Some Interesting Examples of Wang Tilings

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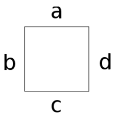
Wang Tilings

- As previously discussed, a Wang tile is a square tile with colored edges.
- In any Wang tiling, the colored edges can be represented by rational numbers:



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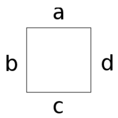

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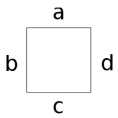

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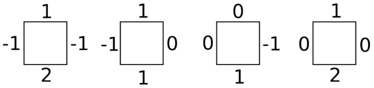
Jarkko Kari, 1995, 14 Tile Aperiodic Tile Set

- T_2



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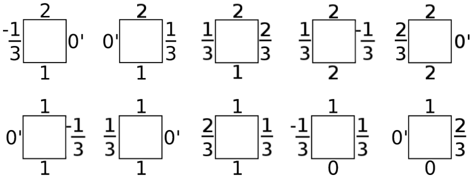

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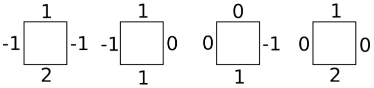

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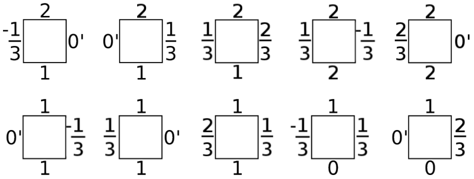
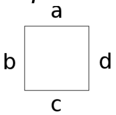


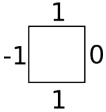
Tiling Rules for this Set

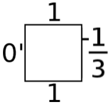
- Define the property that a tile multiplies by q if $aq + b = c + d$ for any Wang tile:



The numbers along the left and right edge are called carries.

- For example:





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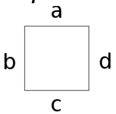

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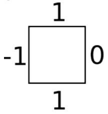

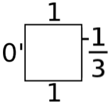

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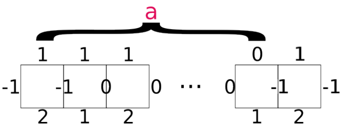
Aperiodicity

- Let $f : \mathbb{Z}^2 \to T$ be a tiling from Kari's tiles. Suppose it is periodic with horizontal period a and vertical period b.
- Let $i \in \mathbb{Z}$, Generate the set of tiles $F_i = \{f(1,i), f(2,i) \cdots , f(a,i)\}$:



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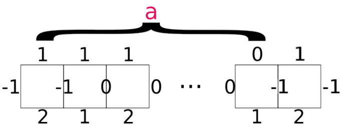

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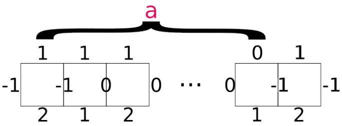

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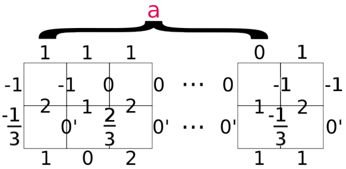
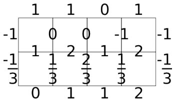
Aperiodicity (part 2)

- F_{i+1}:



- $q_i n_i + \text{left carry}(f(1,i)) = n_{i+1} + \text{right carry}(f(a,i))$, $n_{i+1} = q_i n_i$.
- For example:



- $(1(q) + -1) = (1+0)$, $(1(q)+0) = (2+0)$, $(0+0) = (-1+1)$, $(1(q)+-1) = (2-1) \Rightarrow n_i q = 3q$, $n_{i+1} = 1+2+1+2 = 6, \Rightarrow q = 2$

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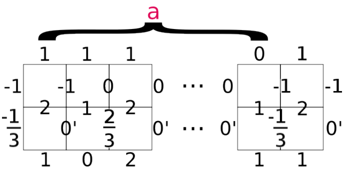

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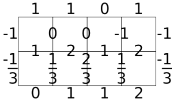

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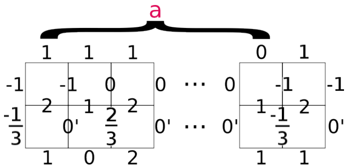

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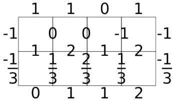

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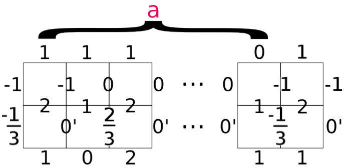

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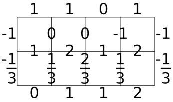

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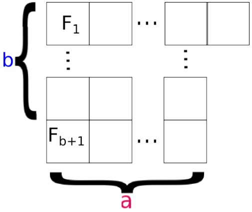
- f is also vertically periodic, F_1 will have the same tiles as F_{b+1}:



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  A contradiction.
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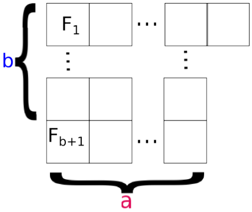

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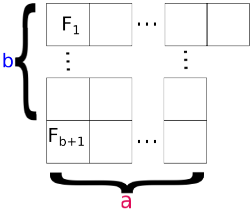

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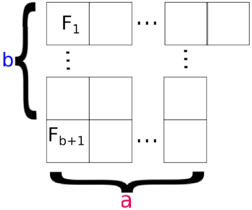

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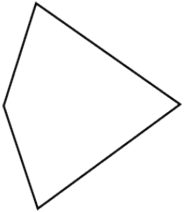
Penrose Mappings

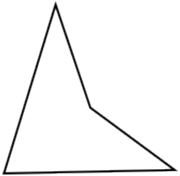
- Robert Penrose suggested that a Wang tiling could be constructed from a Penrose tiling, and Raphael M. Robinson refined a technique to do so.

Penrose P2 Tiling

- The Penrose P2 tiling is an aperiodic tiling consisting of two tiles,
  a kite shaped tile:



  and a dart shaped tile:



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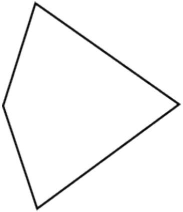

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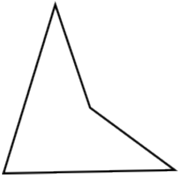
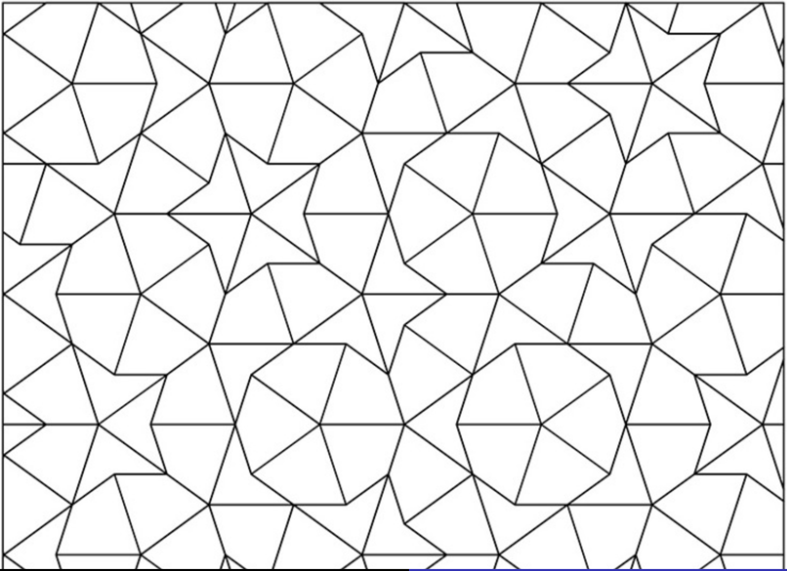
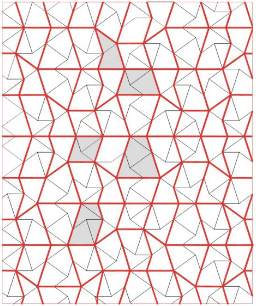


Penrose P2 TIling (Cont.)



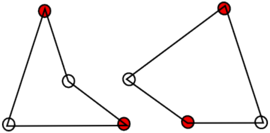
Quadrilateral/Pentagonal Partitioning



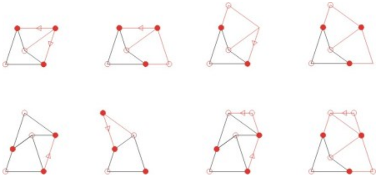
Use the P2 tiling on the previous page to partition the set into quadrilateral and non-regular pentagonal sections as shown above.

Quadrilateral/Pentagonal Partitioning (cont.)

- The Penrose P2 matching rule:



- The partition on the previous page yields the following 8 shapes:



- However, 32 Wang tiles are needed to account for rotation and reflection.

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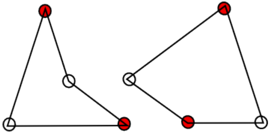

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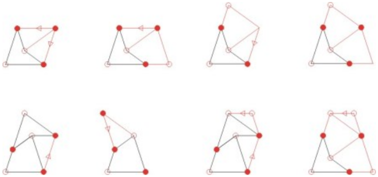

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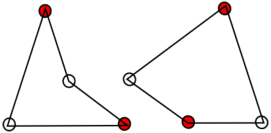

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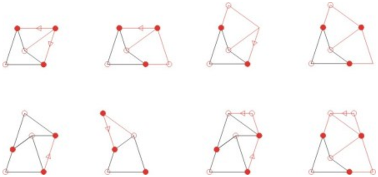

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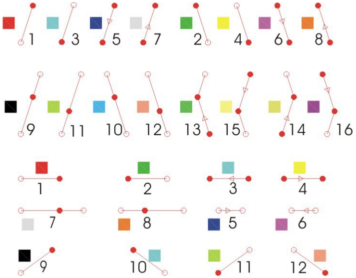
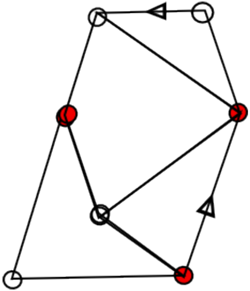
Color Assignment

- Using 16 colors, each horizontal and vertical edge of the polygons are assigned colors as follows:

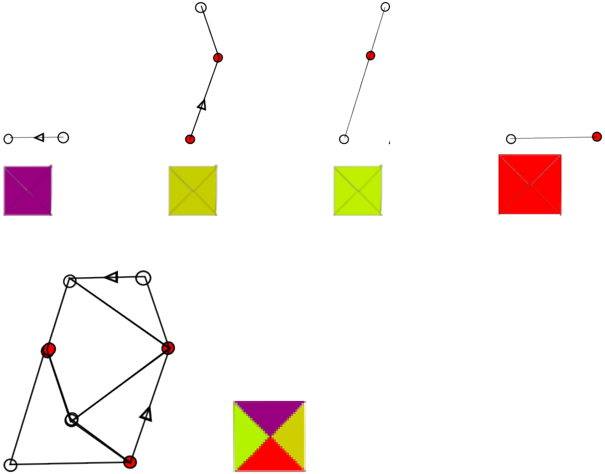


Assign Wang tiles

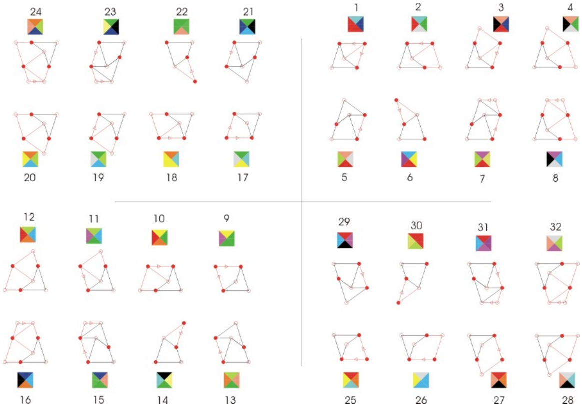
- These 4 colors and orientations for each polygon can be assigned to a Wang tile as follows.
- Using one example polygon:



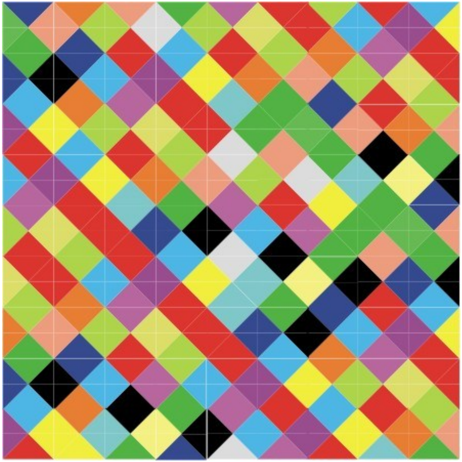
Assign Wang Tiles (Cont.)



The Full Mapping



The 32 Tile Wang Tiling



Summary

- An aperiodic Wang tiling can be generated from the Penrose P2 Tiling.
- Since the P2 tiles are aperiodic, the Wang tile set generated from it is also aperiodic.
- An tiling of the plane consisting of 14 Wang tiles was demonstrated to be aperiodic.

- Further Research
  - Further research can include generating a Wang Tiling, using a similar method, from the Penrose P3 tiling, an aperiodic tiling consisting of two rhombs.

Bibliography I

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A Small Aperiodic Set of Wang Tiles.
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