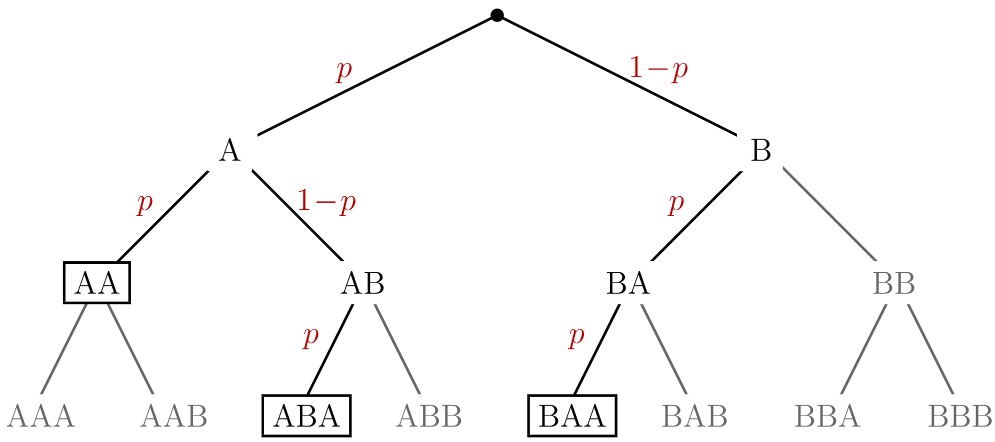
The Probability of Winning a Series

Gregory Quenell

Exercise: Team A and Team B play a series of $2n+1$ games. The first team to win $n+1$ games wins the series. All games are independent, and Team A wins any single game with some fixed probability p.

What is the probability that Team A wins the series?



$$p^2 + p^2(1-p) + p^2(1-p) = 3p^2 - 2p^3$$

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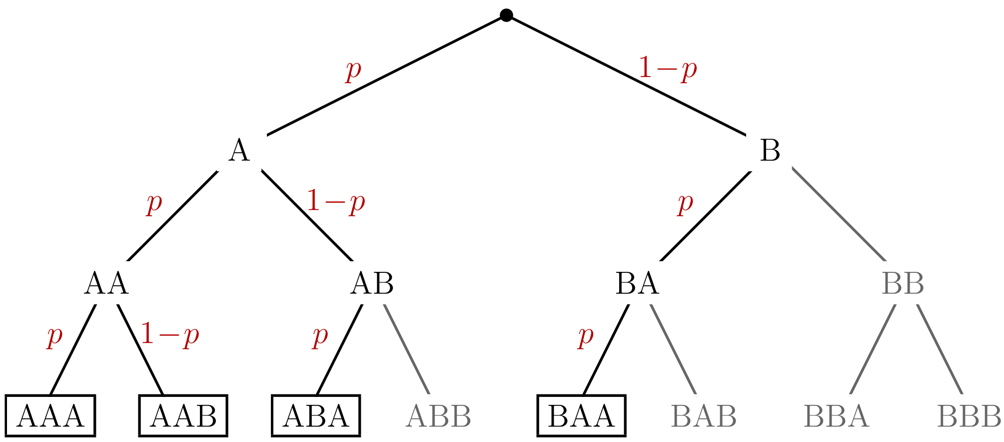


$p^3 + p^2(1-p) + p^2(1-p) \quad + \quad p^2(1-p)$

$= \quad 3p^2 - 2p^3$

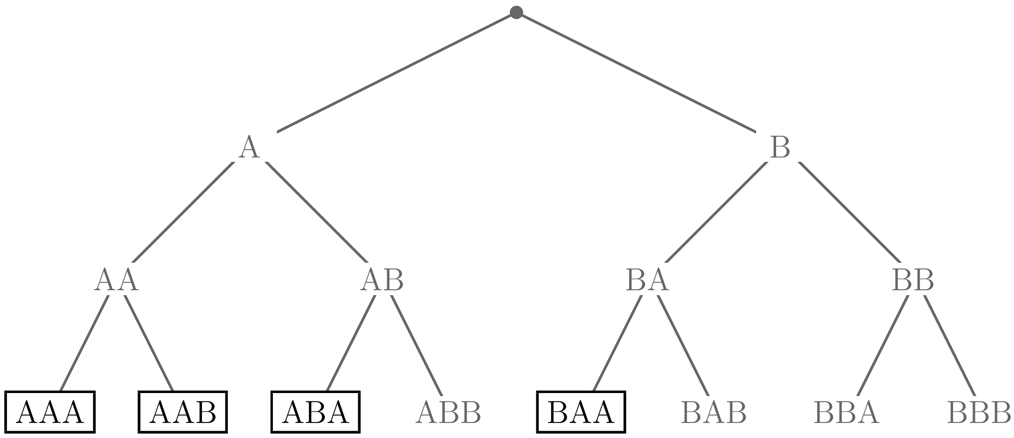
“Vertical” solution for $2n+1$ games:

$$
\begin{aligned}
&P(A \text{ wins the series}) \\
&\quad = \sum_{k=n+1}^{2n+1} P(A \text{ wins the series on the } k^{\text{th}} \text{ game}) \\
&\quad = \sum_{k=n+1}^{2n+1} P(A \text{ wins } n \text{ of the first } k-1 \text{ games}) \cdot P(A \text{ wins the } k^{\text{th}} \text{ game}) \\
&\quad = \sum_{k=n+1}^{2n+1} \binom{k-1}{n} p^n (1-p)^{k-1-n} \cdot p \\
&\quad = p^{n+1} \sum_{k=n+1}^{2n+1} \binom{k-1}{n} (1-p)^{k-(n+1)}
\end{aligned}
$$

“Horizontal” solution for $2n+1$ games:

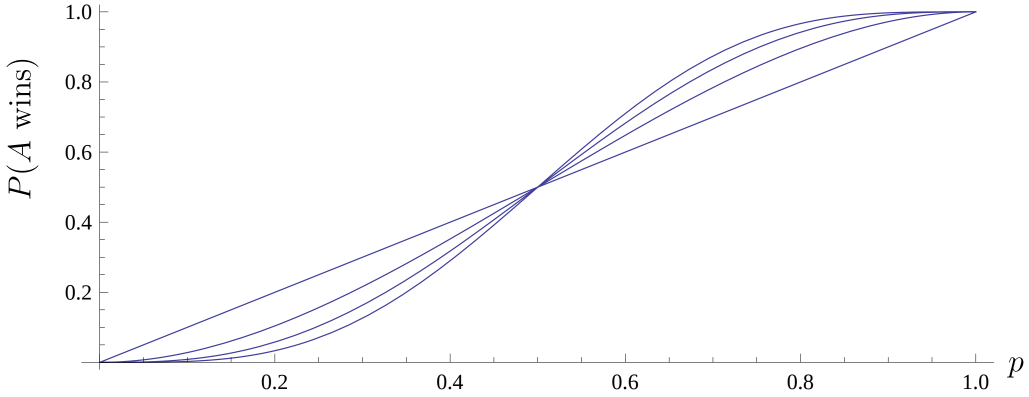
$$P(A \text{ wins the series}) = \sum_{k=n+1}^{2n+1} P(A \text{ wins } k \text{ of the } 2n+1 \text{ games})$$

$$= \sum_{k=n+1}^{2n+1} \binom{2n+1}{k} p^k (1-p)^{2n+1-k}$$



The probability that Team A wins a $(2n+1)$-game series

$p^{n+1} \sum_{k=n+1}^{2n+1} \binom{k-1}{n} (1-p)^{k-(n+1)}$		$\sum_{k=n+1}^{2n+1} \binom{2n+1}{k} p^k (1-p)^{2n+1-k}$
p	$(n=0)$	p
$3p^2 - 2p^3$	$(n=1)$	$3p^2 - 2p^3$
$10p^3 - 15p^4 + 6p^5$	$(n=2)$	$10p^3 - 15p^4 + 6p^5$
$35p^4 - 84p^5 + 70p^6 - 20p^7$	$(n=3)$	$35p^4 - 84p^5 + 70p^6 - 20p^7$



What are the coefficients? After some messy reindexing, we get

$$p^{n+1} \sum_{k=n+1}^{2n+1} \binom{k-1}{n} (1-p)^{k-(n+1)}$$

$$= \sum_{r=n+1}^{2n+1} \left[(-1)^{r-(n+1)} \sum_{k=r}^{2n+1} \binom{k-1}{n} \binom{k-(n+1)}{r-(n+1)} \right] p^r$$

$$\sum_{k=n+1}^{2n+1} \binom{2n+1}{k} p^k (1-p)^{2n+1-k}$$

$$= \sum_{r=n+1}^{2n+1} \left[\sum_{k=n+1}^{r} (-1)^{r-k} \binom{2n+1}{k} \binom{2n+1-k}{r-k} \right] p^r$$

A Pascal's triangle relation

Equating the coefficients of p^r in the "vertical" and "horizontal" polynomials, we get a two-parameter family of relations:

$$\left[(-1)^{r-(n+1)} \sum_{k=r}^{2n+1} \binom{k-1}{n} \binom{k-(n+1)}{r-(n+1)}\right]$$

$$=$$

$$\left[\sum_{k=n+1}^{r} (-1)^{r-k} \binom{2n+1}{k} \binom{2n+1-k}{r-k}\right]$$

for each n and for each r in $\{n+1, n+2, \ldots, 2n+1\}$.

$$(-1)^{r-(n+1)} \sum_{k=r}^{2n+1} \binom{k-1}{n} \binom{k-(n+1)}{r-(n+1)} = \sum_{k=n+1}^{r} (-1)^{r-k} \binom{2n+1}{k} \binom{2n+1-k}{r-k}$$

Example: $n = 3$, $r = 4$.

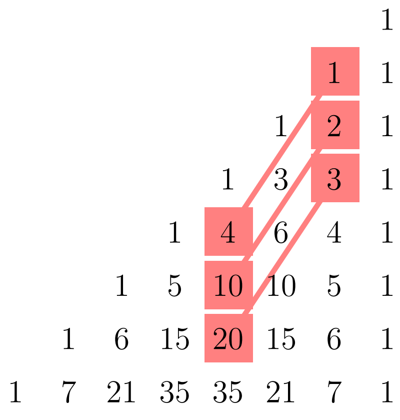
							1
						1	**1**
					1	2	**1**
				1	3	3	**1**
			1	**4**	6	4	1
		1	5	**10**	10	5	1
	1	6	15	**20**	15	6	1
1	7	21	35	35	21	7	1

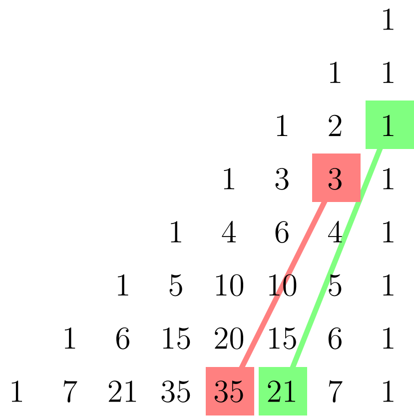
							1
						1	1
					1	2	1
				1	3	3	**1**
			1	4	6	4	1
		1	5	10	10	5	1
	1	6	15	20	15	6	1
1	7	21	35	**35**	21	7	1

$$\binom{3}{3}\binom{0}{0} + \binom{4}{3}\binom{1}{0} + \binom{5}{3}\binom{2}{0} + \binom{6}{3}\binom{3}{0} = 35 = \binom{7}{4}\binom{3}{0}$$

$$(-1)^{r-(n+1)} \sum_{k=r}^{2n+1} \binom{k-1}{n} \binom{k-(n+1)}{r-(n+1)} = \sum_{k=n+1}^{r} (-1)^{r-k} \binom{2n+1}{k} \binom{2n+1-k}{r-k}$$

Example: $n = 3$, $r = 5$.

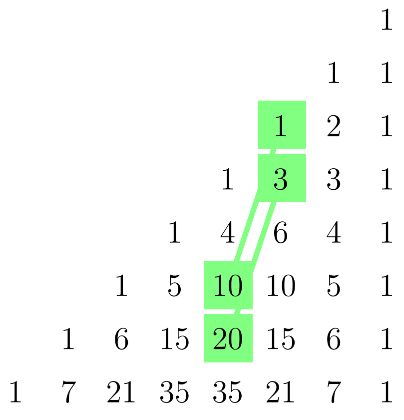


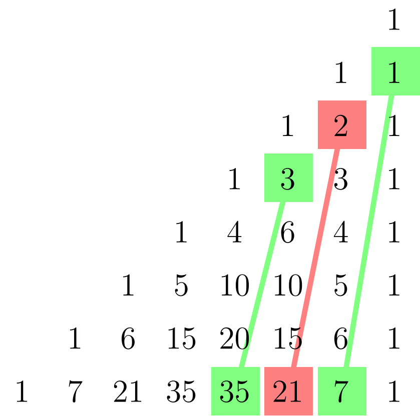


$$-\binom{4}{3}\binom{1}{1} - \binom{4}{3}\binom{2}{1} - \binom{6}{3}\binom{3}{1} \quad = \quad -84 \quad = \quad -\binom{7}{4}\binom{3}{1} + \binom{7}{5}\binom{2}{0}$$

$$(-1)^{r-(n+1)} \sum_{k=r}^{2n+1} \binom{k-1}{n} \binom{k-(n+1)}{r-(n+1)} = \sum_{k=n+1}^{r} (-1)^{r-k} \binom{2n+1}{k} \binom{2n+1-k}{r-k}$$

Example: $n = 3$, $r = 6$.

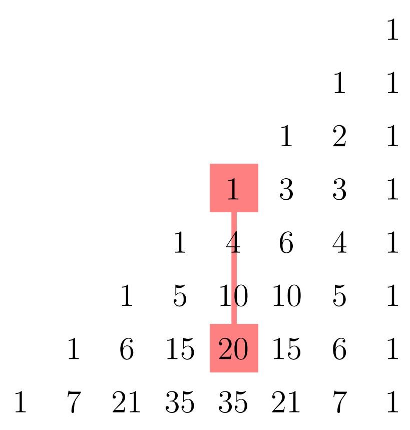


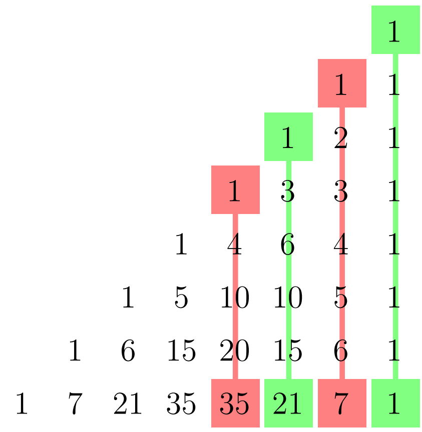


$$\binom{5}{3}\binom{2}{2} + \binom{6}{3}\binom{3}{2} = 70 = \binom{7}{4}\binom{3}{2} - \binom{7}{5}\binom{2}{1} + \binom{7}{6}\binom{1}{0}$$

$$(-1)^{r-(n+1)} \sum_{k=r}^{2n+1} \binom{k-1}{n} \binom{k-(n+1)}{r-(n+1)} = \sum_{k=n+1}^{r} (-1)^{r-k} \binom{2n+1}{k} \binom{2n+1-k}{r-k}$$

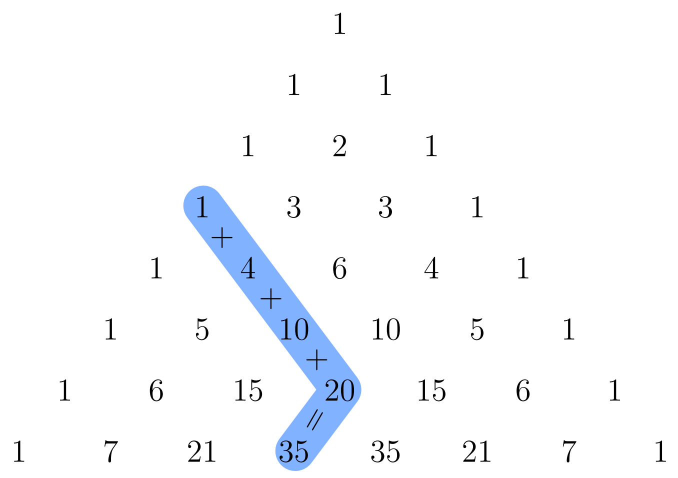
Example: $n = 3$, $r = 7$.





$$-\binom{6}{3}\binom{3}{3} = -20 = -\binom{7}{4}\binom{3}{3} + \binom{7}{5}\binom{2}{2} - \binom{7}{6}\binom{1}{1} + \binom{7}{7}\binom{0}{0}$$

Remark: In the boundary case where $r = n + 1$, we get the "hockey stick" relation.



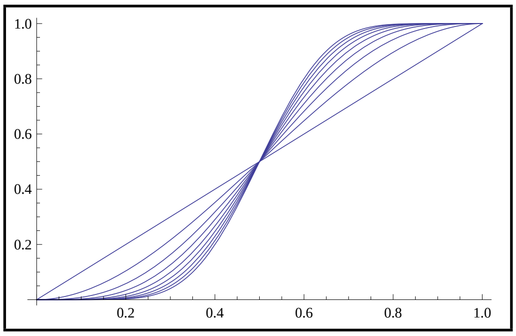
Remark: In the other boundary case where $r = 2n + 1$, we get an alternating hockey stick.

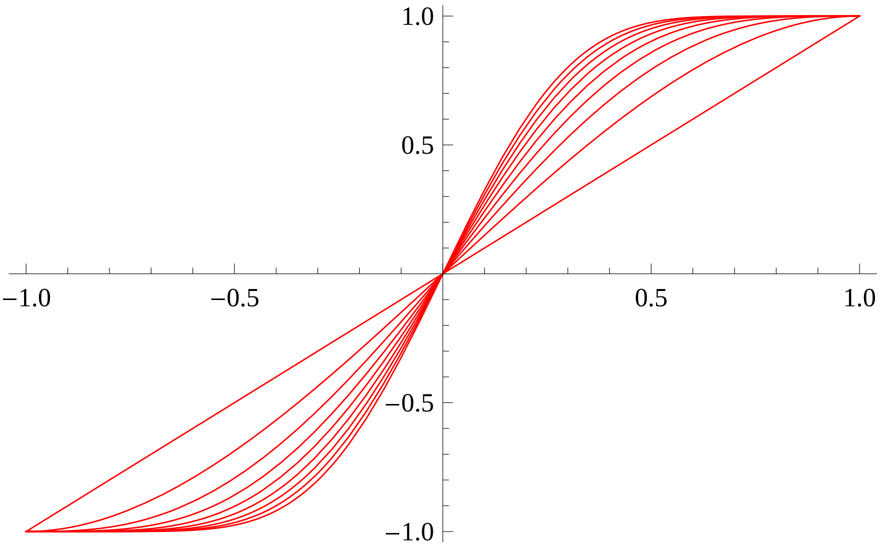
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

$$1 - 7 + 21 - 35 = 20$$

Back to the polynomials

We translate and scale the functions to get a sequence of odd polynomials on $[-1, 1]$

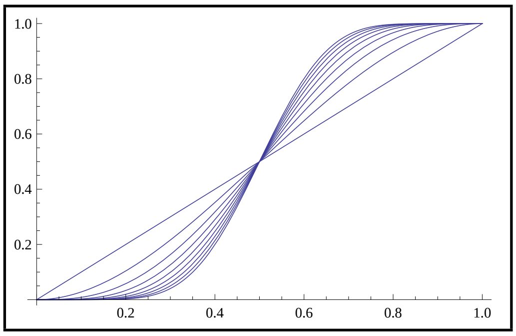


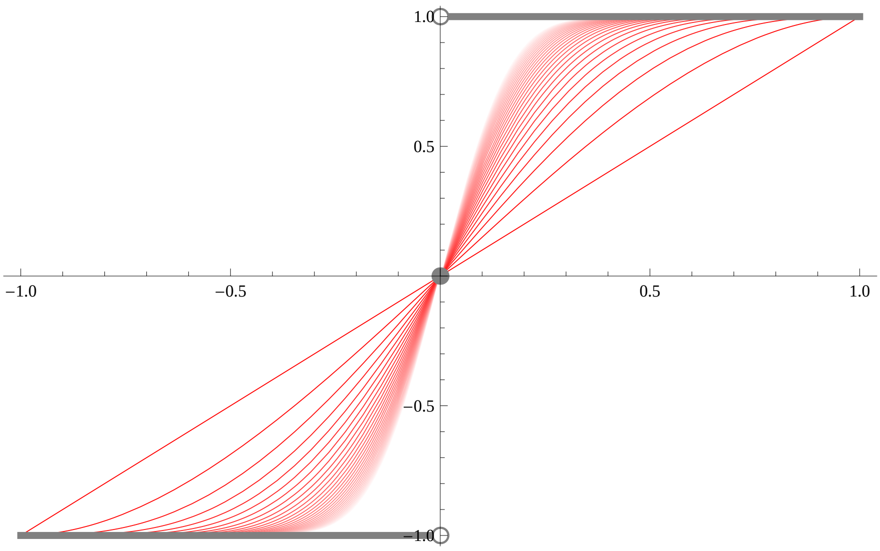


$$
\begin{aligned}
s_0(x) &= x \\
s_1(x) &= \frac{3x - x^3}{2} \\
s_2(x) &= \frac{15x - 10x^3 + 3x^5}{8} \\
&\vdots
\end{aligned}
$$

Observation

This sequence converges pointwise and monotonically to the signum function.





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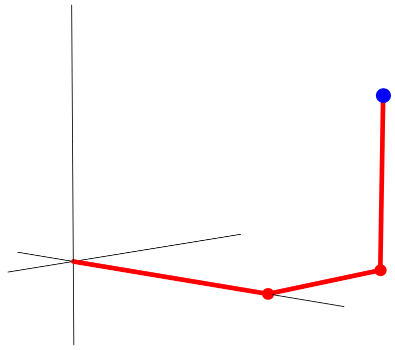
&\vdots
\end{aligned}
$$

Question

Can we make this sequence converge to $\mathrm{sgn}(x)$ in some vector space of functions on $[-1,1]$? That is, can we find a system

$$\{\varphi_0, \varphi_1, \varphi_2, \ldots\}$$

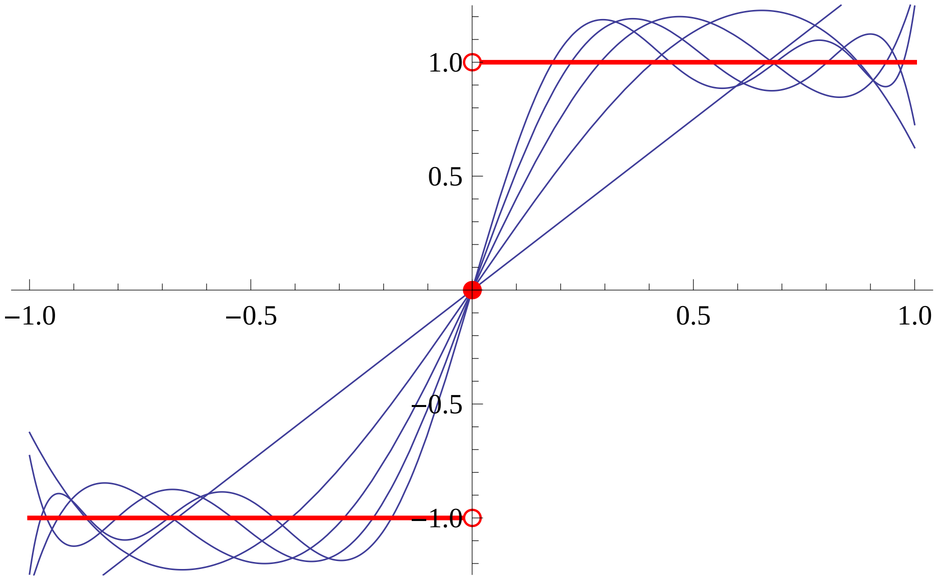
of polynomials on $[-1,1]$ so that



- each φ_k is a polynomial of degree $2k+1$
- there is an inner product $\langle \cdot\,, \cdot \rangle$ with $\langle \varphi_m, \varphi_n \rangle = \delta_{m,n}$
- $\mathrm{sgn}(x) = \displaystyle\sum_{k=0}^{\infty} \langle \mathrm{sgn}, \varphi_k \rangle \, \varphi_k(x)$
- our s_n polynomials are the partial sums of this series: $s_n(x) = \displaystyle\sum_{k=0}^{n} \langle \mathrm{sgn}, \varphi_k \rangle \, \varphi_k(x)$

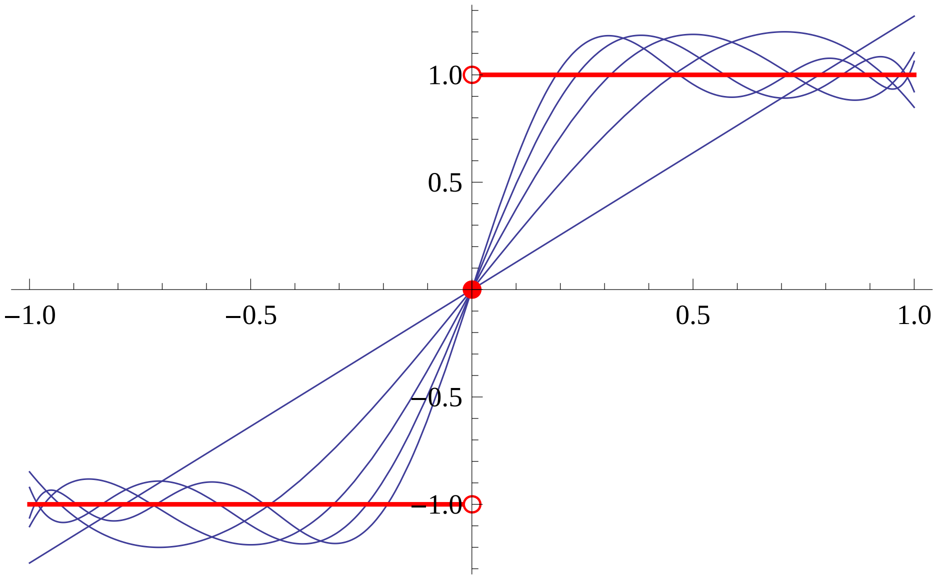
  ?

First attempt: Use the normalized Legendre polynomials.



The partial sums converge to $\operatorname{sgn}(x)$, but they are not our s_n polynomials.

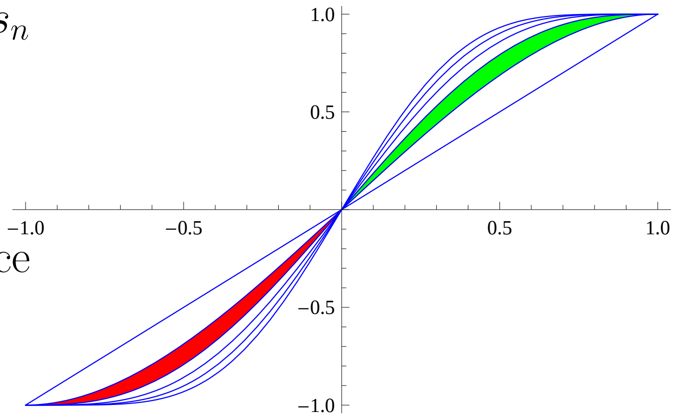
Second attempt: Use the normalized Chebyshev T polynomials.



Again, we do not get the monotonic convergence.

Inconvenient Truth: Our polynomials s_n cannot be the partial sums of a series

$$\sum \langle \mathrm{sgn}, \varphi_k \rangle \, \varphi_k .$$



Reason: In such a series, the difference between two partial sums,

$$s_{n+1} - s_n$$

is a scalar multiple of φ_{n+1}, so it must be orthogonal to s_n.

But in our sequence, $s_n(x)$ and $s_{n+1}(x) - s_n(x)$ always have the same sign, so their inner product

$$\langle s_n, s_{n+1} - s_n \rangle \; = \; \int_{-1}^{1} s_n(x)[s_{n+1}(x) - s_n(x)] w(x) \, dx$$

cannot be zero.

Question: Why does $\langle f, g \rangle$ have the form

$$\int_{-1}^{1} f(x)g(x)w(x)\,dx$$

where w is some non-negative weight function?

In $\mathbb{R}^n$, we can take $\langle \mathbf{x}, \mathbf{y} \rangle$ to be

$$\begin{bmatrix} \mathbf{x}^T \end{bmatrix} \begin{bmatrix} \\ A \\ \\ \end{bmatrix} \begin{bmatrix} \\ \mathbf{y} \\ \\ \end{bmatrix}$$

for any symmetric, positive-definite matrix A (not just a diagonal A).

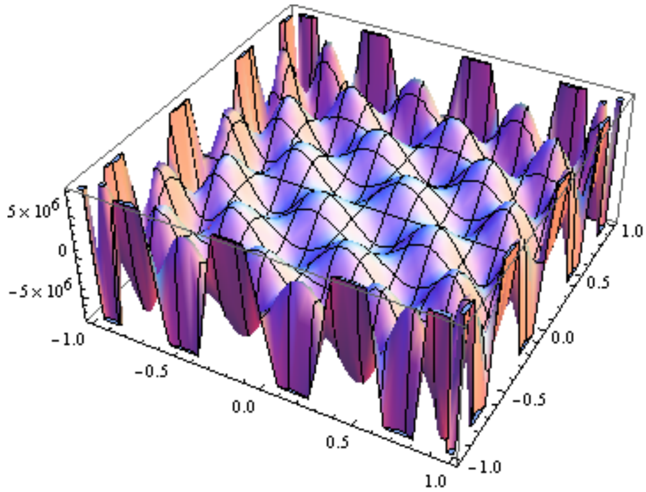
So why not say

$$\langle f, g \rangle = \int_{-1}^{1}\int_{-1}^{1} f(x)A(x,y)g(y)\,dy\,dx$$

where A is symmetric and positive definite?

Try this on

$$
\begin{aligned}
s_0(x) &= x \\
s_1(x) - s_0(x) &= \tfrac{1}{2}x(1-x^2) \\
s_2(x) - s_1(x) &= \tfrac{3}{8}x(1-x^2)^2 \\
s_3(x) - s_2(x) &= \tfrac{5}{16}x(1-x^2)^3 \\
s_4(x) - s_3(x) &= \tfrac{35}{128}x(1-x^2)^4
\end{aligned}
$$



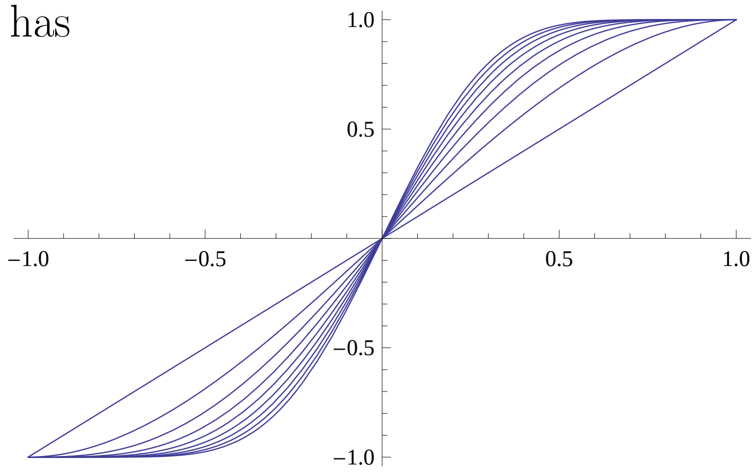
These functions are orthonormal with respect to the inner product

$$\langle f, g \rangle = \int_{-1}^{1} \int_{-1}^{1} f(x) A(x,y) g(y)\, dx\, dy,$$

where A is the function graphed above.

Better news: The polynomial s_n has the following properties:

- s_n is odd of degree $2n+1$
- $s_n(1) = 1$
- $s_n^{(r)}(1) = 0$ for $r = 1, 2, \ldots, n$



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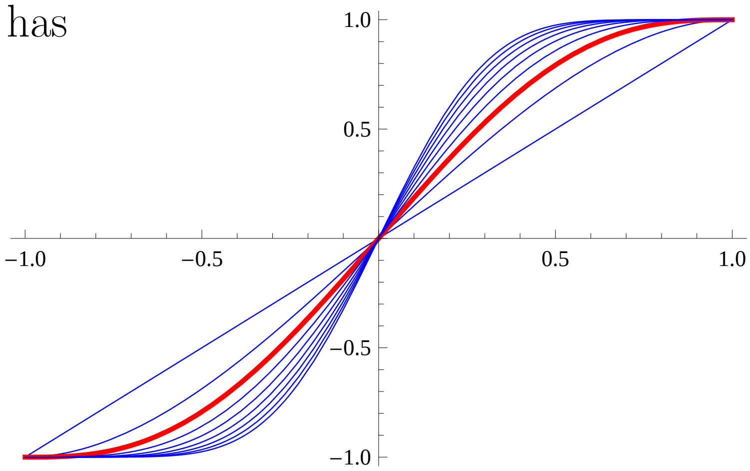

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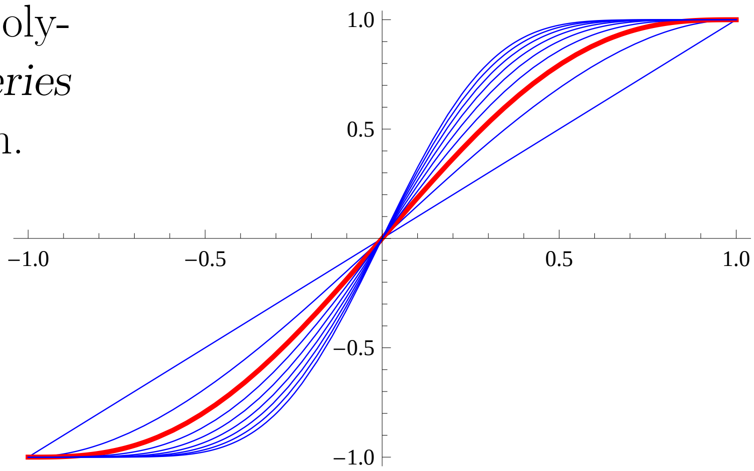
Better still, these properties *determine* s_n.

Example: Write $s_2(x) = a_1 x + a_3 x^3 + a_5 x^5$. Then

$$\left.\begin{array}{rcccl} 1 & = & s_2(1) & = & a_1 + a_3 + a_5 \\ 0 & = & s_2'(1) & = & a_1 + 3a_3 + 5a_5 \\ 0 & = & s_2''(1) & = & 2 \cdot 3a_3 + 4 \cdot 5a_5 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} a_1 & = & 15/8 \\ a_2 & = & -10/8 \\ a_3 & = & 3/8 \end{array}\right.$$

Observation: This makes our s_n polynomials look a lot like *power-series convergents* to the signum function.



Sort of:

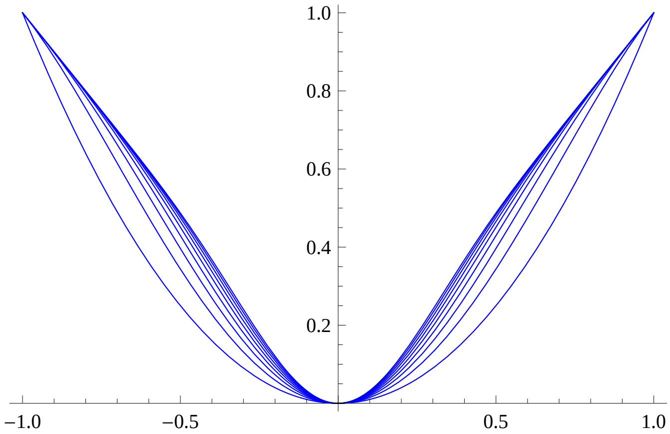
An n-coefficient polynomial approximation to a function f is determined by

- values of f at 0 and 1;
- the right number of derivatives of f at 1; and
- being an odd polynomial.

Something else to play with:

Here are some even polynomials $v_n(x)$ determined by

- $v_n(0) = 0$;
- $v_n(1) = 1$;
- $v_n'(1) = 1$ for $n > 1$;
- $v_n^{(r)} = 0$ for $n > 1$ and $r \geq 2$.



$$
\begin{aligned}
v_1(x) &= x^2 \\
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v_3(x) &= \tfrac{1}{8}(15x^2 - 10x^4 + 3x^6) \\
v_4(x) &= \tfrac{1}{32}(70x^2 - 70x^4 + 42x^6 - 10x^8) \\
&\vdots
\end{aligned}
$$

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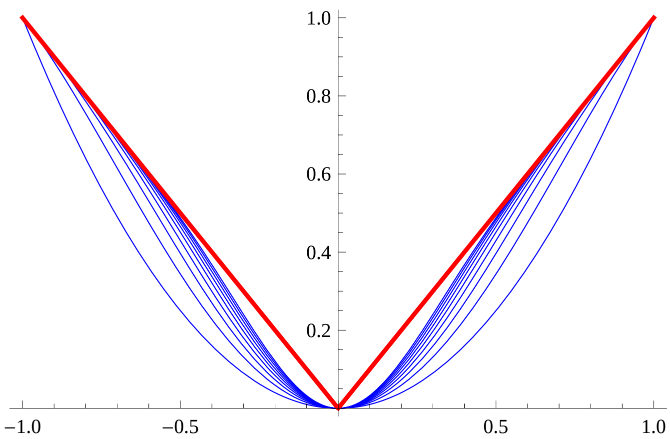
And here is an even function with all those same properties.



$$\begin{aligned} v_1(x) &= x^2 \\ v_2(x) &= \tfrac{1}{2}(3x^2 - x^4) \\ v_3(x) &= \tfrac{1}{8}(15x^2 - 10x^4 + 3x^6) \\ v_4(x) &= \tfrac{1}{32}(70x^2 - 70x^4 + 42x^6 - 10x^8) \\ &\vdots \end{aligned}$$