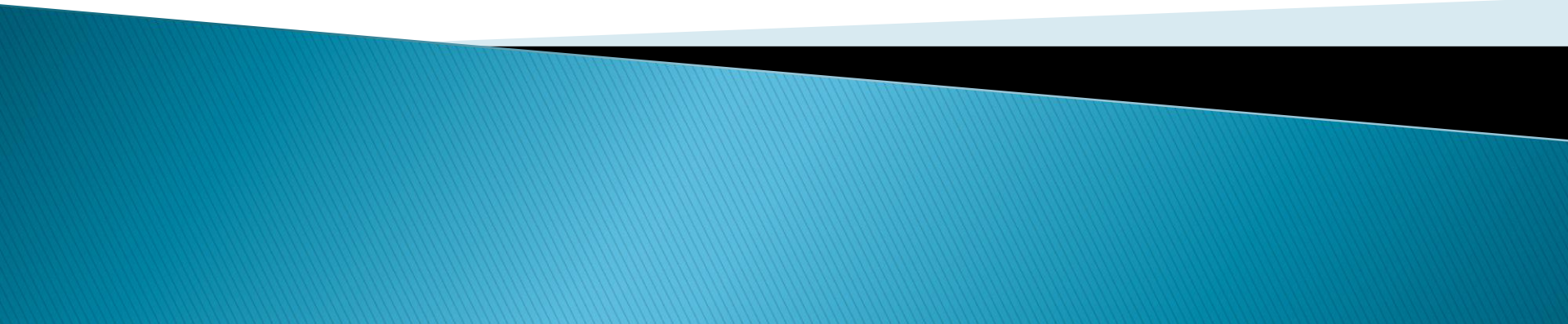


# A Theorem of Block and Thielman

Presented by Samuel Williams



# Commuting Polynomials

- ▶ Certain polynomials, with coefficients in the real or complex numbers, commute under composition
- ▶ Two polynomials,  $f(x)$  and  $g(x)$ , *commute under composition* if
$$(f \circ g)(x) = (g \circ f)(x)$$
or
$$f(g(x)) = g(f(x))$$

# Similarity

- ▶ A polynomial  $f(x)$  is *similar* to a polynomial,  $g(x)$ , if there exists a degree 1 polynomial  $\lambda(x)$  such that

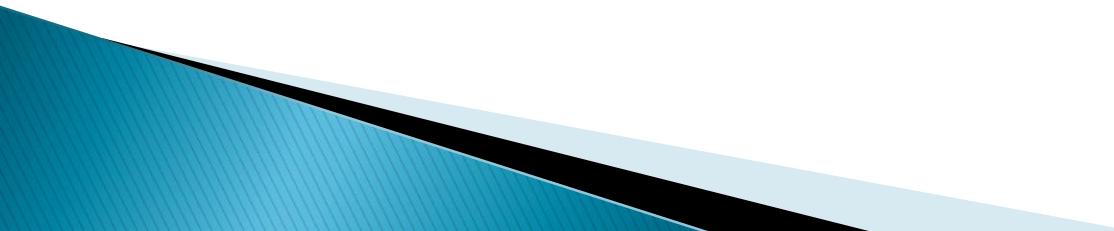
$$g(x) = (\lambda^{-1} \circ f \circ \lambda)(x) \qquad \lambda(x) = ax + b$$
$$\lambda^{-1}(x) = \frac{x - b}{a}$$

- ▶ Similarity is an equivalence relation

# Similarity Theorem

- ▶ Take two polynomials,  $f(x)$  and  $g(x)$
- ▶ Assume  $f(x)$  commutes with  $g(x)$
- ▶ Then  $(\lambda^{-1} \circ f \circ \lambda)(x)$  commutes with  $(\lambda^{-1} \circ g \circ \lambda)(x)$
- ▶ This is our first helper theorem

# Degree 2 Theorem

- ▶ There is, at most, one polynomial of any degree (greater than 1) that commutes with a given degree 2 polynomial
  - ▶ One may consult Rivlin for further information
  - ▶ This is our second helper theorem
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# Chains

- ▶ A *chain* is a sequence of polynomials which
  - contains one polynomial of each positive degree
  - such that every polynomial commutes with any polynomial in the chain

# Major Examples of Chains

- ▶ Power monomials
  - Given by  $\{x^n, n = 1, 2, 3, \dots\}$
- ▶ Chebyshev polynomials
  - Given by  $\{T_n(x), n = 1, 2, 3, \dots\}$ 
    - $T_n(x) = \cos n(\cos^{-1}(x))$
    - $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$

$$T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

# Constructing New Chains Through Similarity

- ▶ We can construct new chains from the two major chains

$$\lambda(x) = x + 1 \quad \lambda^{-1}(x) = x - 1$$

- ▶  $(\lambda^{-1} \circ (x^n) \circ \lambda)(x)$  is a chain

$$\begin{aligned}x &\rightarrow x \\x^2 &\rightarrow x^2 + 2x \\x^3 &\rightarrow x^3 + 3x^2 + 3x \\x^4 &\rightarrow x^4 + 4x^3 + 6x^2 + 4x.\end{aligned}$$

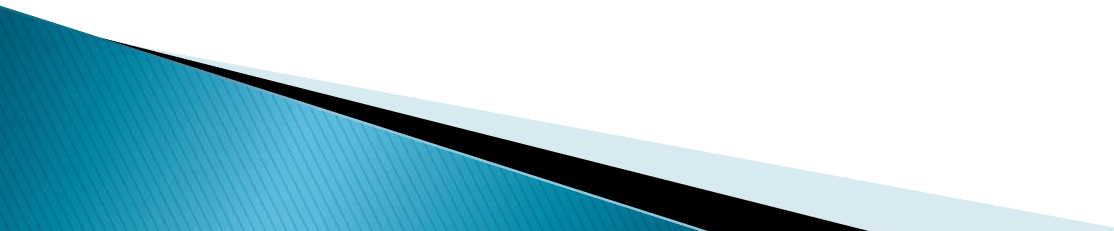
- ▶  $(\lambda^{-1} \circ (T_n) \circ \lambda)(x)$  is also a chain



# Recall

- ▶ A polynomial  $f(x)$  is even if and only if  $f(-x)=f(x)$ 
  - All odd degree coefficients in an even polynomial are 0
- ▶ A polynomial  $f(x)$  is odd if and only if  $f(-x)=-f(x)$ 
  - All even degree coefficients in an odd polynomial are 0

# Block Thielman Theorem

- ▶ All chains are similar to either the power monomials or the Chebyshev polynomials
  - ▶ The power monomials and the Chebyshev polynomials are the only two chains, up to similarity
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# Proof

- ▶ Let  $\{p_n(x), n = 1, 2, 3, \dots\}$  be a chain  
$$p_2(x) = a_2x^2 + a_1x + a_0$$
- ▶ Let  $\{q_j(x), j = 1, 2, 3, \dots\}$  be a chain similar to  $\{p_n(x)\}$  via 
$$\lambda(x) = \frac{x}{a_2} - \frac{a_1}{2a_2}$$
- ▶  $q_2(x) = x^2 + c$
- ▶ We know that  $q_2(x)$  commutes with  $q_3(x)$

# Proof Continued

- ▶ So, by definition

$$q_3(x^2 + c) = q_3^2(x) + c \quad (*)$$

- ▶ We can see that

$$q_3(x^2 + c) - c = q_3^2(x) + c - c = q_3^2(x)$$

- ▶ Which means that

$$\begin{aligned} q_3^2(-x) &= q_3((-x)^2 + c) - c \\ &= q_3(x^2 + c) - c \\ &= q_3^2(x), \end{aligned}$$

- ▶ So

$$q_3(-x) = \pm q_3(x)$$

# Proof Continued

- ▶  $q_3(x)$  is a degree 3 polynomial
  - The degree 3 coefficient cannot be 0
  - Thus,  $q_3(x)$  cannot be even
  - So  $q_3(-x) = -q_3(x)$ , and  $q_3(x)$  is odd
- ▶ This implies that  $q_3(x) = b_3x^3 + b_1x$
- ▶ Because  $q_2(x)$  is monic,  $q_3(x)$  is also monic  
 $b_3 = 1$

# Proof Continued

- ▶ We substitute  $q_3(x)$  back into equation (\*)

$$(x^2 + c)^3 + b_1(x^2 + c) = (x^3 + b_1x)^2 + c$$

$$(x^6 + 3cx^4 + 3c^2x^2 + c^3) + b_1(x^2 + c) = x^6 + 2b_1x^4 + b_1^2x^2 + c$$

$$x^6 + 3cx^4 + (3c^2 + b_1)x^2 + c^3 + b_1c = x^6 + 2b_1x^4 + b_1^2x^2 + c$$

# Compare Coefficients of $x^4$

$$3c = 2b_1$$

So

$$b_1 = \frac{3}{2}c$$

# Compare Coefficients of $x^2$

$$3c^2 + b_1 = (b_1)^2$$

$$3c^2 + \frac{3}{2}c = \left(\frac{3}{2}c\right)^2$$

$$3c^2 + \frac{3}{2}c = \frac{9}{4}c^2$$

$$\frac{3}{4}c^2 + \frac{3}{2}c = 0$$

$$3c^2 + 6c = 0$$

$$c^2 + 2c = 0$$

$$c(c + 2) = 0.$$



# Compare Constants

$$c^3 + \frac{3}{2}c^2 = c$$

$$2c^3 + 3c^2 = 2c$$

$$2c^3 + 3c^2 - 2c = 0$$

$$(c + 2)(2c^2 - c) = 0$$

$$c(c + 2)(2c - 1) = 0.$$

► Thus,  $c = -2$  or  $c = 0$

# Case 1: $c = 0$

- ▶ Then  $q_2(x) = x^2 + c = x^2$
- ▶  $q_2(x)$  is a power monomial
  - The only polynomials that commute with  $x^2$  are the power monomials by the second helper theorem
- ▶  $\{q_j(x), j = 1, 2, 3, \dots\}$  must be the power monomials
- ▶ So  $\{p_n(x), n = 1, 2, 3, \dots\}$  is similar to the power monomials

## Case 2: $c = -2$

- ▶ Then  $q_2(x) = x^2 + c = x^2 - 2$
- ▶ Consider

$$\alpha(x) = 2x \quad \alpha^{-1}(x) = \frac{1}{2}x$$

- ▶ Then

$$\begin{aligned}(\alpha^{-1} \circ q_2 \circ \alpha)(x) &= \alpha^{-1}((2x)^2 - 2) \\&= \alpha^{-1}(4x^2 - 2) \\&= \frac{1}{2}(4x^2 - 2) \\&= 2x^2 - 1 \\&= T_2(x).\end{aligned}$$

# Case 2 Continued

- ▶ We know that

$$\{(\alpha^{-1} \circ q_j \circ \alpha)(x), \quad j = 1, 2, 3, \dots\}$$

is a chain from our first helper theorem

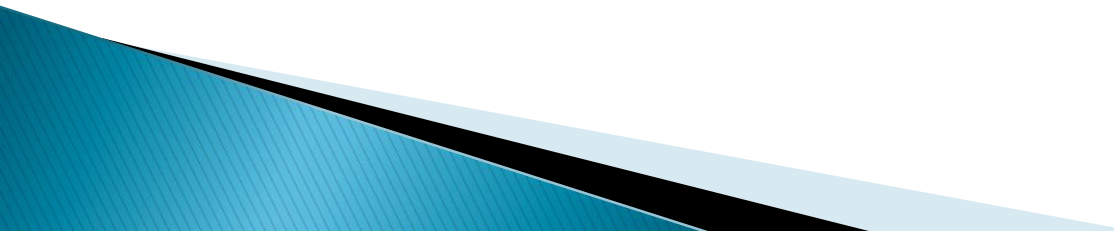
- ▶ We know that this chain is actually the Chebyshev polynomials by our second helper theorem
- ▶ Thus,  $\{p_n(x), n = 1, 2, 3, \dots\}$  is similar to the Chebyshev polynomials, as similarity is transitive

# References

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T. J. Rivlin, Chebyshev Polynomials, Wiley Interscience, 1971, pp 194.



# Questions?

- ▶ Thank you!