Just a Coincidence?

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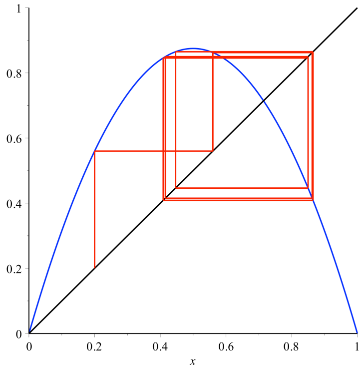
Logistic family of quadratic functions

$$f_r(x) = rx(1-x) \quad (0 \leq r \leq 4)$$

We'll consider the behavior of sequences $s, f_r(s), f_r^{(2)}(s) = f(f(s)), f_r^{(3)}(s), \ldots$, where s is some number between 0 and 1.

For example, a cobweb diagram illustrates what happens when $r = 3.5$ and $s = 0.2$:

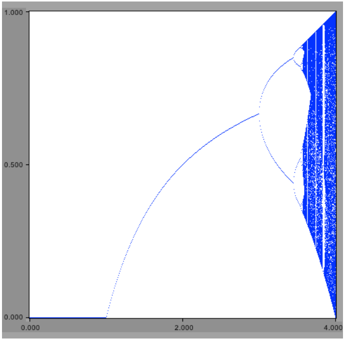


The sequence $0.2, f_{3.5}(0.2), \ldots$ starts repeating (if rounded to 4 decimal places) in the pattern $0.3828, 0.8269, 0.5009, 0.8750$ over and over.

Furthermore, it can be shown that for "almost every" choice of s in $[0, 1]$, the same cycle of four numbers will eventually emerge. Exceptions include (and might be limited to) $s = 0$, $s = 1$, and (cycle of length two). So the overall story of $f_{3.5}$ is "sequence tends to a cycle of length four". In practice, we usually pay more attention to r than to s.

Bifurcation diagrams: an overview

The horizontal axis gives r-values, from 0 to 4. The vertical axis gives x-values, from 0 to 1.



- For $0 < r < 1$, $f_r^{(k)}(x) \to 0$ as $k \to \infty$, for all $x \in [0,1]$.
- For $1 < r < 3$, $0 < \lim_{k\to\infty} f_r^{(k)}(x) < 1$, for almost all $x \in [0,1]$.
- For $3 < r < 1 + \sqrt{6} \approx 3.4495$,

$$\lim_{k\to\infty} f_r^{(2k)}(x) \neq \lim_{k\to\infty} f_r^{(2k+1)}(x).$$

- Then period 4, then period 8, ...
- And then things start *really* getting interesting.

$$f_r(x) = rx(1-x)$$

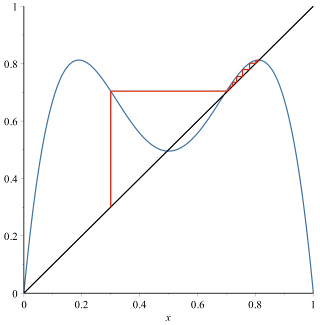
Notice that when $r > 4$, f_r no longer sends all points of the interval $[0,1]$ back inside $[0,1]$, since then $r^2 - 4r > 0$ and $f_r(x) > 1$ when

$$\frac{1}{2} - \frac{\sqrt{r^2-4r}}{2} < x < \frac{1}{2} + \frac{\sqrt{r^2-4r}}{2}$$

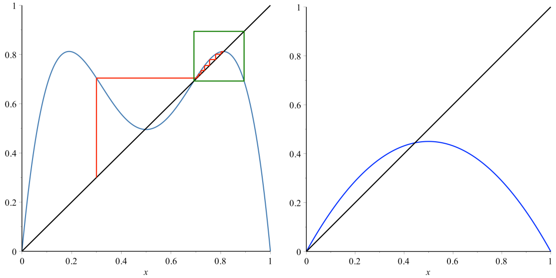
For example, $f_5(1/2) = 5/4$, $f_5^{(2)}(1/2) = -25/16$, $f_5^{(3)}(1/2) = -5125/256$, and $f_5^{(k)}(1/2) \to -\infty$ as $k \to \infty$.

So $r = 4$ is an important transition point in the family of logistic functions.

To better understand the behavior of logistic functions yielding period-two behavior, when $3 < r < 1 + \sqrt{6}$, we can look at the cobweb diagram for $f_r^{(2)}(x)$. Below is an example for $r = 3.25$ and $s = 0.3$, converging to 0.8124:



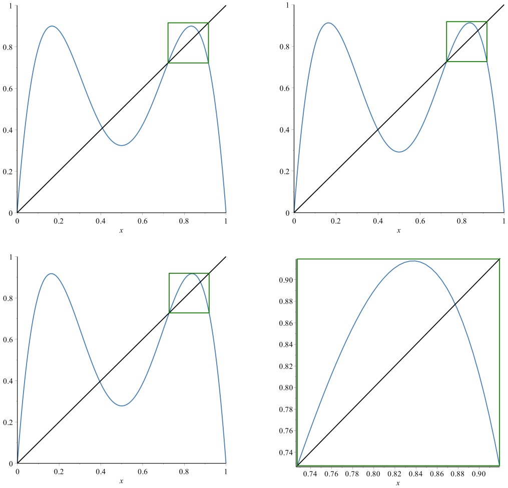
We add a “renormalization box”, which highlights the similarity between the behavior of $f_{3.25}^{(2)}(x)$ on the interval $[0.692, 0.894]$ and the behavior of a function like $f_{1.8}(x)$ on $[0, 1]$.



The comparison is not exact (the piece of graph inside the renormalization box is not symmetric about a vertical line splitting the box in two), but the qualitative behavior is close.

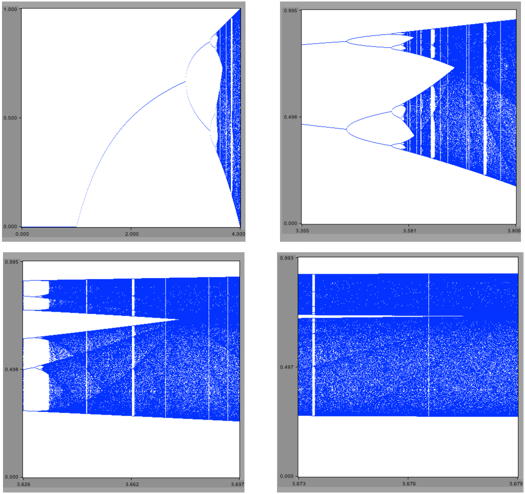
Now I can ask the questions that motivated this talk. For what value of r does the graph of $f_r^{(2)}(x)$ reach the top of the renormalization box, and what (if anything) is going on in the bifurcation diagram when r takes that value?

This was originally a homework problem for an ODE course. The students had a MATLAB program that would draw graphs and renormalization boxes, and could use a Java applet on the Internet to zoom into the bifurcation diagram as much as they liked.



First row: $r = 3.6, 3.65$; second row: $r = 3.67$, with zoom.

Zooming in on the bifurcation diagram: r-intervals are $[0, 4]$, $[3.355, 3.806]$, $[3.626, 3.697]$, $[3.673, 3.679]$



The thick bands correspond to "chaotic" (non-periodic) behavior, and they appear to merge into one band (reading from left to right) at about 3.677. Are this *exactly* the same value of r as we found for the bifurcation box? How could we prove this?

First check that $f_r(x) = x$ for $x = 0, 1 - \frac{1}{r}$. The latter, when $r \approx 3.6$ gives the x and y coordinates of the lower left corner of the renormalization box.

To find the lower right corner of the box, we need the solution to $f_r^{(2)}(x) = 1 - \frac{1}{r}$ which is larger than $1 - \frac{1}{r}$. This is a quartic equation,

$$r(rx(1-x))(1-(rx(1-x))) = 1 - 1/r$$

but we know (why?) that $x = 1 - \frac{1}{r}$ and $x = \frac{1}{r}$ are roots, so it can be reduced to a quadratic equation.

The solution we want is

$$x = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{r^2}}$$

That number, by construction, is also the height of the top of the box, and so the condition on r to obtain a local maximum tangent to the top of the box is:

$$f_r^{(2)}(x) = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{r^2}} \text{ and } {f_r^{(2)}}'(x) = 0$$

This looks awful, but fortunately the maximum value of $f_r^{(2)}(x)$ will equal the maximum value of $f_r(x)$, as long as $1/2$ is in the range of $f_r(x)$, which is true for $r \geq 2$. That common maximum value is $r/4$.

$$\begin{aligned}
\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{r^2}} &= \frac{r}{4} \\
\sqrt{\frac{1}{4} - \frac{1}{r^2}} &= \frac{r-2}{4} \\
\frac{1}{4} - \frac{1}{r^2} &= \frac{1}{16}(r-2)^2 \\
4r^2 - 16 &= r^2(r-2)^2 \\
4(r-2)(r+2) &= r^2(r-2)^2 \\
4r + 8 &= r^3 - 2r^2 \quad (\text{we know } r \neq 2) \\
0 &= r^3 - 2r^2 - 4r - 8
\end{aligned}$$

The cubic has one real root,

$$r = \tfrac{2}{3}(1 + (19 + 3\sqrt{33})^{1/3}) + \frac{8}{3(19 + 3\sqrt{33})^{1/3}} \approx 3.6786$$

Neidinger and Annen, in 1996, described a method for finding polynomials which would "outline" the bands in the bifurcation diagram of a closely related family of quadratic functions:

$$g_c(z) = z^2 + c$$

(when z is a complex variable, this is the family used to define the Mandelbrot set). In order to convert their results into the context of the logistic family, we'll need to find a correspondence between x and z that converts g_c to f_r and vice versa. Technically speaking, we want to find a "conjugating map".

It is not hard to check that $z = \phi(x) = -rx + (r/2)$ will satisfy $g_c \circ \phi = \phi \circ f_r$, when $c = (r/2) - (r^2/4)$.

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{f_r} & \mathbb{R} \\ \phi \downarrow & & \downarrow \phi \\ \mathbb{R} & \xrightarrow{g_c} & \mathbb{R} \end{array}$$

$$(-rx + r/2)^2 + (r/2 - r^2/4) \stackrel{?}{=} -r(rx(1-x)) + r/2$$

The following figure from Neidinger and Annen shows the bifurcation diagram for $g_c(z) = z^2 + c$, with graphs of several polynomials superimposed. We're looking for an intersection point of $Q_3(c) = g_c^{(3)}(0)$ and $Q_4(c) = g_c^{(4)}(0)$.

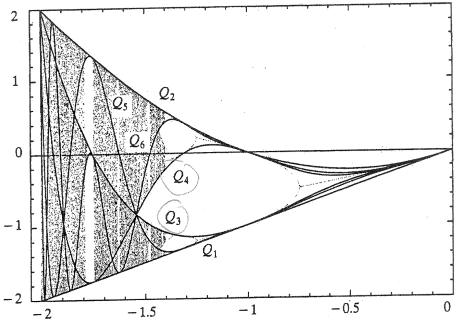


Figure 2. The first six Q-curves reveal dynamics and shapes within the diagram.

$$(c^2+c)^2+c = ((c^2+c)^2+c)^2+c$$
$$(c^2+c)^2+c = (c^4+2c^3+c^2+c)^2+c$$
$$\pm(c^2+c) = c^4+2c^3+c^2+c$$
$$0 = c^3+2c^2+2c+2 \text{ or } c=0 \text{ or } c=-2$$

Using the substitution $c = (r/2) - (r^2/4)$ in the cubic polynomial, we get the corresponding equation for r.

$$0 = -\frac{1}{64}r^6 + \frac{3}{32}r^5 - \frac{1}{16}r^4 - \frac{3}{8}r^3 + r + 2$$
$$0 = -\frac{1}{64}(r^3-2r^2-4r-8)(r^3-4r^2+16)$$

After checking that $r^3 - 4r^2 + 16$ has no roots in$[0, 4]$, we can conclude that r for the renormalization box equals r for the bands in the bifurcation diagram.

$$r = \frac{2}{3}(1 + (19 + 3\sqrt{33})^{1/3}) + \frac{8}{3(19 + 3\sqrt{33})^{1/3}}$$

$$c = -\frac{1}{3}(2 + (17 + 3\sqrt{33})^{1/3}) + \frac{2}{3(17 + 3\sqrt{33})^{1/3}}$$

$$\approx -1.543689012$$

Which leads to . . .

The c-value on the previous slide is an example of a *Misiurewicz point*, namely the sequence $\{g_c(0), g_c^{(2)}(0), g_c^{(3)}(0), \ldots\}$ eventually becomes periodic, but is not itself periodic.

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Thanks for coming to my talk!

References

1. Chaos Lab, `http://math.fau.edu/MLogan/Pattern_Exploration/Chaos_Lab/BifFamily.html`
2. R.D. Neidinger and R.J. Annen, "The Road to Chaos is Filled with Polynomial Curves", American Mathematical Monthly, 103 #8, 640–653.
3. Bodil Branner, "The Mandelbrot Set", in Chaos and Fractals, AMS, 1989.