WHAT IS A PERMUTAHEDRON, AND HOW IS IT USED TO COUNT TIES IN AN ELECTION?

By Adam Z. Margulies

VOTING THEORY BASICS

A = set of all *candidates*
|A| = m

V = set of all *voters* in the electorate
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Ballot = linear ordering on A
L(A) = is the set of all ballots

Example: Election for student body president:
A = {Polly ("P"), Quincy ("Q"), and Raleigh ("R")},
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L(A) = {(P, Q, R), (P, R, Q), (Q, P, R), (Q, R, P), (R, P, Q), (R, Q, P)}

PROFILES

Profile = record of all ballots cast

Definition – function that outputs a ballot for each voter *OR*
vectors representing ballots that receive votes *OR*
simple list of ballots and voters:

Ex.

v_1	v_2	v_3	v_4
P	P	Q	R
Q	Q	P	P
R	R	R	Q

CHOOSING A WINNER

If $m \geq 3$, many voting rules are possible. Our focus from now on is the ***Borda Count***.

- Jean-Charles de Borda (18th Century French mathematician , political scientist)
- Each voter awards

  m – 1 points to top choice

  m – 2 points to second choice

  ...

  0 points to least preferred candidate
- Winner = candidate with greatest point total

Ex. Previous profile: P gets 6 points; Q gets 4 points; R get 2 points.

MOTIVATION: M-WAY TIES

- One purpose of election – to choose winner(s) from set of candidates
- Different voting rules have different likelihood of ties
- Most extreme Borda tie: all candidates get same number of points (“m-way tie”)

Ex. m = 3; two voter profile.

v_1	v_2
P	Q
R	R
Q	P

P, Q, R all get 2 points - every candidate wins.

WHAT DOES THIS HAVE TO DO WITH A PERMUTAHEDRON…AND WHAT IS A PERMUTAHEDRON ANYWAY?

Borda count has an equivalent geometrical interpretation…

First some definitions:

General case, $A = <a_1, a_2 \ldots, a_m>$, a_i and a_j any two candidates in A

$a_i >_\sigma a_j$ if ballot σ has alternative a_i above a_j

For given σ, *rank* $\rho(a_j)$ = number of alternatives a_k in A satisfying $a_j >_\sigma a_k$

rank vector $\rho(\sigma)$= m-tuple $(\rho(a_1), \rho(a_2), \ldots, \rho(a_m))$.

Example:

A = <P, Q, R>

If σ = (Q, P, R), then:

$\rho(P) = 1$

$\rho(Q) = 2$

$\rho(R) = 0$

So $\rho(\sigma) = (1, 2, 0)$

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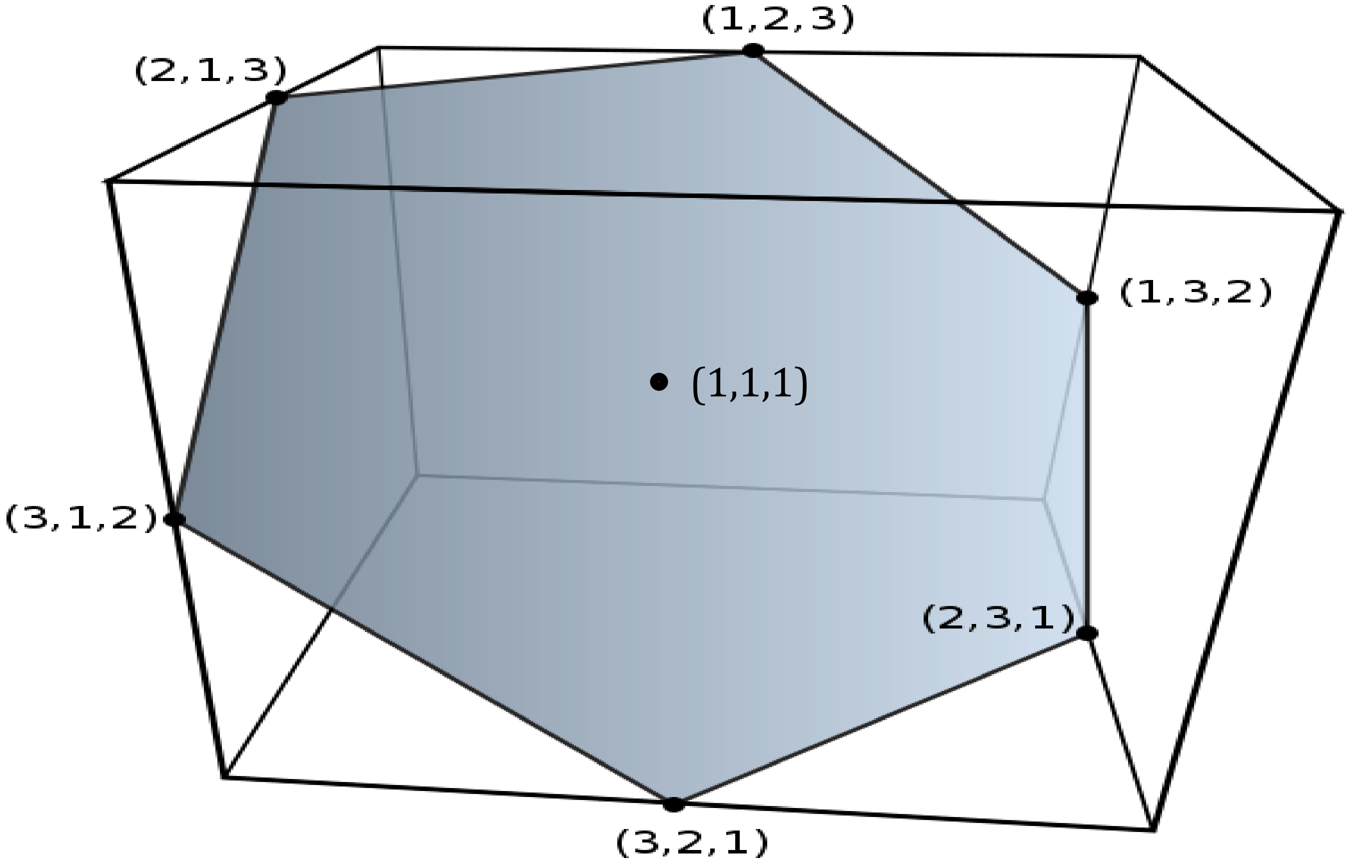
The 3 Permutahedron!



THE PERMUTA-MEAN RULE

Borda winner = point on permutahedron closest to the mean of all rank vectors taken from a profile (Zwicker, 2008).

So… an m-way tie will always result in the mean of rank vectors at the center of the permutahedron!

Ex. A = <P, Q, R>:

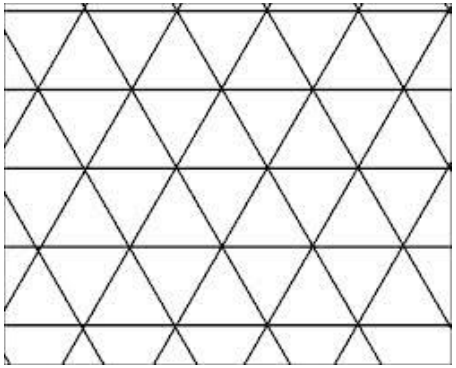
v_1	v_2
P	Q
R	R
Q	P

Rank vectors = (2, 0, 1) and (0, 2, 1).

So mean point = (1, 1, 1) = m-way tie.

COUNTING TIES

- Random walks on a special lattice (Marchant)



- Ehrhart Theory

- Combinatorial brute force

EHRHART THEORY

- Used to count the number of integer lattice points L in a dilated polytope as a function of an integer dilation factor, n

- L(n) is given as an *Ehrhart quasi-polynomial*, or a polynomial with modular coefficients

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Thus $L(4) = 1(4^2) + 4(4) = 32$

EHRHART THEORY CONTINUED (DAI, 2008)

- Dai modeled conditions for 3-way Borda tie as a system of linear equations to form a convex polyhedron
- The number of integer lattice points contained within the polyhedron represented the number of profiles that would give a three-way tie
- "Dilation factor" n = number of voters in the electorate
- Answer was computed with computer software LattE, short for Lattice point Enumeration.
- The answer:

$$L(n) = \frac{n^3 + 9n^2 + \ll 42, 15 \gg n + \ll 72, 25, -88, 9, 56, 7 \gg}{72}$$

RESULT CONFIRMED WITH "BRUTE FORCE" COMBINATORICS

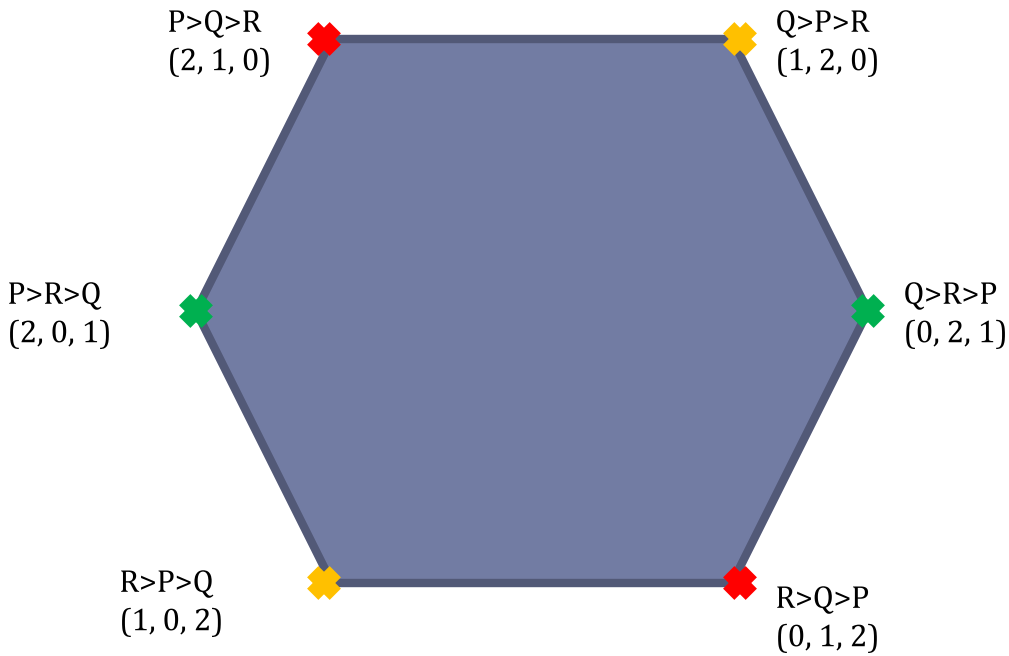
- Dai also classified profiles and used combinatorial formulas to count all that produced an m-way tie…

Definition: A profile is *central* if it results in an m-way tie.

Definition: A profile is *elementary* if it is central AND contains no smaller central profiles

ELEMENTARY PROFILES FOR M = 3

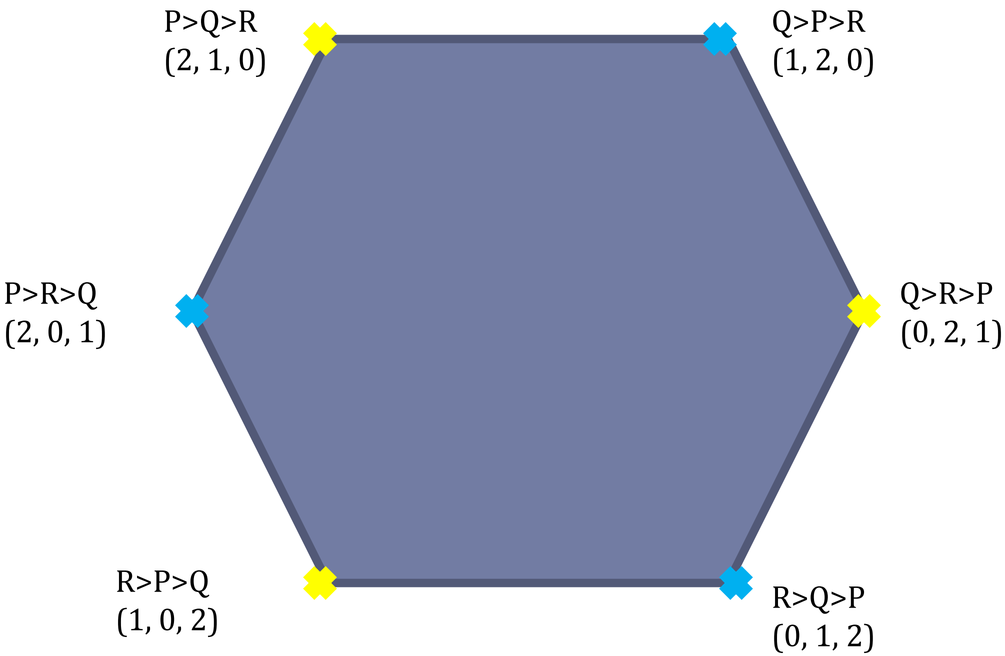
Reference enumeration: A = <P, Q, R>



3 "elementary reversals":

P	*Q*	*R*	*Q*	*R*	*R*
R	*R*	*P*	*P*	*Q*	*P*
Q	*P*	*Q*	*R*	*P*	*Q*

ELEMENTARY PROFILES FOR M =3, CONTINUED…



2 elementary "cycles":

P	Q	R
R	P	Q
Q	R	P

P	R	Q
Q	P	R
R	Q	P

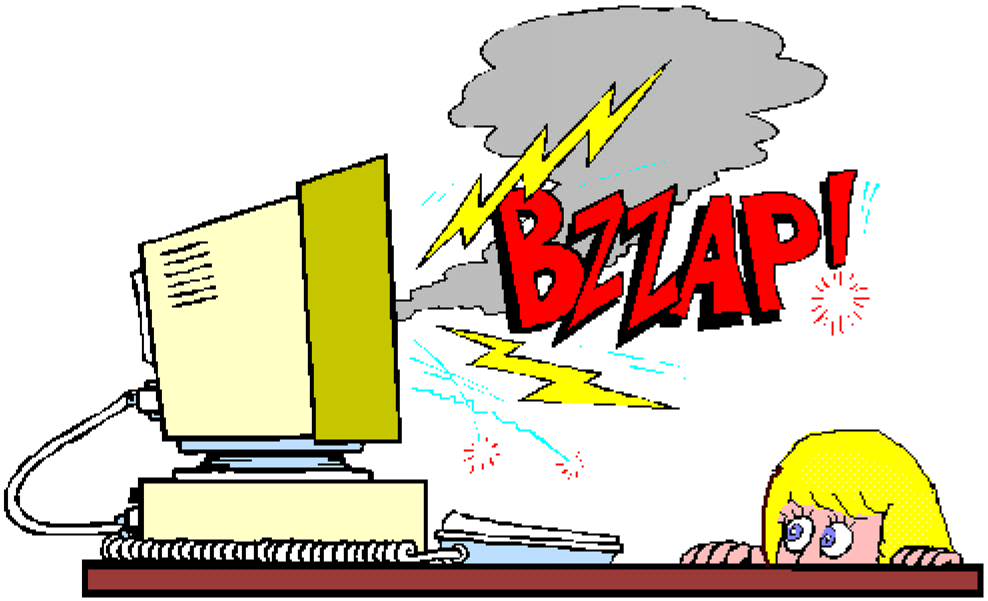
PROBLEM: DOUBLE COUNTING

The same central profile can be constructed from different sums of elementary profiles

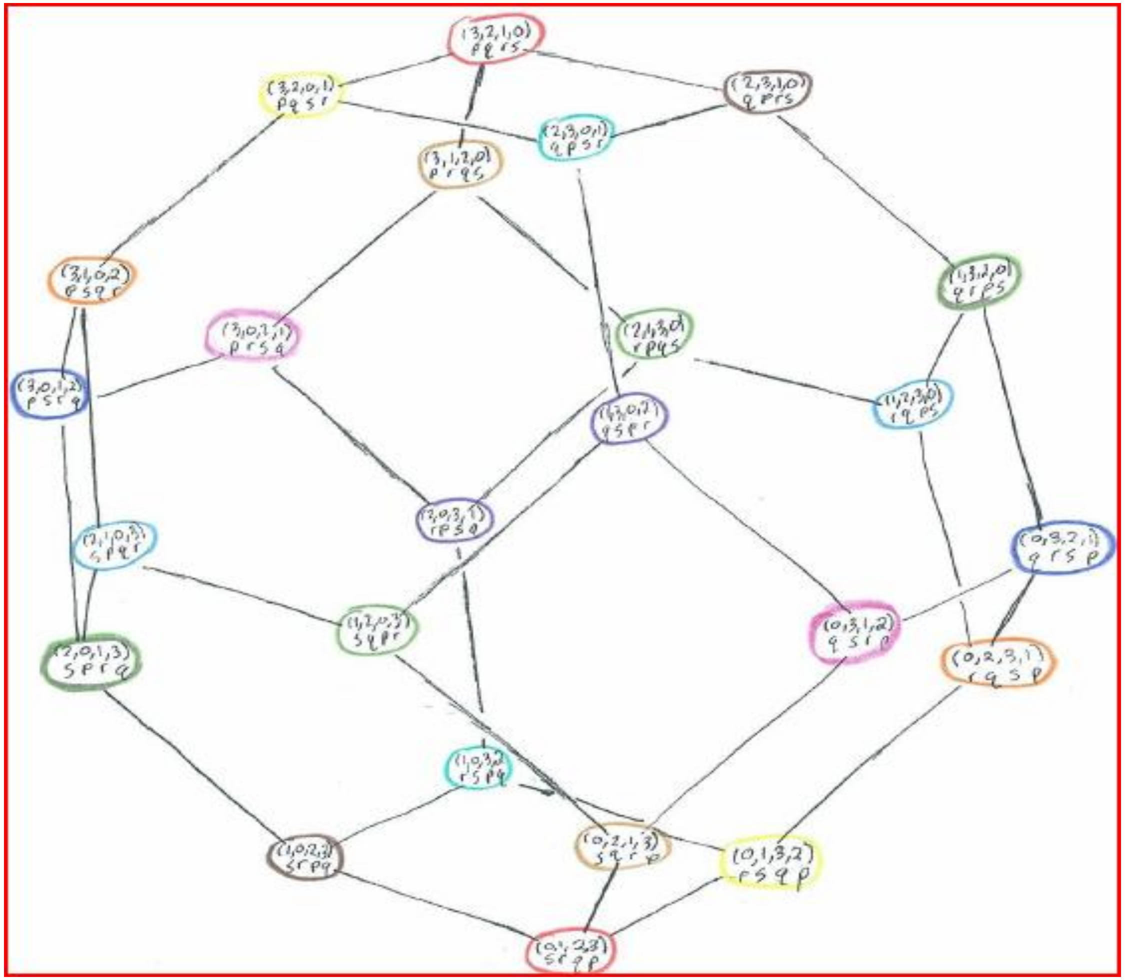
- Dai proved that the 5 elementary profiles were the <u>only</u> elementary profiles for 3 candidates.
- He also proved that every central profile could be expressed as a positive integer linear combination of elementary reversals and one of the elementary cycles.
- These results were necessary to account for double counting

4 CANDIDATES AND BEYOND

Problem: Due to the complexity of the system of linear constraints for 4 or more alternatives, Ehrhart theory is currently unable to calculate the number of 4-way ties.



BRUTE FORCE WITH 4 ALTERNATIVES: THE 4-PERMUTAHEDRON



Properties: A truncated octahedron living in 4-space; 24 vertices; 36 edges; 6 square faces, 8 hexagonal faces.

HOW COMPLICATED IS COMPLICATED?

We wrote a computer program to find all elementary profiles for 4 candidates:

Voters	2	4	6	8	10	12
Elementary profiles	12*	36	532	2076	5664	???

*See previous slide

We went from 5 elementary profiles for 3 candidates, to 8320 (and counting) elementary profiles for 4 candidates!

FUTURE WORK

- Find the set of elementary profiles that will produce all central profiles via a positive integer linear combination
- Classify elementary profiles into types similar to the notion of “reversals” and “cycles” for 3 candidates.
- Ultimate goal: find quasi-polynomial that gives the number of central profiles (m-way ties) for m = 4 as a function of n

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