Two Distinct Fractal Versions of the Sphinx Substitution Tiling

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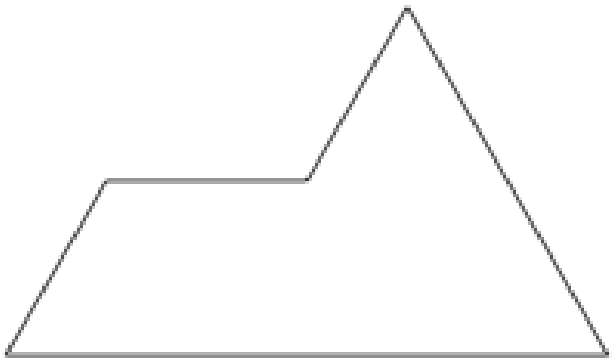
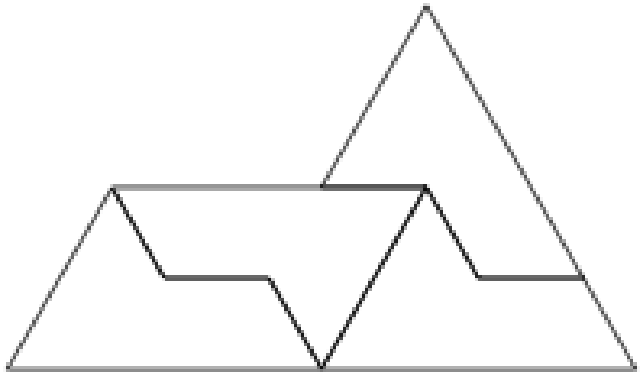
Vassar College

April 6th - Hudson River Undergraduate Mathematics Conference

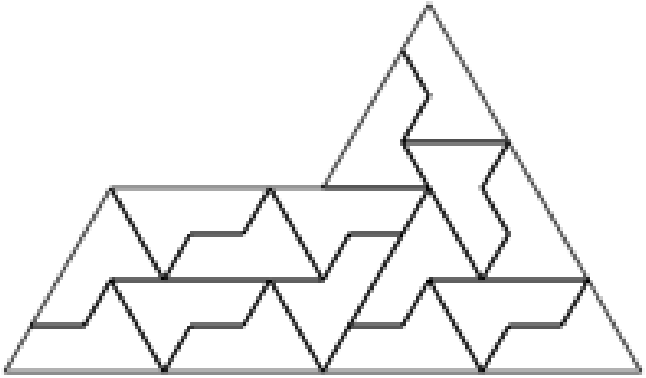
The Sphinx and its Substitution Rule



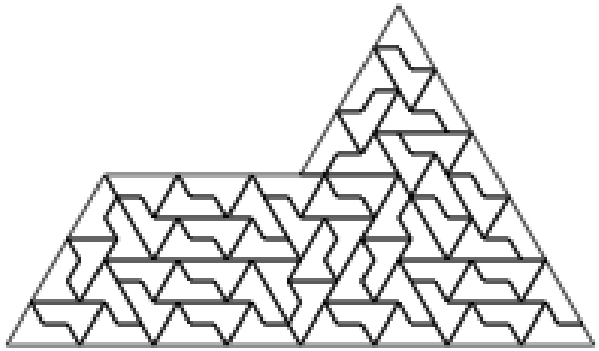
Level 0



Level 1

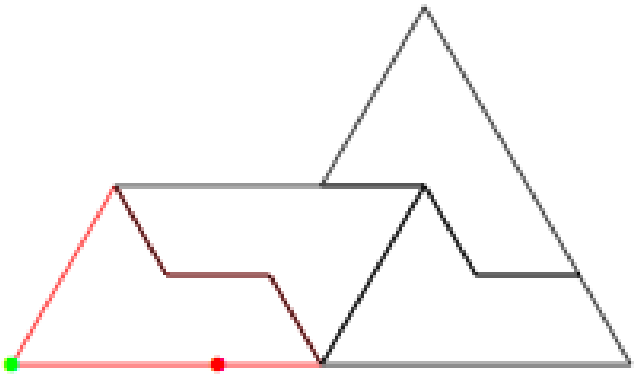


Level 2

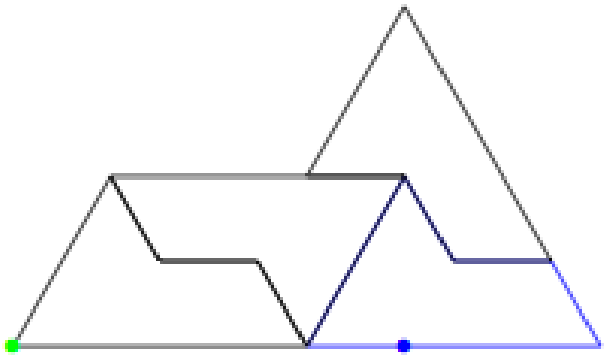


Level 3

Identifying Exterior Fixed Points



$f_1(x) = -1/2x + 3/2$
$\Rightarrow f_2(1) = 1$



$f_2(x) = -1/2x + 3$
$\Rightarrow f_2(2) = 2$

There are 2 exterior fixed points.

Note: there are actually 4 fixed points but the others are interior fixed points.

?

These points determine the two distinct fractals!

Creating Fractal Arcs

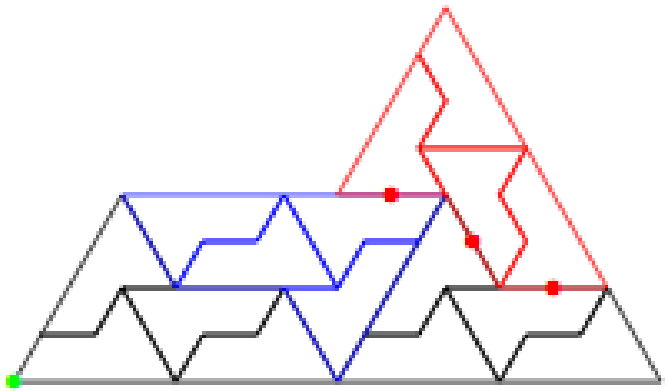
We need a way to "break up" the Sphinx in a meaningful way. That is, one that can be iterated with a substitution much like that of the sphinx.

Some things we need:

- Other Important "anchor" points of the sphinx.
- A way to connect them that...
  - when iterated connects the important exterior points.
  - gives the same combinatorics at each vertex.

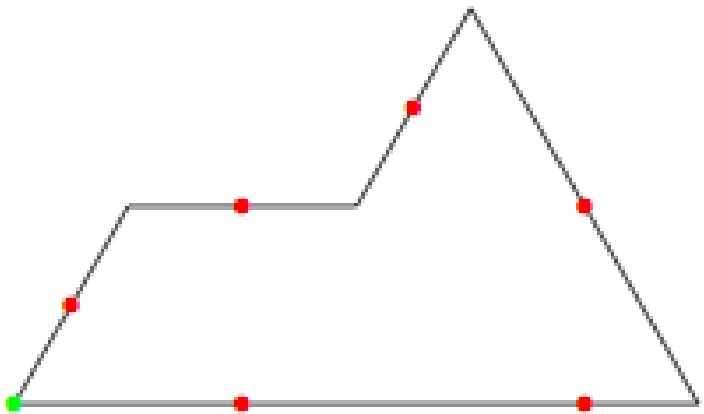
Important Anchor Points

Thinking ahead, we want our exterior points to be those that allow the fractal to pass from one sphinx to another in different levels of iteration.
So considering the fixed point at (1,0),

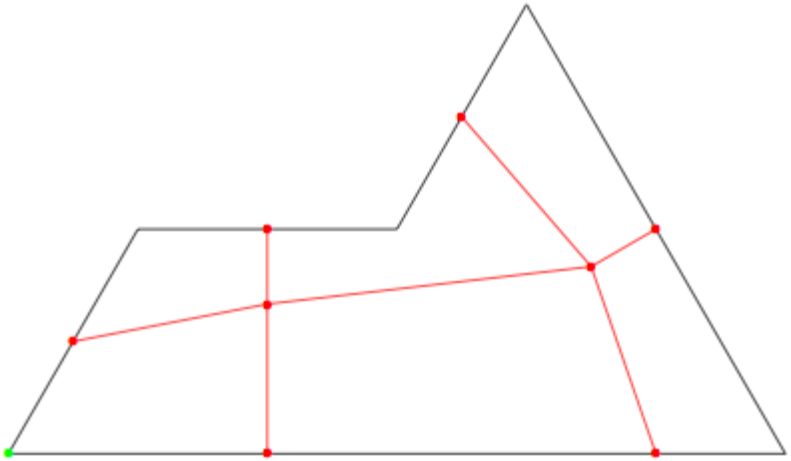


Important Anchor Points

Then we have our complete set of anchor points. Importantly, upon iteration of the sphinx, these anchor points only meet at each other.



Connecting The Anchor Points



Important things to notice

1. Upon iteration, These points will always be able to be connected
2. Using interior points, we ensure that upon iteration the edge of the level 0 sphinx is not touched where we don't already have anchor points-This will create new points of contact which is BAD!
3. We have made sure that all vertexes of the original fractal have the same combinatorics as the level 0 upon iteration including interior vertices.

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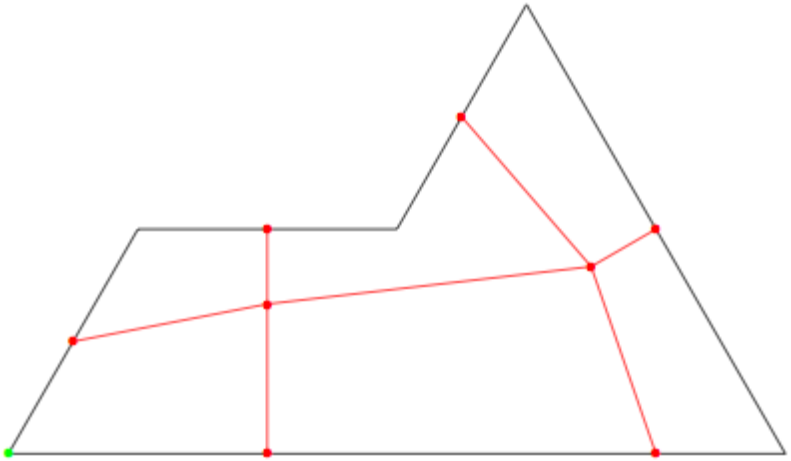

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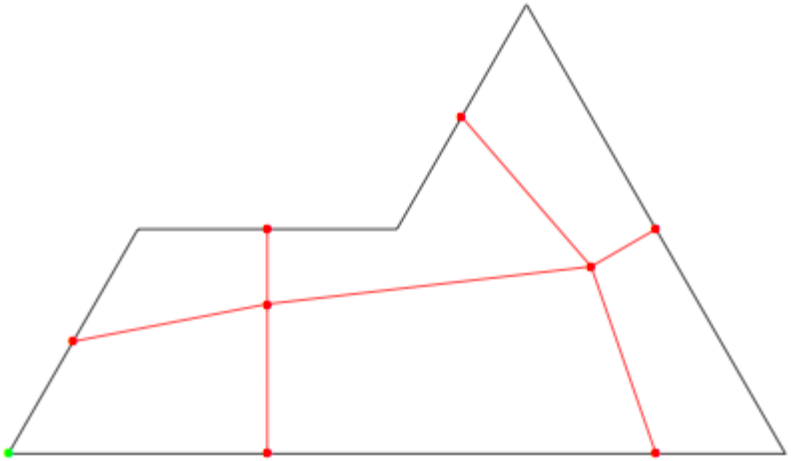

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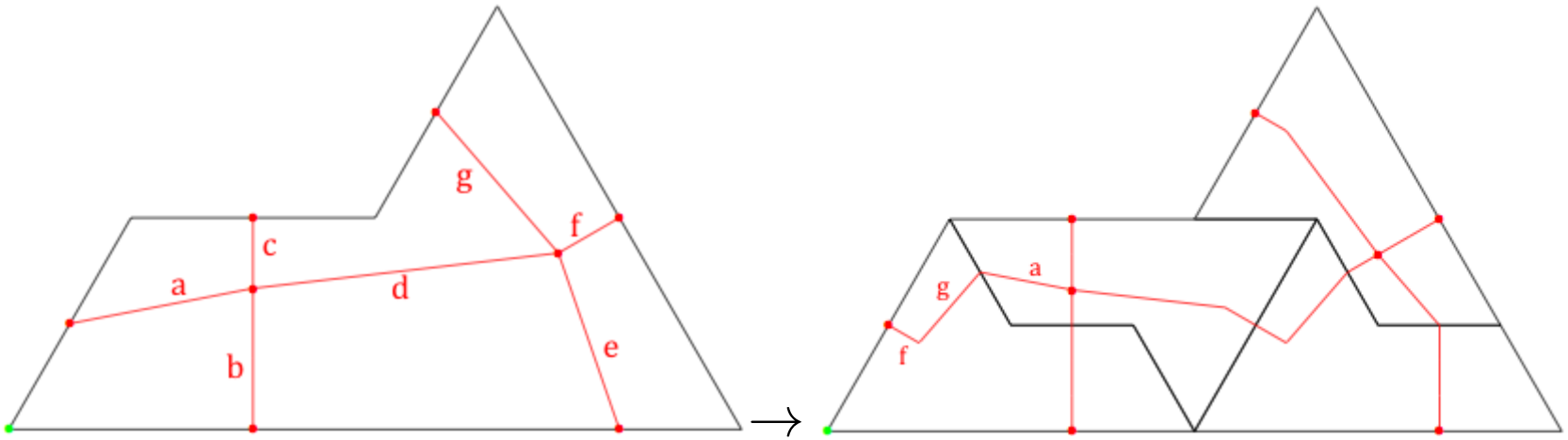
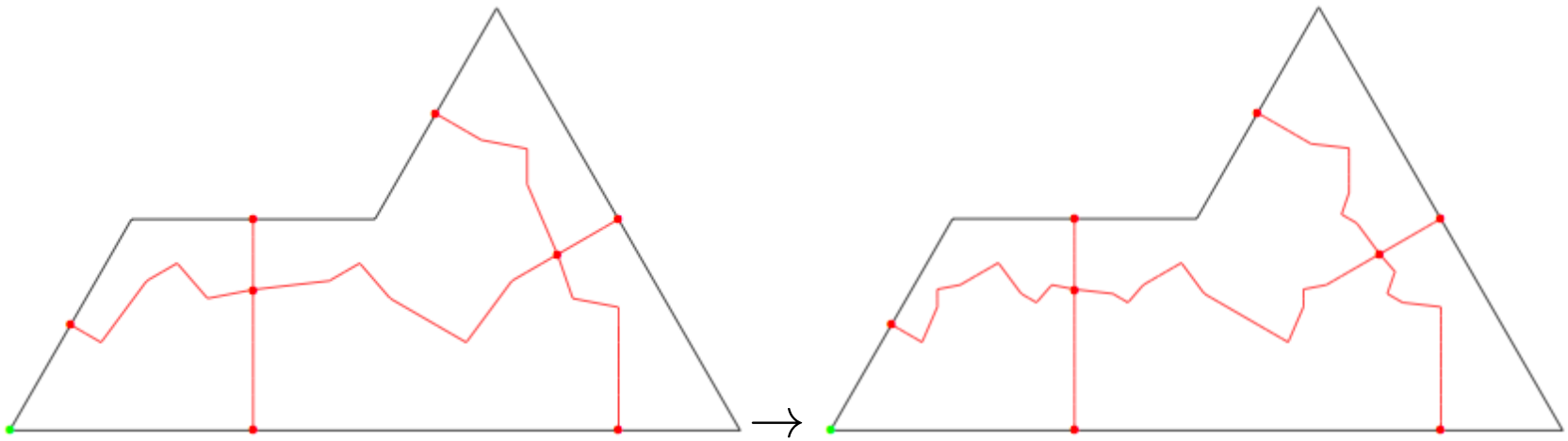
Fractal Aorta

Now that we have an aorta, we can begin to employ an iterated function system.



Fractal Aorta



This gives us a fractal aorta, much like the aorta of the fractal pinwheel.

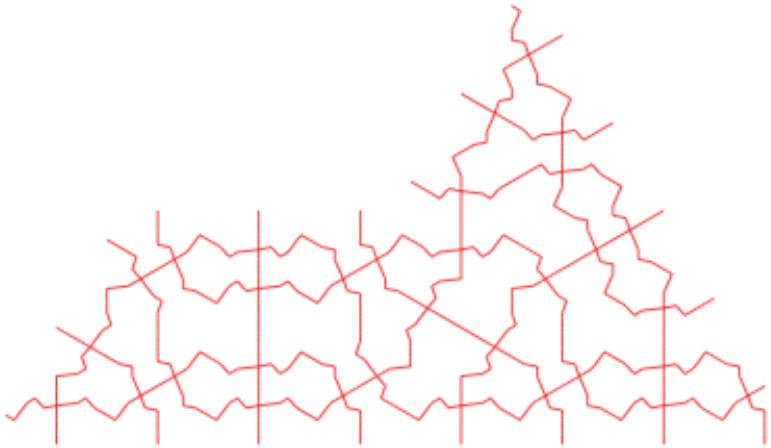
All the vertices are the same place as in the original and have the same combinatoric properties, as was desired.

Creating the Fractal

Now we take the fractal aorta and iterate it with the same substitution rule as the sphinx.

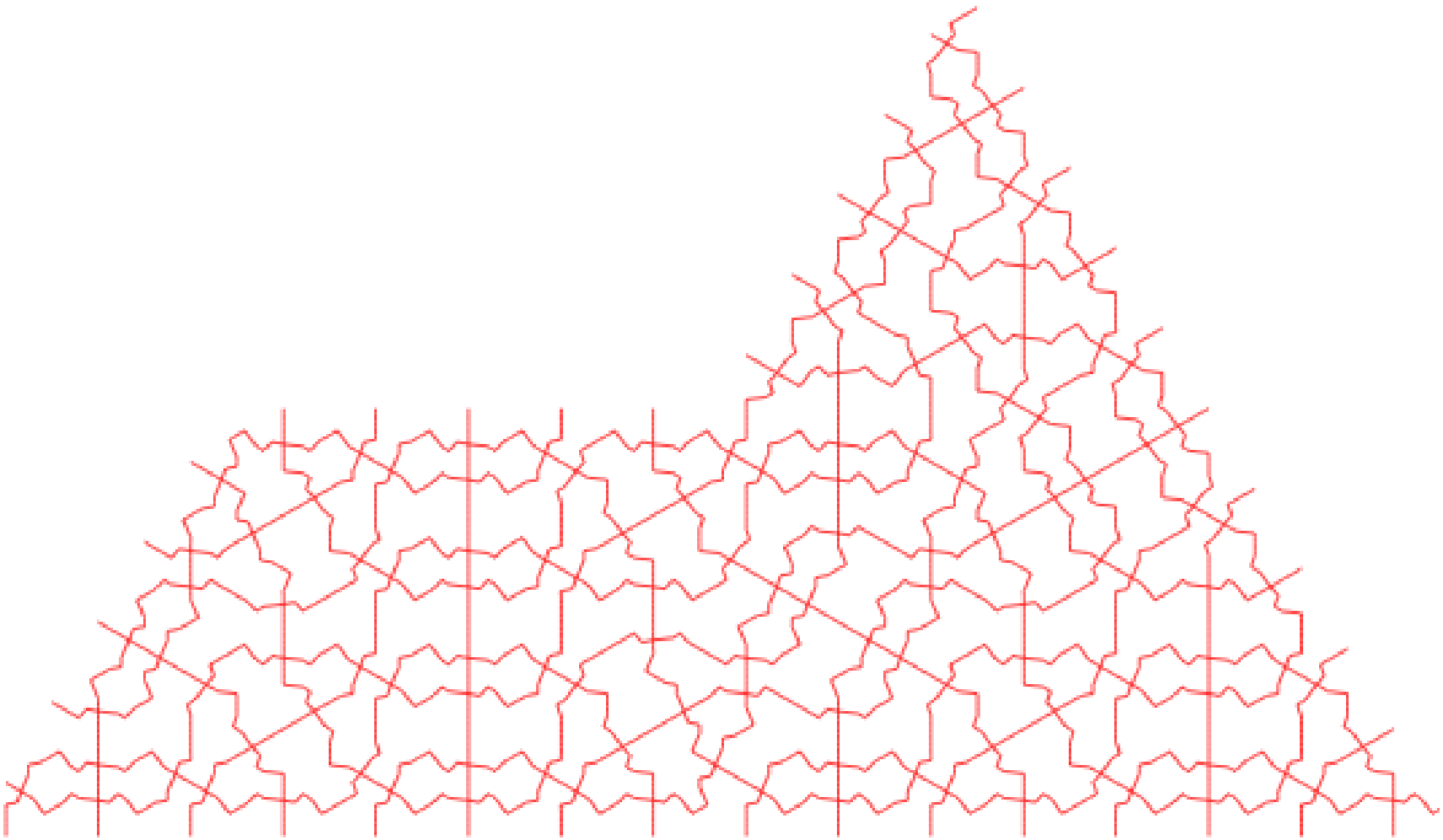
Level 0

Level 1



Level 2

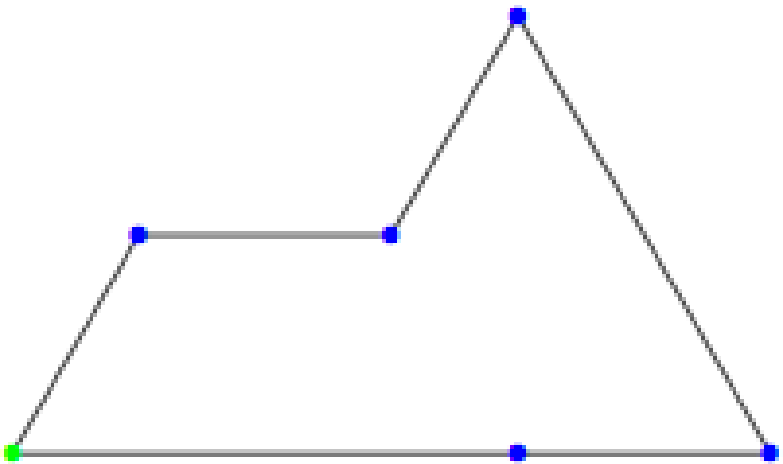
Creating the Fractal



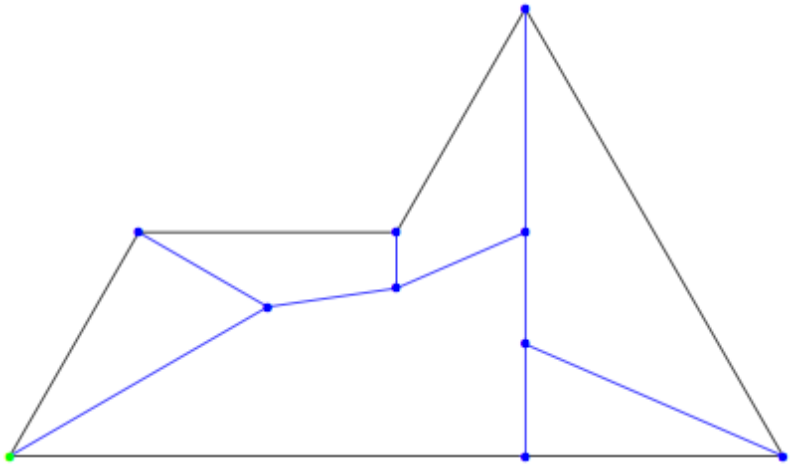
These tiles make a fractal version of the sphinx!

Second Version of Fractal Sphinx

Anchor points:



Fractal arc:



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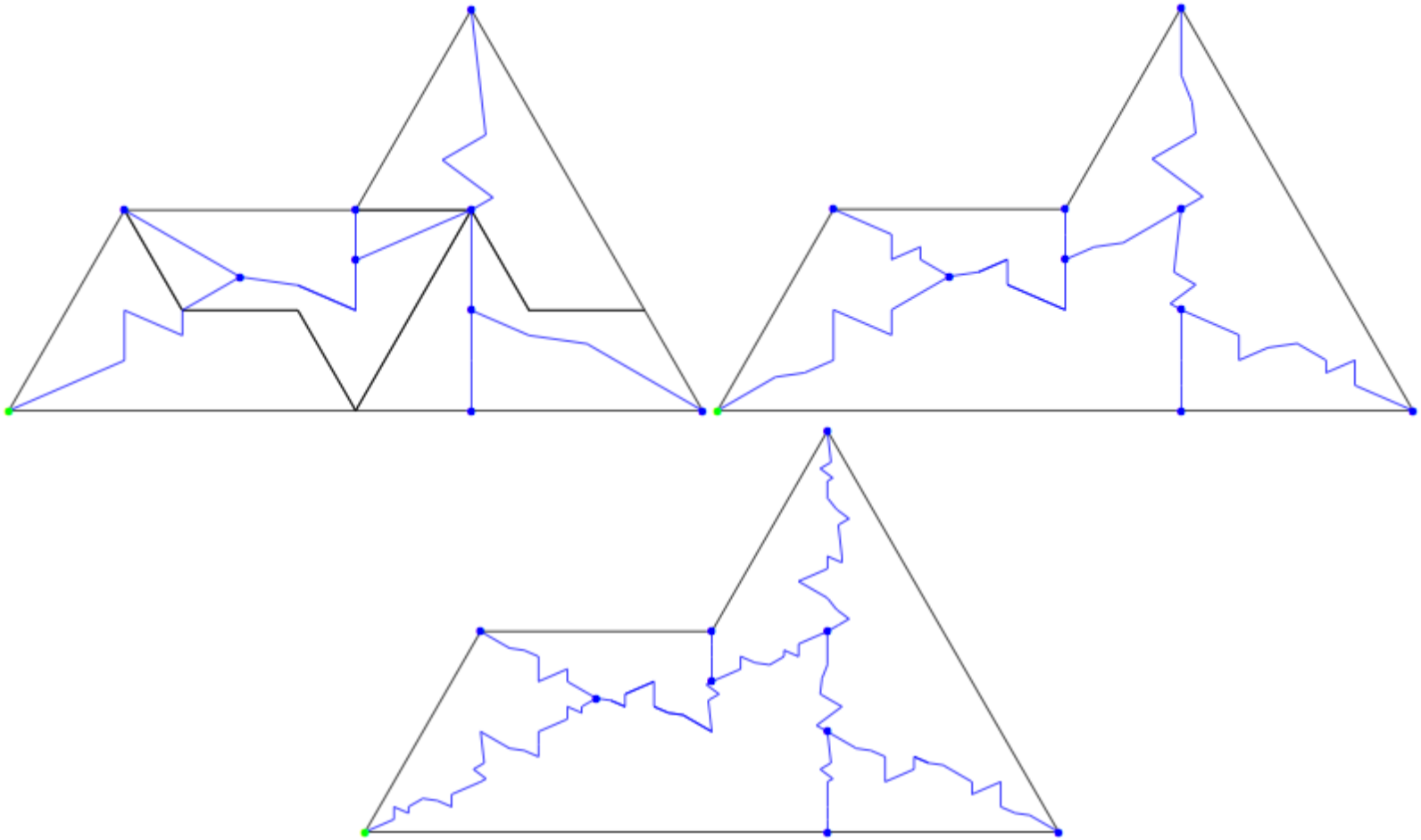

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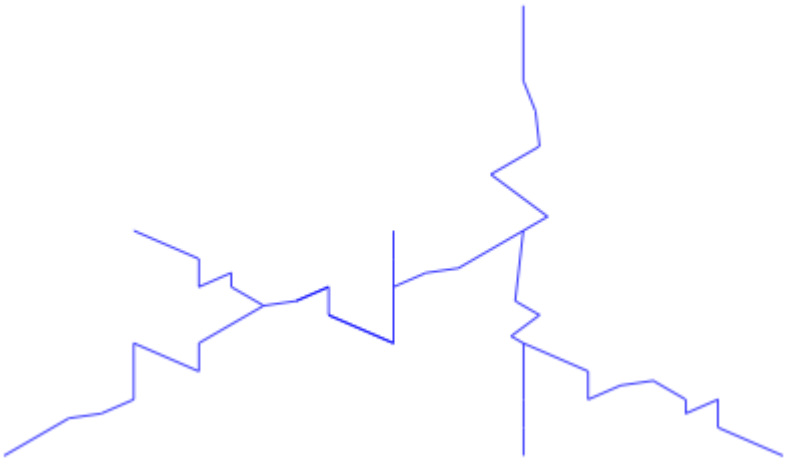

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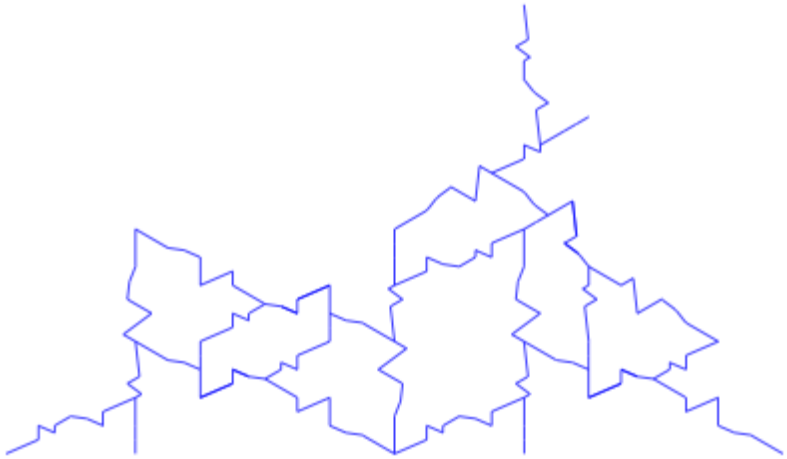

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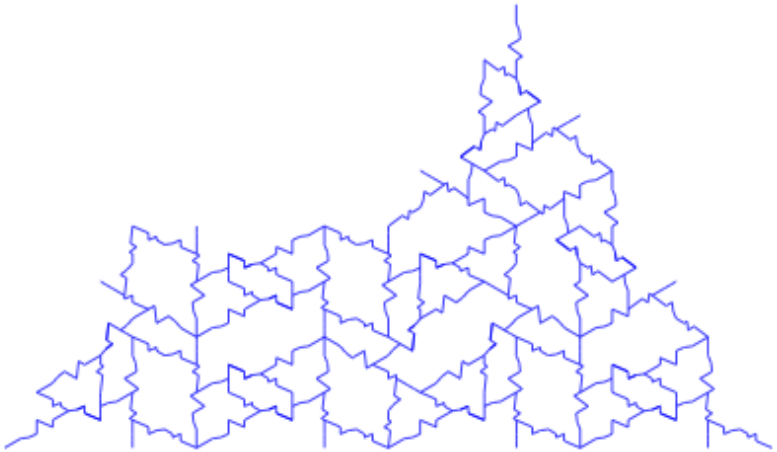
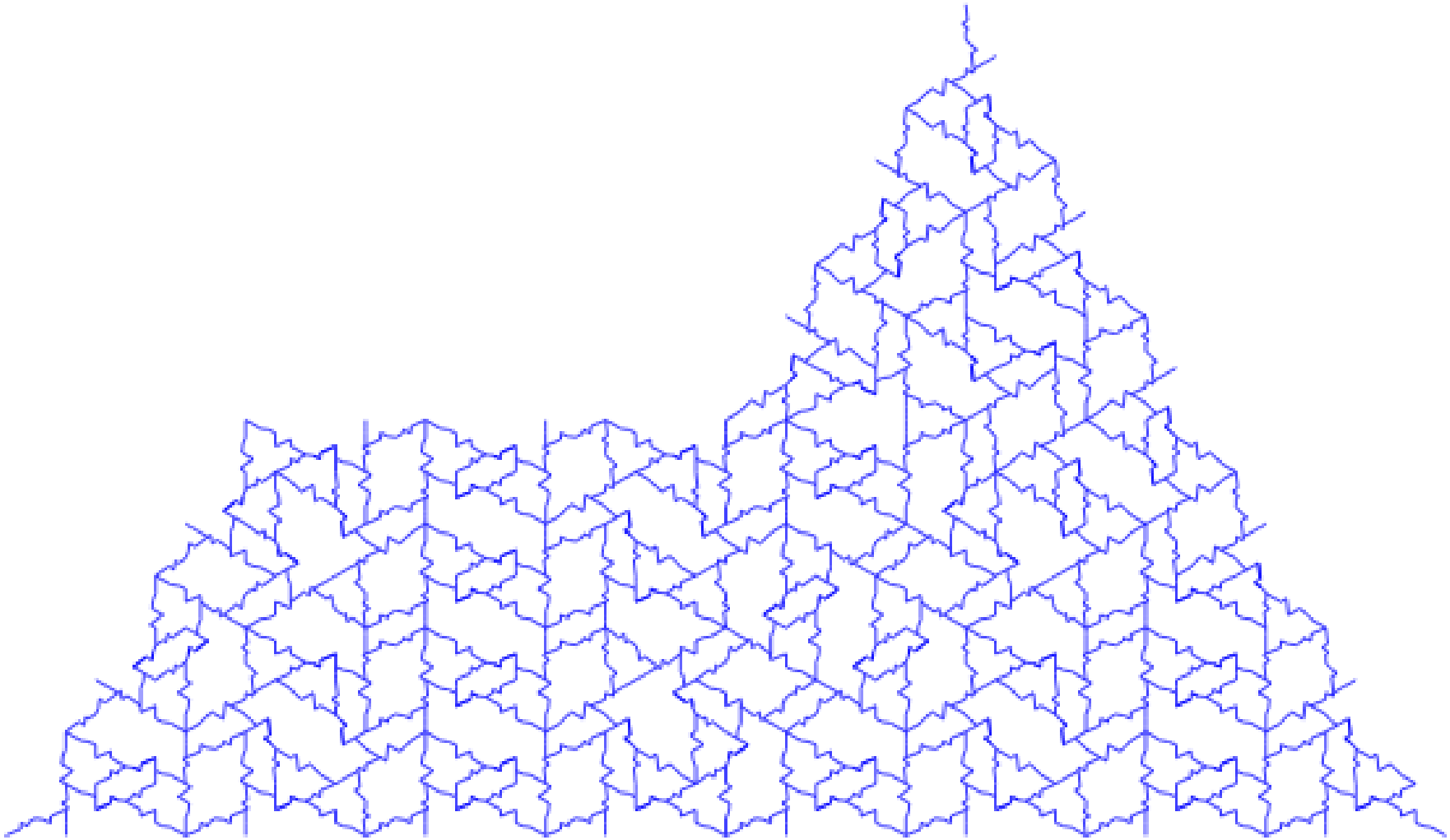


Level 2

Second Version of Fractal Sphinx



These tiles make a second fractal version of the sphinx!

Where to go from here?

1. This has never been seen before.
2. Comparison of the two fractals:
   - Number of tiles that make up each version.
   - Do certain tiles correspond to each other?
3. What is special about the sphinx that gives it these two distinct tilings?
4. What are certain properties of the tiling (e.g. what can repeating elements of a tiling-worms- tell us about the level of that tiling)

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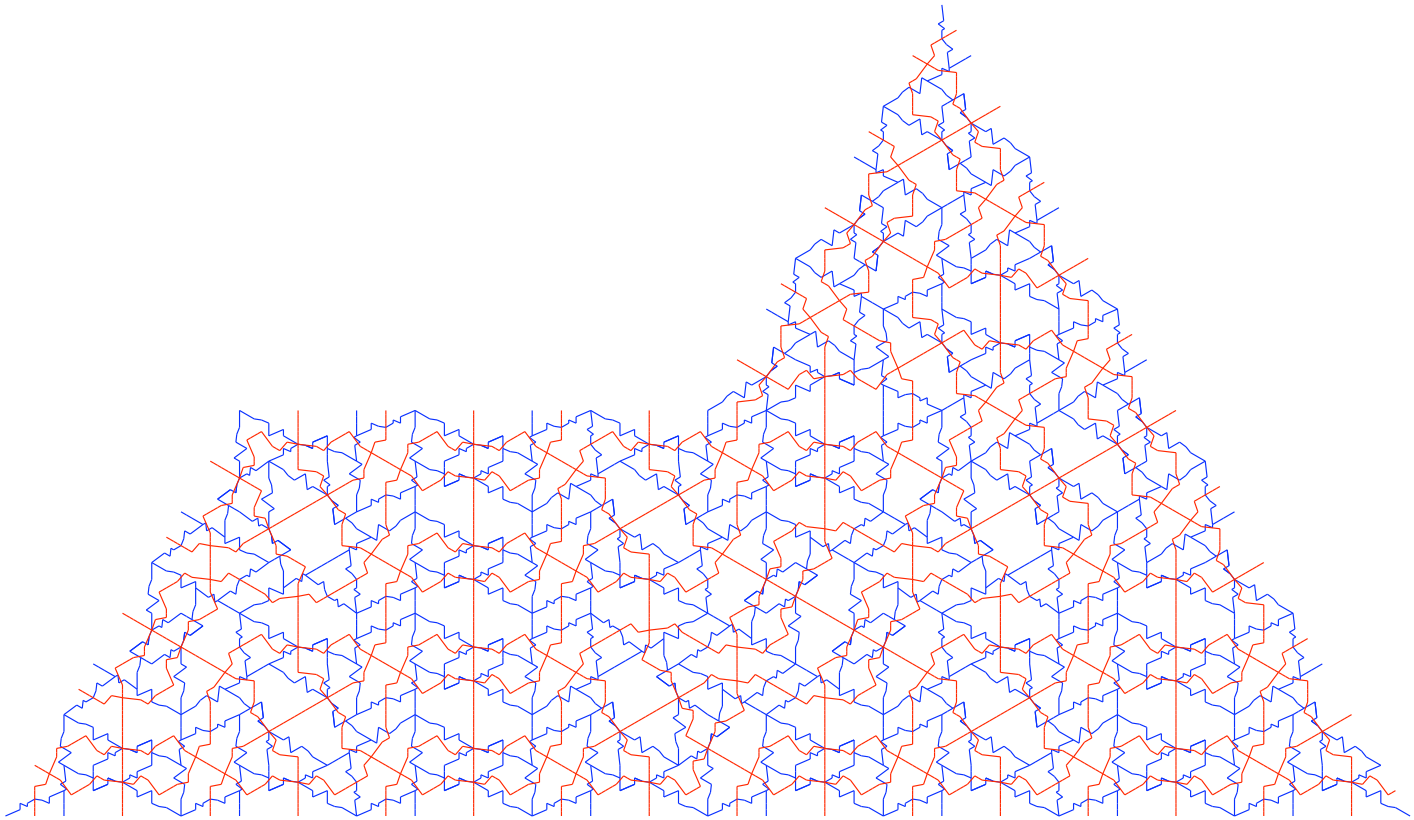
Overlaying 2 Sphinxes



References

- Prof. Natalie Priebe Frank. Vassar College Department of Mathematics.
- Prof. Michael F. Wittaker. University of Wollongong School of Mathmatics and Applied Statistics.