Honors 1: Undergraduate Math Lab*

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Abstract

Note on conditions arising in non-trivial bound on sum of fourth powers of Kloosterman sums (Heath-Brown review).

Please refer to Sarnak lecture of 10/16/02, and Heath-Brown's review article. There are six summations inherent in the desired sum

$$\sum_{a=0}^{p-1}\sum_{b=0}^{p-1} |\mathrm{Kl}(a,b,p)|^4 , \tag{1}$$

namely the two sums shown and one internal sum in each of the Kl's. Using the result

$$\sum_{a=0}^{p-1} e\left(\frac{an}{p}\right) = \begin{cases} p-1, & n=0 \pmod{p} \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

twice, turns the two $e(\cdot)$ functions into counting conditions on the variables internal to the four Kl's. Calling these variables w, x, y, z, we want the total number of ways that, with $1 \le w, x, y, z \le p-1$ (the four surviving sums), we can obey both

$$w + x - y - z = 0 \ \ (p) \tag{3}$$

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and

$$\bar{w} + \bar{x} - \bar{y} - \bar{z} = 0 \ \ (p). \tag{4}$$

Multiplying Eq. 4 by $wxyz$ gives

$$(w + x)yz - (y + z)wx \ = \ 0 \ \ (p). \tag{5}$$

Substituting Eq. 3 gives

$$(w + x)(yz - wx) \ = \ 0 \ \ (p). \tag{6}$$

So either $w + x = 0 \ \ (p)$ or $yz - wx = 0 \ \ (p)$, or both.

The first set of cases has $x = -w$ from (3) giving $z = -y$, so there are $(p-1)^2$ choices of w and y. For each choice x and z are fixed uniquely. Therefore these cases contribute $(p-1)^2$ ways.

For the second set, we have two equations

$$\begin{aligned} y + z \ &= \ w + x \ \ (p) && (7) \\ yz \ &= \ wx \ \ (p) && (8) \end{aligned}$$

for two unknowns y, z, for any of the arbitrary choices of w, x. You could combine these equations into the single quadratic

$$y^2 - y(w + x) + wx \ = 0 \ \ (p). \tag{9}$$

Two solutions for y are $y = w$ and $y = x$ (check by substitution). Since it is a quadratic, these are the only two solutions. Therefore the number of ways contributed is at most 2 times the $(p-1)^2$ ways of choosing w, x.

Overcounting due to the and/or is at least of order p or smaller, but also can only reduce the number of ways. Therefore the total number of ways $\leq 3(p-1)^2$, which is $O(p^2)$. From this follows the bound on the sum given in the article and lecture.