) \sum_{n=0}^{\infty} n\, x^{n-1} t^{n-1}(1-t)^{n-1} dt$$

Integrals

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \text{For } |z| < 1.$$

- Taking the derivative gives

$$\sum_{n=0}^{\infty} n z^{n-1} = \frac{1}{(1-z)^2} \quad , |z| < 1$$

- Set $z = (1-t)xt$,

$$\boxed{\sum_{n=0}^{\infty} \frac{x^n}{\binom{2n}{n}} = \int_0^1 \frac{x(1-t)}{\left(1 - xt(1-t)\right)^2} dt}$$

We have our integral!

Radius of Convergence

- If x is outside the radius of convergence, the equation will not hold.

$$|xt(1-t)| < 1, 0 \leq t \leq 1$$

$$t(1-t) \text{ maximized at } t = \frac{1}{2}$$

$$\left|\left(\frac{1}{4}\right)x\right| < 1$$

$$|x| < 4$$

Functions

- To solve the integral, we used
  - u substitution
  - Trigonometric substitution
  - Integration by parts

$$\sum_{n=0}^{\infty} \frac{x^n}{\binom{2n}{n}} = \int_0^1 \frac{x(1-t)}{\left(1-xt(1-t)\right)^2} dt = \frac{4\sqrt{x}}{(4-x)^{3/2}} \arctan\left(\sqrt{\frac{x}{4-x}}\right) + \frac{x}{4-x}$$

Results

$$\sum_{n=0}^{\infty} \frac{x^n}{\binom{2n}{n}} = \int_0^1 \frac{x(1-t)}{\big(1-xt(1-t)\big)^2}\,dt = \frac{4\sqrt{x}}{(4-x)^{3/2}} \arctan\left(\sqrt{\frac{x}{4-x}}\right) + \frac{x}{4-x}$$

$$\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}} = \int_0^1 \frac{(1-t)dt}{(t^2-t+1)^2} = \frac{9+2\pi\sqrt{3}}{27}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\binom{2n}{n}} = \int_0^1 \frac{-(1-t)}{\big(1+t(1-t)\big)^2}\,dt = -\frac{1}{5} + \frac{4\sqrt{5}}{25}\left(\ln\left[\frac{1+\sqrt{5}}{2}\right]\right)$$

Leibniz Rule

- Allows us to derive and integrate under the integral sign

$f(t, x)$ is on the rectangle

$R = [0,1] \times [-4,4]$

$\frac{\partial f}{\partial x}(t, x)dt$ must be continuous on R

$$\frac{d}{dx}\int_a^b f(t, x)dt = \int_a^b \frac{\partial f}{\partial x}(t, x)dt$$

Series with n in the Denominator

$$\sum_{n=1}^{\infty} \frac{x^n}{n\binom{2n}{n}} = \int_0^1 \frac{x - tx}{1 + (-1+t)tx} dt = \frac{2\sqrt{x}\text{ArcTan}[\frac{\sqrt{x}}{\sqrt{4-x}}]}{\sqrt{4-x}}$$

$$\sum_{n=1}^{\infty} \frac{1}{n\binom{2n}{n}} = \int_0^1 \frac{1 - t}{1 + (-1+t)t} dt = \frac{\pi}{3\sqrt{3}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n\binom{2n}{n}} = \int_0^1 \frac{t - 1}{1 - (-1+t)t} dt = \frac{1}{\sqrt{5}} \text{Ln}\left(\frac{3+\sqrt{5}}{2}\right)$$

Series with n in the Denominator

$$\sum_{n=1}^{\infty} \frac{2^n}{n\binom{2n}{n}} = \int_0^1 \frac{2(1-t)}{1+(-1+t)2t}\,dt = \frac{\pi}{2}$$

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n\binom{2n}{n}} = \int_0^1 \frac{-2(1-t)}{1-(-1+t)2t}\,dt = \frac{1}{\sqrt{3}}\text{Ln}\left(-2+\sqrt{3}\right)$$

$$\sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n\binom{2n}{n}} = \int_0^1 \frac{\left(\frac{1}{2}\right)(1-t)}{1+(-1+t)\left(\frac{1}{2}\right)t}\,dt = \frac{2\text{ArcTan}[\frac{1}{\sqrt{7}}]}{\sqrt{7}}$$

$$\sum_{n=1}^{\infty} \frac{\left(-\frac{1}{2}\right)^n}{n\binom{2n}{n}} = \int_0^1 \frac{\left(-\frac{1}{2}\right)(1-t)}{1-(-1+t)\left(\frac{1}{2}\right)t}\,dt = -\frac{\text{Ln}[2]}{3}$$

Series with n^2 in the Denominator

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2 \binom{2n}{n}} = \int_0^1 -\frac{\text{Ln}[1 + (-1 + t)tx]}{t} dt = 2\text{ArcTan}\left[\frac{1}{\sqrt{-1 + \frac{4}{x}}}\right]^2$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2 \binom{2n}{n}} = \int_0^1 -\frac{\text{Ln}[1 + (-1 + t)t]}{t} dt = \frac{\pi^2}{18}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 \binom{2n}{n}} = \int_0^1 -\frac{\text{Ln}[1 - (-1 + t)t]}{t} dt = -2\left(\text{Ln}\left(\frac{1 + \sqrt{5}}{2}\right)\right)^2$$

Series with n^2 in the Denominator

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2 \binom{2n}{n}} = \int_0^1 -\frac{\text{Ln}[1 + (-1 + t)2t]}{t} dt = \frac{\pi^2}{8}$$

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2 \binom{2n}{n}} = \int_0^1 -\frac{\text{Ln}[1 - (-1 + t)2t]}{t} dt = \frac{1}{2}\left[\text{Ln}\left(2 + \sqrt{3}\right)\right]^2$$

$$\sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n^2 \binom{2n}{n}} = \int_0^1 -\frac{\text{Ln}\left[1 + (-1 + t)t\left(\frac{1}{2}\right)\right]}{t} dt = 2\text{ArcTan}\left[\frac{1}{\sqrt{7}}\right]^2$$

$$\sum_{n=1}^{\infty} \frac{\left(-\frac{1}{2}\right)^n}{n^2 \binom{2n}{n}} = \int_0^1 -\frac{\text{Ln}\left[1 - (-1 + t)t\left(\frac{1}{2}\right)\right]}{t} dt = -\text{Ln}[2]$$

Series with n^3 in the Denominator

- Lehmer calls this a "higher transcendent"

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \quad \text{For } |z| < 1$$

$$\sum_{n=1}^{\infty} \frac{(x)^n}{n^3 \binom{2n}{n}} = \int_0^1 \frac{\mathrm{Li}_2((1-t)tx)}{t} \, dt$$

Series with n^3 in the Denominator

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3 \binom{2n}{n}} = \int_0^1 \frac{\mathrm{Li}_2\big((1-t)(-t)\big)}{t} dt = -\frac{2}{5}\zeta(3)$$

$$\sum_{n=1}^{\infty} \frac{\left(-\frac{1}{2}\right)^n}{n^3 \binom{2n}{n}} = \int_0^1 \frac{\mathrm{Li}_2\left((1-t)\left(-\frac{1}{2}t\right)\right)}{t} dt = \frac{\mathrm{Ln}[2]^3}{6} - \frac{\zeta(3)}{4}$$

Series with n^4 in the Denominator

$$(\arcsin(x))^2 = \frac{1}{2}\sum_{n=1}^{\infty}\frac{2^{2n}}{n^2\binom{2n}{n}}x^{2n}$$

- Multiply both sides by $\frac{2}{x}$

$$\frac{2(\arcsin(x))^2}{x} = 2\sum_{n=1}^{\infty}\frac{2x^{2n-1}}{n^2\binom{2n}{n}}$$

$$2\int_0^u \frac{(\arcsin(x))^2}{x}dx = \sum_{n=1}^{\infty}\frac{2^{2n-1}u^{2n}}{n^3\binom{2n}{n}}$$

Series with n^4 in the Denominator

$$2\int_0^{\frac{1}{2}}\frac{du}{u}\int_0^u\frac{(\arcsin(x))^2}{x}dx=\int_0^{\frac{1}{2}}\sum_{n=1}^{\infty}\frac{2^{2n-1}u^{2n-1}}{n^3\binom{2n}{n}}du=\frac{1}{4}\sum_{n=1}^{\infty}\frac{2^{2n}\left(\frac{1}{2}\right)^{2n}}{n^4\binom{2n}{n}}$$

- The numerator becomes 1
- Multiply both sides by 4

$$8\int_0^{\frac{1}{2}}\frac{du}{u}\int_0^u\frac{(\arcsin(x))^2}{x}dx=\sum_{n=1}^{\infty}\frac{1}{n^4\binom{2n}{n}}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4\binom{2n}{n}}=\frac{17\pi}{3240}$$

Series with n in the Numerator

$$\sum_{n=1}^{\infty} \frac{n(x)^n}{\binom{2n}{n}} = \int_0^1 x\left(\frac{2(1-t)^2 tx}{(1-(1-t)tx)^3} + \frac{1-t}{(1-(1-t)tx)^2}\right)dt = \frac{x(6\sqrt{-(-4+x)x} + 4(2+x)\mathrm{ArcTan}[\frac{\sqrt{x}}{\sqrt{4-x}}])}{(-4+x)^2\sqrt{-(-4+x)x}}$$

$$\sum_{n=1}^{\infty} \frac{n}{\binom{2n}{n}} = \int_0^1 \left(\frac{2(1-t)^2 t}{(1-(1-t)t)^3} + \frac{1-t}{(1-(1-t)t)^2}\right)dt = \frac{6\sqrt{3}+2\pi}{9\sqrt{3}}$$

$$\sum_{n=1}^{\infty} \frac{n(-1)^n}{\binom{2n}{n}} = \int_0^1 -\left(\frac{-2(1-t)^2 t}{(1+(1-t)t)^3} + \frac{1-t}{(1+(1-t)t)^2}\right)dt = -\frac{6}{25} + \frac{2\mathrm{Ln}\left[\frac{\left(1-\frac{1}{\sqrt{5}}\right)}{\left(1+\frac{1}{\sqrt{5}}\right)}\right]}{25\sqrt{5}}$$

Series with n in the Numerator

$$\sum_{n=1}^{\infty} \frac{n(2)^n}{\binom{2n}{n}} = \int_0^1 2\left(\frac{4(1-t)^2 t}{(1-(1-t)2t)^3} + \frac{1-t}{(1-(1-t)2t)^2}\right) dt = \frac{1}{4}(12+4\pi)$$

$$\sum_{n=1}^{\infty} \frac{n(-2)^n}{\binom{2n}{n}} = \int_0^1 -2\left(\frac{-4(1-t)^2 t}{(1+(1-t)2t)^3} + \frac{1-t}{(1+(1-t)2t)^2}\right) dt = -\frac{1}{3}$$

$$\sum_{n=1}^{\infty} \frac{n\left(\frac{1}{2}\right)^n}{\binom{2n}{n}} = \int_0^1 \left(\frac{1}{2}\right)\left(\frac{(1-t)^2 t}{\left(1-(1-t)\left(\frac{1}{2}\right)t\right)^3} + \frac{1-t}{\left(1-(1-t)\left(\frac{1}{2}\right)t\right)^2}\right) dt = \frac{4(3\sqrt{7} + 10\text{ArcTan}[\frac{1}{\sqrt{7}}])}{49\sqrt{7}}$$

$$\sum_{n=1}^{\infty} \frac{n\left(-\frac{1}{2}\right)^n}{\binom{2n}{n}} = \int_0^1 -\frac{1}{2}\left(\frac{-(1-t)^2 t}{\left(1+(1-t)\left(\frac{1}{2}\right)t\right)^3} + \frac{1-t}{\left(1+(1-t)\left(\frac{1}{2}\right)t\right)^2}\right) dt = \frac{-4}{81}(3+\text{Ln}[2])$$

Series with n^2 in the Numerator

$$\sum_{n=1}^{\infty} \frac{n^2 (x)^n}{\binom{2n}{n}} = \int_0^1 x\left(\frac{2(1-t)^2 tx}{(1-(1-t)tx)^3} + \frac{1-t}{(1-(1-t)tx)^2} + x\left(\frac{6(1-t)^3 t^2 x}{(1-(1-t)tx)^4} + \frac{4(1-t)^2 t}{(1-(1-t)tx)^3}\right)\right) dt$$

$$= \frac{4x(\sqrt{-(-4+x)x}(7+2x) + (4+x(10+x))\mathrm{ArcTan}[\frac{\sqrt{x}}{\sqrt{4-x}}])}{(-4+x)^3\sqrt{-(-4+x)x}}$$

$$\sum_{n=1}^{\infty} \frac{n^2}{\binom{2n}{n}} = \int_0^1 \left(\frac{2(1-t)^2 t}{(1-(1-t)t)^3} + \frac{1-t}{(1-(1-t)t)^2} + \left(\frac{6(1-t)^3 t^2}{(1-(1-t)t)^4} + \frac{4(1-t)^2 t}{(1-(1-t)t)^3}\right)\right) dt = \frac{2}{81}(54 + 5\sqrt{3}\pi)$$

$$\sum_{n=1}^{\infty} \frac{n^2 (x)^n}{\binom{2n}{n}} = \int_0^1 -\left(\frac{-2(1-t)^2 t}{(1+(1-t)t)^3} + \frac{1-t}{(1+(1-t)t)^2} - \left(\frac{-6(1-t)^3 t^2}{(1+(1-t)t)^4} + \frac{4(1-t)^2 t}{(1+(1-t)t)^3}\right)\right) dt = -\frac{4}{25} - \frac{2\mathrm{Ln}\left[\frac{\left(1-\frac{1}{\sqrt{5}}\right)}{\left(1+\frac{1}{\sqrt{5}}\right)}\right]}{25\sqrt{5}}$$

Series with n^2 in the Numerator

$$\sum_{n=1}^{\infty} \frac{n^2 (2)^n}{\binom{2n}{n}} = \int_0^1 2\left(\frac{4(1-t)^2 t}{(1-(1-t)2t)^3} + \frac{1-t}{(1-(1-t)2t)^2} + 2\left(\frac{12(1-t)^3 t^2}{(1-(1-t)2t)^4} + \frac{8(1-t)^2}{(1-(1-t)2t)^3}\right)\right) dt = \frac{1}{2}(22+7\pi)$$

$$\sum_{n=1}^{\infty} \frac{n^2 (-2)^n}{\binom{2n}{n}} = \int_0^1 -2\left(\frac{-4(1-t)^2 t}{(1+(1-t)2t)^3} + \frac{1-t}{(1+(1-t)2t)^2} - 2\left(\frac{-12(1-t)^3 t^2}{(1+(1-t)2t)^4} - \frac{8(1-t)^2}{(1+(1-t)2t)^3}\right)\right) dt = \frac{1}{9} + \frac{\mathrm{Ln}[1-\frac{1}{\sqrt{3}}]}{9\sqrt{3}} - \frac{\mathrm{Ln}[1+\frac{1}{\sqrt{3}}]}{9\sqrt{3}}$$

$$\sum_{n=1}^{\infty} \frac{n^2 \left(\frac{1}{2}\right)^n}{\binom{2n}{n}} = \int_0^1 \frac{1}{2}\left(\frac{(1-t)^2 t}{\left(1-(1-t)\left(\frac{1}{2}\right)t\right)^3} + \frac{1-t}{\left(1-(1-t)\left(\frac{1}{2}\right)t\right)^2} + \left(\frac{1}{2}\right)\left(\frac{3(1-t)^3 t^2}{\left(1-(1-t)\left(\frac{1}{2}\right)t\right)^4} + \frac{2(1-t)^2}{\left(1-(1-t)\left(\frac{1}{2}\right)t\right)^3}\right)\right) dt$$

$$= \frac{32(4\sqrt{7} + \frac{37}{4}\mathrm{ArcTan}[\frac{1}{\sqrt{7}}])}{343\sqrt{7}}$$

$$\sum_{n=1}^{\infty} \frac{n^2 \left(-\frac{1}{2}\right)^n}{\binom{2n}{n}} = \int_0^1 -\frac{1}{2}\left(\frac{-(1-t)^2 t}{\left(1+(1-t)\left(\frac{1}{2}\right)t\right)^3} + \frac{1-t}{\left(1+(1-t)\left(\frac{1}{2}\right)t\right)^2} - \left(\frac{1}{2}\right)\left(\frac{-3(1-t)^3 t^2}{\left(1+(1-t)\left(\frac{1}{2}\right)t\right)^4} - \frac{2(1-t)^2}{\left(1+(1-t)\left(\frac{1}{2}\right)t\right)^3}\right)\right) dt$$

$$= \frac{4}{729}(-24 + \mathrm{Ln}[2])$$

Series with n^3 in the Numerator

$$\sum_{n=1}^{\infty}\frac{n^3(x)^n}{\binom{2n}{n}}=\int_0^1 x\left(\frac{2(1-t)^2tx}{(1-(1-t)tx)^3}+\frac{1-t}{(1-(1-t)tx)^2}+x\left(\frac{6(1-t)^3t^2x}{(1-(1-t)tx)^4}+\frac{4(1-t)^2t}{(1-(1-t)tx)^3}\right)\right.$$

$$\left.+x\left(\frac{12(1-t)^3t^2x}{(1-(1-t)tx)^4}+\frac{8(1-t)^2t}{(1-(1-t)tx)^3}+x\left(\frac{24(1-t)^4t^3x}{(1-(1-t)tx)^5}+\frac{18(1-t)^3t^2}{(1-(1-t)tx)^4}\right)\right)\right)dt$$

$$=\frac{2x\left(5\sqrt{-(-4+x)x}(12+x(14+x))+2\left(8+x(72+x(30+x))\right)\mathrm{ArcTan}\left[\frac{\sqrt{x}}{\sqrt{4-x}}\right]\right)}{(-4+x)^4\sqrt{-(-4+x)x}}$$

$$\sum_{n=1}^{\infty}\frac{n^3(x)^n}{\binom{2n}{n}}=\frac{10}{3}+\frac{74\pi}{81\sqrt{3}}$$

$$\sum_{n=1}^{\infty}\frac{n^3(x)^n}{\binom{2n}{n}}=\frac{2}{125}-\frac{14\mathrm{Ln}[1-\frac{1}{\sqrt{5}}]}{125\sqrt{5}}+\frac{14\mathrm{Ln}[1+\frac{1}{\sqrt{5}}]}{125\sqrt{5}}$$

Series with n^3 in the Numerator

$$\sum_{n=1}^{\infty} \frac{n^3 (2)^n}{\binom{2n}{n}} = 55 + \frac{35\pi}{2}$$

$$\sum_{n=1}^{\infty} \frac{n^3 (-2)^n}{\binom{2n}{n}} = \frac{1}{81}(15 - \sqrt{3}\mathrm{Ln}[3 - \sqrt{3}] + \sqrt{3}\mathrm{Ln}[3 + \sqrt{3}])$$

$$\sum_{n=1}^{\infty} \frac{n^3 \left(\frac{1}{2}\right)^n}{\binom{2n}{n}} = \frac{4(385 + 118\sqrt{7}\mathrm{ArcTan}[\frac{1}{\sqrt{7}}])}{2401}$$

$$\sum_{n=1}^{\infty} \frac{n^3 \left(-\frac{1}{2}\right)^n}{\binom{2n}{n}} = \frac{20(-21 + \mathrm{Ln}[2048])}{6561}$$

Correction

$$\sum_{n=1}^{\infty} \frac{n^3(-1)^n}{\binom{2n}{n}}$$

Lehmer's value of $\frac{2}{625}\left[28\sqrt{5}\ln\left[\frac{1+\sqrt{5}}{2}\right]\right]$

should be $\frac{2}{625}\left[14\sqrt{5}\ln\left[\frac{1+\sqrt{5}}{2}\right]\right]$

Conjectures

- We noticed a pattern emerge for series divided by the central binomial coefficient
- The different values of x caused reoccurring portions in the value of the sums.

$$x = 1 \text{ and } \pi\sqrt{3}$$

$$x = -1 \text{ and } \ln\left[\frac{1+\sqrt{5}}{2}\right]$$

$$x = 2 \text{ and } \frac{\pi}{2}$$

$$x = -2 \text{ and } \ln\left[\frac{1}{\sqrt{3}}\right]$$

$$x = \frac{1}{2} \text{ and } \mathrm{ArcTan}\left[\frac{1}{\sqrt{7}}\right]$$

$$x = -\frac{1}{2} \text{ and } \ln[2]$$

Future Research

- Attempt to find patterns for values of x
- Series of the form

$$\sum_{n=1}^{\infty} \frac{(x)^n}{\binom{3n}{n}}$$

- General form

$$\sum_{n=1}^{\infty} \frac{(x)^n}{\binom{kn}{n}}$$

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