A Theorem of Block and Thielman

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Commuting Polynomials

- Certain polynomials, with coefficients in the real or complex numbers, commute under composition

- Two polynomials, f(x) and g(x), *commute under composition* if

$$(f \circ g)(x) = (g \circ f)(x)$$

or

$$f(g(x)) = g(f(x))$$

Similarity

- A polynomial f(x) is *similar* to a polynomial, g(x), if there exists a degree 1 polynomial λ(x) such that

$$g(x) = (\lambda^{-1} \circ f \circ \lambda)(x)$$

$$\lambda(x) = ax + b$$

$$\lambda^{-1}(x) = \frac{x-b}{a}$$

- Similarity is an equivalence relation

Similarity Theorem

- Take two polynomials, f(x) and g(x)
- Assume f(x) commutes with g(x)
- Then $(\lambda^{-1} \circ f \circ \lambda)(x)$ commutes with $(\lambda^{-1} \circ g \circ \lambda)(x)$
- This is our first helper theorem

Degree 2 Theorem

- There is, at most, one polynomial of any degree (greater than 1) that commutes with a given degree 2 polynomial
- One may consult Rivlin for further information
- This is our second helper theorem

Chains

- A *chain* is a sequence of polynomials which
  - contains one polynomial of each positive degree
  - such that every polynomial commutes with any polynomial in the chain

Major Examples of Chains

- Power monomials
  - Given by $\{x^n, n = 1, 2, 3, \ldots\}$
- Chebyshev polynomials
  - Given by $\{T_n(x), n = 1, 2, 3, \ldots\}$
    - $T_n(x) = \cos n(\cos^{-1}(x))$
    - $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$

$$
\begin{aligned}
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x
\end{aligned}
$$

Constructing New Chains Through Similarity

- We can construct new chains from the two major chains
  $\lambda(x) = x + 1 \quad \lambda^{-1}(x) = x - 1$
- $(\lambda^{-1} \circ (x^n) \circ \lambda)(x)$ is a chain

$$
\begin{aligned}
x &\rightarrow x \\
x^2 &\rightarrow x^2 + 2x \\
x^3 &\rightarrow x^3 + 3x^2 + 3x \\
x^4 &\rightarrow x^4 + 4x^3 + 6x^2 + 4x.
\end{aligned}
$$

- $(\lambda^{-1} \circ (T_n) \circ \lambda)(x)$ is also a chain

Recall

- A polynomial $f(x)$ is even if and only if $f(-x)=f(x)$
  - All odd degree coefficients in an even polynomial are 0

- A polynomial $f(x)$ is odd if and only if $f(-x)=-f(x)$
  - All even degree coefficients in an odd polynomial are 0

Block Thielman Theorem

- All chains are similar to either the power monomials or the Chebyshev polynomials

- The power monomials and the Chebyshev polynomials are the only two chains, up to similarity

Proof

- Let $\{p_n(x), n = 1, 2, 3, \ldots\}$ be a chain
  $p_2(x) = a_2x^2+a_1x+a_0$

- Let $\{q_j(x), j = 1, 2, 3, \ldots\}$ be a chain similar to $\{p_n(x)\}$ via $\lambda(x) = \frac{x}{a_2} - \frac{a_1}{2a_2}$

- $q_2(x) = x^2+c$

- We know that $q_2(x)$ commutes with $q_3(x)$

Proof Continued

- So, by definition

$$q_3(x^2+c) = {q_3}^2(x) + c \qquad (*)$$

- We can see that

$$q_3(x^2+c) - c = {q_3}^2(x) + c - c = {q_3}^2(x)$$

- Which means that

$$\begin{aligned} {q_3}^2(-x) &= q_3((-x)^2+c) - c \\ &= q_3(x^2+c) - c \\ &= {q_3}^2(x), \end{aligned}$$

- So

$$q_3(-x) = \pm q_3(x)$$

Proof Continued

- $q_3(x)$ is a degree 3 polynomial
  - The degree 3 coefficient cannot be 0
  - Thus, $q_3(x)$ cannot be even
  - So $q_3(-x) = - q_3(x)$, and $q_3(x)$ is odd

- This implies that $q_3(x) = b_3x^3+b_1x$

- Because $q_2(x)$ is monic, $q_3(x)$ is also monic

  $b_3=1$

Proof Continued

- We substitute $q_3(x)$ back into equation (*)

$$(x^2 + c)^3 + b_1(x^2 + c) = (x^3 + b_1 x)^2 + c$$

$$(x^6 + 3cx^4 + 3c^2x^2 + c^3) + b_1(x^2 + c) = x^6 + 2b_1x^4 + {b_1}^2x^2 + c$$

$$x^6 + 3cx^4 + (3c^2 + b_1)x^2 + c^3 + b_1c = x^6 + 2b_1x^4 + {b_1}^2x^2 + c$$

Compare Coefficients of x^4

$$3c = 2b_1$$

So

$$b_1 = \frac{3}{2}c$$

Compare Coefficients of x^2

$$
\begin{aligned}
3c^2 + b_1 &= (b_1)^2 \\
3c^2 + \frac{3}{2}c &= (\frac{3}{2}c)^2 \\
3c^2 + \frac{3}{2}c &= \frac{9}{4}c^2 \\
\frac{3}{4}c^2 + \frac{3}{2}c &= 0 \\
3c^2 + 6c &= 0 \\
c^2 + 2c &= 0 \\
c(c+2) &= 0.
\end{aligned}
$$

Compare Constants

$$
\begin{aligned}
c^3 + \frac{3}{2}c^2 &= c \\
2c^3 + 3c^2 &= 2c \\
2c^3 + 3c^2 - 2c &= 0 \\
(c+2)(2c^2 - c) &= 0 \\
c(c+2)(2c-1) &= 0.
\end{aligned}
$$

- Thus, $c = -2$ or $c = 0$

Case 1: $c = 0$

- Then $q_2(x) = x^2 + c = x^2$
- $q_2(x)$ is a power monomial
  - The only polynomials that commute with x^2 are the power monomials by the second helper theorem
- $\{q_j(x), j = 1, 2, 3, \ldots\}$ must be the power monomials
- So $\{p_n(x), n = 1, 2, 3, \ldots\}$ is similar to the power monomials

Case 2: $c = -2$

- Then $q_2(x) = x^2+c = x^2-2$
- Consider

$$\alpha(x) = 2x \qquad \alpha^{-1}(x) = \frac{1}{2}x$$

- Then

$$\begin{aligned}(\alpha^{-1} \circ q_2 \circ \alpha)(x) &= \alpha^{-1}((2x)^2 - 2) \\ &= \alpha^{-1}(4x^2 - 2) \\ &= \frac{1}{2}(4x^2 - 2) \\ &= 2x^2 - 1 \\ &= T_2(x).\end{aligned}$$

Case 2 Continued

- We know that

$$\{(\alpha^{-1} \circ q_j \circ \alpha)(x), \quad j = 1, 2, 3, \ldots\}$$

is a chain from our first helper theorem

- We know that this chain is actually the Chebyshev polynomials by our second helper theorem
- Thus, $\{p_n(x), n = 1, 2, 3, \ldots\}$ is similar to the Chebyshev polynomials, as similarity is transitive

References

H. D. Block and H. P. Thielman, Commutative polynomials, Quart. J. Math. Oxford Ser., (2), 2 (1951) 241-243.

E. J. Jacobsthal, Über vertauschbare polynome, Math. Z., 63 (1955) 243-276.

T. J. Rivlin, Chebyshev Polynomials, Wiley Interscience, 1971, pp 194.

Questions?

- Thank you!