Abstract

1. We prove Liouville's Theorem for the order of approximation by rationals of real algebraic numbers.

2. We construct several transcendental numbers.

3. We define Poissonian Behaviour, and study the spacings between the ordered fractional parts of $\{n^k\alpha\}$.

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Contents

Chapter 1

Liouville's Theorem Constructing Transcendentals

1.1 Review of Approximating by Rationals

Definition 1.1.1 (Approximated by rationals to order n). *A real number x is approximated by rationals to order n if there exist a constant $k(x)$ (possibly depending on x) such that there are infinitely many rational $\frac{p}{q}$ with*

$$\left|x - \frac{p}{q}\right| < \frac{k(x)}{q^n}. \tag{1.1}$$

Recall that Dirichlet's Box Principle gaves us:

$$\left|x - \frac{p}{q}\right| < \frac{1}{q^2} \tag{1.2}$$

for infintely many fractions $\frac{p}{q}$. This was proved by choosing a large parameter Q, and considering the $Q + 1$ fractionary parts $\{qx\} \in [0, 1)$ for $q \in \{0, \ldots, Q\}$. The box principle ensures us that there must be two different q's, say:

$$0 \leq q_1 < q_2 \leq Q \tag{1.3}$$

such that both $\{q_1 x\}$ and $\{q_2 x\}$ belong to the same interval $[\frac{a}{Q}, \frac{a+1}{Q})$, for some $0 \leq a \leq Q - 1$. Note that there are exactly Q such intervals partitioning $[0, 1)$, and $Q + 1$ fractionary parts! Now, the length of such an interval is $\frac{1}{Q}$ so we get

$$|\{q_2x\} - \{q_1x\}| < \frac{1}{Q}. \tag{1.4}$$

There exist integers p_1 and p_2 such that

$$\{q_1x\} = q_1x - p,\ \{q_2x\} = q_2x - p. \tag{1.5}$$

Letting $p = p_2 - p_1$ we find

$$|(q_2 - q_1)x - p| \leq \frac{1}{Q} \tag{1.6}$$

Let $q = q_2 - q_1$, so $1 \leq q \leq Q$, and the previous equation can be rewriten as

$$\left|x - \frac{p}{q}\right| < \frac{1}{qQ} \leq \frac{1}{q^2} \tag{1.7}$$

Now, letting $Q \to \infty$, we get an infinite collection of rational fractions $\frac{p}{q}$ satisfying the above equation. If this collection contains only finitely many distinct fractions, then one of these fractions, say $\frac{p_0}{q_0}$, would occur for infintely many choices Q_k of Q, thus giving us:

$$\left|x - \frac{p_0}{q_0}\right| < \frac{1}{qQ_k} \to 0, \tag{1.8}$$

as $k \to \infty$. This implies that $x = \frac{p_0}{q_0} \in \mathbb{Q}$. So, unles x is a rational number, we can find infinitely many *distinct* rational numbers $\frac{p}{q}$ satisfying Equation 1.7. This means that any real, irrational number can be approximated to order $n = 2$ by rational numbers.

1.2 Liouville's Theorem

Theorem 1.2.1 (Liouville's Theorem). *Let x be a real algebraic number of degree n. Then x is approximated by rationals to order at most n.*

Proof. Let

$$f(X) \ = \ a_nX^n + \cdots a_1X + a_0 \tag{1.9}$$

be the polynomial with integer coefficients of smallest degree (minimal polynomial) such that x satisfies

$$f(x) = 0. \tag{1.10}$$

Note that $\deg x = \deg f$ and the condition of minimality implies that $f(X)$ is irreducible over $\mathbb{Z}$. Further, a well known result from algebra states that a polynomial irreducible over $\mathbb{Z}$ is also irreducible over $\mathbb{Q}$.

In particular, as $f(X)$ is irreducible over $\mathbb{Q}$, $f(X)$ does not have any rational roots. If it did, then $f(X)$ would be divisible by a linear polynomial $(X - \frac{a}{b})$. Let $G(X) = \frac{f(X)}{X - \frac{a}{b}}$. Clear denominators (multiply throughout by b), and let $g(X) = bG(X)$. Then $\deg g = \deg f - 1$, and $g(x) = 0$. This contradicts the minimality of f (we choose f to be a polynomial of smallest degree such that $f(x) = 0$). Therefore, f is non-zero at every rational.

Let

$$M = \sup_{|z-x|<1} |f'(z)|. \tag{1.11}$$

Let now $\frac{p}{q}$ be a rational such that $\left|x - \frac{p}{q}\right| < 1$. The Mean Value Theorem gives us that

$$\left|f\left(\frac{p}{q}\right) - f(x)\right| = \left|f'(c)\left(x - \frac{p}{q}\right)\right| \leq M\left|x - \frac{p}{q}\right| \tag{1.12}$$

where c is some real number between x and $\frac{p}{q}$; $|c - x| < 1$ for $\frac{p}{q}$ moderately close to x.

Now we use the fact that $f(X)$ does not have any rational roots:

$$0 \neq f\left(\frac{p}{q}\right) = a_n\left(\frac{p}{q}\right)^n + \cdots + a_0 = \frac{a_n p^n + \cdots a_1 p^{n-1} q + a_0 q^n}{q^n} \tag{1.13}$$

The numerator of the last term is a nonzero integer, hence it has absolute value at least 1. Since we also know that $f(x) = 0$ it follows that

$$\left|f\left(\frac{p}{q}\right) - f(x)\right| = \left|f\left(\frac{p}{q}\right)\right| = \frac{|a_n p^n + \cdots a_1 p^{n-1} q + a_0 q^n|}{q^n} \geq \frac{1}{q^n}. \tag{1.14}$$

Combining the equations 1.12 and 1.14, we get:

$$\frac{1}{q^n} \leq M\left|x - \frac{p}{q}\right| \Rightarrow \frac{1}{Mq^n} \leq \left|x - \frac{p}{q}\right| \tag{1.15}$$

whenever $|x - \frac{p}{q}| < 1$. This last equation shows us that x can be approximated by rationals to order at most n. For assume it was otherwise, namely that x can be approximated to order $n + \epsilon$. Then we would have an infinite sequence of distinct rational numbers $\{\frac{p_i}{q_i}\}_{i \geq 1}$ and a constant $k(x)$ depending only on x such that

$$\left| x - \frac{p_i}{q_i} \right| \; < \; \frac{k(x)}{q_i^{n+\epsilon}}. \tag{1.16}$$

Since the numbers $\frac{p_i}{q_i}$ converge to x we can assume that they already are in the interval $(x-1, x+1)$. Hence they also satisfy Equation 1.15:

$$\frac{1}{q_i^n} \leq M \left| x - \frac{p_i}{q_i} \right|. \tag{1.17}$$

Combining the last two equations we get

$$\frac{1}{Mq_i^n} \leq \left| x - \frac{p_i}{q_i} \right| \; < \; \frac{k(x)}{q_i^{n+\epsilon}}, \tag{1.18}$$

hence

$$q_i^{\epsilon} \; < \; M \tag{1.19}$$

and this is clearly impossible for arbitrarily large q since $\epsilon > 0$ and $q_i \to \infty$.

□

Exercise 1.2.2. *Justify the fact that if* $\{\frac{p_i}{q_i}\}_{i \geq 1}$ *is a rational approximation to order* $n \geq 1$ *of* x*, then* $q_i \to \infty$.

Remark 1.2.3. *So far we have seen that the order to which an algebraic number can be approximated by rationals is bounded by its degree. Hence if a real, irrational number* $\alpha \notin \mathbb{Q}$ *can be approximated by rationals to an arbitrary large order, then* α *must be transcendental! This provides us with a recipe for constructing transcendental numbers.*

1.3 Constructing Transcendental Numbers

1.3.1 $\sum_m 10^{-m!}$

The following construction of transcendental numbers is due to Liouville.

Theorem 1.3.1. *The number*

$$x = \sum_{m=1}^{\infty} \frac{1}{10^{m!}} \tag{1.20}$$

is transcendental.

Proof. The series defining x is convergent, since it is dominated by the geometric series $\sum \frac{1}{10^m}$. In fact, the series converges very rapidly and it is this high rate of convergence that will yield x is transcendental.

Fix N large, and let $n > N$. Write

$$\frac{p_n}{q_n} = \sum_{m=1}^{n} \frac{1}{10^{m!}} \tag{1.21}$$

with $p_n, q_n > 0$ and $(p_n, q_n) = 1$. Then $\{\frac{p_n}{q_n}\}_{n\geq 1}$ is a monotone increasing sequence converging to x. In particular, all these rational numbers are distinct. Not also that q_n must divide $10^{n!}$, which implies

$$q_n \leq 10^{n!}. \tag{1.22}$$

Using this, we get

$$\begin{aligned} 0 < x - \frac{p_n}{q_n} &= \sum_{m>n} \frac{1}{10^{m!}} = \frac{1}{10^{(n+1)!}} \left(1 + \frac{1}{10^{n+2}} + \frac{1}{10^{(n+2)(n+3)}} + \cdots\right) \\ &< \frac{2}{10^{(n+1)!}} = \frac{2}{(10^{n!})^{n+1}} \\ &< \frac{2}{q_n^{n+1}} \leq \frac{2}{q_n^N}. \end{aligned} \tag{1.23}$$

This gives an approximation by rationals of order N of x. Since N can be chosen arbitrarily large, this implies that x can be approximated by rationals to arbitrary order. We can conclude, in view of our precious remark 1.2.3 that x is transcendental. □

1.3.2 $[10^{1!}, 10^{2!}, \ldots]$

Theorem 1.3.2. *The number*

$$y = [10^{1!}, 10^{2!}, \dots] \tag{1.24}$$

is transcendental.

Proof. Let $\frac{p_n}{q_n}$ be the continued fraction of $[10^{1!} \cdots 10^{n!}]$. Then

$$\begin{aligned} \left| y - \frac{p_n}{q_n} \right| &= \frac{1}{q_n q'_{n+1}} = \frac{1}{q_n(a'_{n+1} q_n + q_{n-1})} \\ &< \frac{1}{a_{n+1}} = \frac{1}{10^{(n+1)!}}. \end{aligned} \tag{1.25}$$

Since $q_k = a_n q_{k-1} + q_{n-2}$, it implies that $q_k > q_{k-1}$ Also, $q_{k+1} = a_{k+1} q_n + q_{k-1}$, so we get

$$\frac{q_{k+1}}{q_k} = a_{k+1} + \frac{q_{k-1}}{q_k} < a_{k+1} + 1. \tag{1.26}$$

Hence writing this inequality for $k = 1, \cdots, n-1$ we obtain

$$\begin{aligned} q_n = q_1 \frac{q_2}{q_1} \frac{q_3}{q_2} \cdots \frac{q_n}{q_{n-1}} &< (a_1 + 1)(a_2 + 1) \cdots (a_n + 1) \\ &= (1 + \frac{1}{a_1}) \cdots (1 + \frac{1}{a_n}) a_1 \cdots a_n \\ &< 2^n a_1 \cdots a_n = 2^n 10^{1! + \cdots + n!} \\ &< 10^{2n!} = a_n^2 \end{aligned} \tag{1.27}$$

Combining equations 1.25 and 1.27 we get:

$$\begin{aligned} \left| y - \frac{p_n}{q_n} \right| &< \frac{1}{a_{n+1}} = \frac{1}{a_n^{n+1}} \\ &< \left(\frac{1}{a_n^2} \right)^{\frac{n}{2}} < \left(\frac{1}{q_n^2} \right)^{\frac{n}{2}} \\ &= \frac{1}{q_n^{n/2}}. \end{aligned} \tag{1.28}$$

In this way we get, just as in the previous theorem, an approximation of y by rationals to arbitrary order. This proves that y is transcendental.

□

1.3.3 Buffon's Needle and π

Consider a collection of infinitely long parallel lines in the plane, where the spacing between any two adjacent lines is d. Let the lines be located at $x = 0, \pm d, \pm 2d, \dots$. Consider a rod of length l, where for convenience we assume $l < d$.

If we were to *randomly* throw the rod on the plane, what is the probability it hits a line? This question was first asked by Buffon in 1733.

Because of the vertical symmetry, we may assume the center of the rod lies on the line $x = 0$, as shifting the rod (without rotating it) up or down will not alter the number of intersections. By the horizontal symmetry, we may assume $-\frac{d}{2} \le x < \frac{d}{2}$. We posit that all values of x are equally likely. As x is continuous distributed, we may add in $x = \frac{d}{2}$ without changing the probability. The probability density function of x is $\frac{dx}{d}$.

Let θ be the angle the rod makes with the x-axis. As each angle is equally likely, the probability density function of θ is $\frac{d\theta}{2\pi}$.

We assume that x and θ are chosen independently. Thus, the probability density for (x, θ) is $\frac{dxd\theta}{d \cdot 2\pi}$.

The projection of the rod (making an angle of θ with the x-axis) along the x-axis is $l \cdot |\cos\theta|$. If $|x| \le l \cdot |\cos\theta|$, then the rod hits exactly one vertical line exactly once; if $x > l \cdot |\cos\theta|$, the rod does not hit a vertical line. Note that if $l > d$, a rod could hit multiple lines, making the arguments more involved.

Thus, the probability a rod hits a line is

$$\begin{aligned} p &= \int_{\theta=0}^{2\pi} \int_{x=-l\cdot|\cos\theta|}^{l\cdot|\cos\theta|} \frac{dxd\theta}{d \cdot 2\pi} \\ &= \int_{\theta=0}^{2\pi} \frac{l \cdot |\cos\theta|}{d} \frac{d\theta}{2\pi} \\ &= \frac{2l}{\pi d}. \end{aligned} \tag{1.29}$$

Exercise 1.3.3. *Show*

$$\frac{1}{2\pi} \int_0^{2\pi} |\cos\theta| d\theta = \frac{2}{\pi}. \tag{1.30}$$

Let A be the random variable which is the number of intersections of a rod of length l thrown against parallel vertical lines separated by $d > l$ units. Then

$$A = \begin{cases} 1 & \text{with probability } \frac{2l}{\pi d} \\ 0 & \text{with probability } 1 - \frac{2l}{\pi d} \end{cases}. \tag{1.31}$$

If we were to throw N rods independently, since the expected value of a sum is the sum of the expected values (Lemma **??**), we expect to observe

$$N \cdot \frac{2l}{\pi d} \tag{1.32}$$

intersections.

Turning this around, let us throw N rods, and let I be the number of observed intersections of the rods with the vertical lines. Then

$$I \;\approx\; N \cdot \frac{2l}{\pi d} \quad \rightarrow \quad \pi \;\approx\; \frac{N}{I} \cdot \frac{2l}{d}. \tag{1.33}$$

The above is an *experimental* formula for π!

Chapter 2

Poissonian Behavior and $\{n^k\alpha\}$

2.1 Equidistribution

We say a sequence of number $x_n \in [0, 1)$ is equidistributed if

$$\lim_{N\to\infty} \frac{\#\{n : 1 \le n \le N \text{ and } x_n \in [a, b]\}}{N} = b - a \tag{2.1}$$

for any subinterval $[a, b]$ of $[0, 1]$.

Recall Weyl's Result, Theorem **??**: If $\alpha \notin \mathbb{Q}$, then the fractional parts $\{n\alpha\}$ are equidistributed. Equivalently, $n\alpha \bmod 1$ is equidistributed.

Similarly, one can show that for any integer k, $\{n^k\alpha\}$ is equidistributed. See Robert Lipshitz's paper for more details.

2.2 Point Masses and Induced Probability Measures

Recall from physics the concept of a unit point mass located at $x = a$. Such a point mass has no length (or, in higher dimensions, width or height), but finite mass. As mass is the integral of the density over space, a finite mass in zero volume (or zero length on the line) implies an infinite density.

We can make this more precise by the notion of an Approximation to the Identity. See also Theorem **??**.

Definition 2.2.1 (Approximation to the Identity). *A sequence of functions $g_n(x)$ is an approximation to the identity (at the origin) if*

1. $g_n(x) \ge 0$.

2. $\int g_n(x)dx = 1$.

3. *Given* $\epsilon, \delta > 0$ *there exists* $N > 0$ *such that for all* $n > N$, $\int_{|x|>\delta} g_n(x)dx < \epsilon$.

We represent the limit of any such family of $g_n(x)$*s by* $\delta(x)$.

If $f(x)$ is a nice function (say near the origin its Taylor Series converges) then

$$\int f(x)\delta(x)dx \;=\; \lim_{n\to\infty}\int f(x)g_n(x) \;=\; f(0). \tag{2.2}$$

Exercise 2.2.2. *Prove Equation 2.2.*

Thus, in the limit the functions g_n are acting like point masses. We can consider the probability densities $g_n(x)dx$ and $\delta(x)dx$. For $g_n(x)dx$, as $n \to \infty$, almost all the probability is concentrated in a narrower and narrower band about the origin; $\delta(x)dx$ is the limit with all the mass at one point. It is a discrete (as opposed to continuous) probability measure.

Note that $\delta(x-a)$ acts like a point mass; however, instead of having its mass concentrated at the origin, it is now concentrated at a.

Exercise 2.2.3. *Let*

$$g_n(x) \;=\; \begin{cases} n & \textit{if } |x| \leq \frac{1}{2n} \\ 0 & \textit{otherwise} \end{cases} \tag{2.3}$$

Prove $g_n(x)$ *is an approximation to the identity at the origin.*

Exercise 2.2.4. *Let*

$$g_n(x) \;=\; c\frac{\frac{1}{n}}{\frac{1}{n^2}+x^2}. \tag{2.4}$$

Find c *such that the above is an approximation to the identity at the origin.*

Given N point masses located at x_1, $x_2, \dots, x_N$, we can form a probability measure

$$\mu_N(x)dx = \frac{1}{N}\sum_{n=1}^{N}\delta(x-x_n)dx. \tag{2.5}$$

Note $\int \mu_N(x)dx = 1$, and if $f(x)$ is a nice function,

$$\int f(x)\mu_N(x)dx \;=\; \frac{1}{N}\sum_{n=1}^{N} f(x_n). \tag{2.6}$$

Exercise 2.2.5. *Prove Equation 2.6 for nice $f(x)$.*

Note the right hand side of Equation 2.6 looks like a Riemann sum. Or it *would* look like a Riemann sum if the x_ns were equidistributed. In general the x_ns will not be equidistributed, but assume for any interval $[a,b]$ that as $N \to \infty$, the fraction of x_ns ($1 \leq n \leq N$) in $[a,b]$ goes to $\int_a^b p(x)dx$ for some nice function $p(x)$:

$$\lim_{N\to\infty} \frac{\#\{n : 1 \leq n \leq N \text{ and } x_n \in [a,b]\}}{N} \to \int_a^b p(x)dx. \tag{2.7}$$

In this case, if $f(x)$ is nice (say twice differentiable, with first derivative uniformly bounded), then

$$\begin{aligned}\int f(x)\mu_N(x)dx &= \frac{1}{N}\sum_{n=1}^{N} f(x_n) \\ &\approx \sum_{k=-\infty}^{\infty} f\left(\frac{k}{N}\right)\frac{\#\{n : 1 \leq n \leq N \text{ and } x_n \in \left[\frac{k}{N}, \frac{k+1}{N}\right]\}}{N} \\ &\to \int f(x)p(x)dx. \end{aligned} \tag{2.8}$$

Definition 2.2.6 (Convergence to $p(x)$). *If the sequence of points x_n satisfies Equation 2.7 for some nice function $p(x)$, we say the probability measures $\mu_N(x)dx$ converge to $p(x)dx$.*

2.3 Neighbor Spacings

We now consider finer questions. Let α_n be a collection of points in $[0,1)$. We order them by size:

$$0 \leq \alpha_{\sigma(1)} \leq \alpha_{\sigma(2)} \leq \cdots \leq \alpha_{\sigma(N)}, \tag{2.9}$$

where σ is a permutation of $123\cdots N$. Note the ordering depends crucially on N. Let $\beta_j = \alpha_{\sigma(j)}$.

We consider how the differences $\beta_{j+1} - \beta_j$ are distributed. We will use a slightly different definition of distance, however.

Recall $[0,1)$ is equivalent to the unit circle under the map $x \to e^{2\pi i x}$. Thus, the numbers $.999$ and $.001$ are actually very close; however, if we used the standard definition of distance, then $|.999 - .001| = .998$, which is quite large. Wrapping $[0,1)$ on itself (identifying 0 and 1), we see that $.999$ and $.001$ are separated by $.002$.

Definition 2.3.1 (mod 1 distance). *Let $x, y \in [0,1)$. We define the mod 1 distance from x to y, $||x-y||$, by*

$$||x-y|| \;=\; \min\Big\{|x-y|,\; 1-|x-y|\Big\}. \tag{2.10}$$

Exercise 2.3.2. *Show that the mod 1 distance between any two numbers in $[0,1)$ is at most $\frac{1}{2}$.*

In looking at spacings between the β_js, we have $N-1$ pairs of neighbors:

$$(\beta_2, \beta_1),\ (\beta_3, \beta_2),\ \dots,\ (\beta_N, \beta_{N-1}). \tag{2.11}$$

These pairs give rise to spacings $\beta_{j+1} - \beta_j \in [0,1)$.

We can also consider the pair (β_1, β_N). This gives rise to the spacing $\beta_1 - \beta_N \in [-1,0)$; however, as we are studying this sequence mod 1, this is equivalent to $\beta_1 - \beta_N + 1 \in [0,1)$.

Henceforth, whenever we perform any arithmetic operation, we always mean mod 1; thus, our answers always live in $[0,1)$

Definition 2.3.3 (Neighbor Spacings). *Given a sequence of numbers α_n in $[0,1)$, fix an N and arrange the numbers α_n ($n \le N$) in increasing order. Label the new sequence β_j; note the ordering will depend on N. Let $\beta_{-j} = \beta_{N-j}$ and $\beta_{N+j} = \beta_j$.*

1. *The nearest neighbor spacings are the numbers $\beta_{j+1} - \beta_j$, $j = 1$ to N.*

2. *The k^{th}-neighbor spacings are the numbers $\beta_{j+k} - \beta_j$, $j = 1$ to N.*

Remember to take the differences $\beta_{j+k} - \beta_j$ mod 1.

Exercise 2.3.4. *Let $\alpha = \sqrt{2}$, and let $\alpha_n = \{n\alpha\}$ or $\{n^2\alpha\}$. Calculate the nearest neighbor and the next-nearest neighbor spacings in each case for $N = 10$.*

Definition 2.3.5 (wrapped unit interval). *We call $[0,1)$, when all arithmetic operations are done mod 1, the wrapped unit interval.*

2.4 Poissonian Behavior

Let $\alpha \notin \mathbb{Q}$. Fix a positive integer k, and let $\alpha_n = \{n^k\alpha\}$. As $N \to \infty$, look at the ordered α_ns, denoted by β_n. How are the nearest neighbor spacings of β_n distributed? How does this depend on k? On α? On N?

Before discussing this problem, we consider a simpler case. Fix N, and consider N independent random variables x_n. Each random variable is chosen from the uniform distribution on $[0, 1)$; thus, the probability that $x_n \in [a, b)$ is $b - a$.

Let y_n be the x_ns arranged in increasing order. How do the neighbor spacings behave?

First, we need to decide what is the correct scale to use for our investigations. As we have N objects on the wrapped unit interval, we have N nearest neighbor spacings. Thus, we expect the average spacing to be $\frac{1}{N}$.

Definition 2.4.1 (Unfolding). *Let $z_n = Ny_n$. The numbers $z_n = Ny_n$ have unit mean spacing. Thus, while we expect the average spacing between adjacent y_ns to be $\frac{1}{N}$ units, we expect the average spacing between adjacent z_ns to be 1 unit.*

So, the probability of observing a spacing as large as $\frac{1}{2}$ between adjacent y_ns becomes negligible as $N \to \infty$. What we should ask is what is the probability of observing a nearest neighbor spacing of adjacent y_ns that is *half* the average spacing. In terms of the z_ns, this will correspond to a spacing between adjacent z_ns of $\frac{1}{2}$ a unit.

2.4.1 Nearest Neighbor Spacings

By symmetry, on the wrapped unit interval the expected nearest neighbor spacing is independent of j. Explicitly, we expect $\beta_{j+1} - \beta_j$ to have the same distribution as $\beta_{i+1} - \beta_i$.

What is the probability that, when we order the x_ns in increasing order, the next x_n after x_1 is located between $\frac{t}{N}$ and $\frac{t+\Delta t}{N}$? Let the x_ns in increasing order be labeled $y_1 \leq y_2 \leq \cdots \leq y_N$, $y_n = x_{\sigma(n)}$.

As we are choosing the x_ns independently, there are $\binom{N-1}{1}$ choices of subscript n such that x_n is nearest to x_1. This can also be seen by symmetry, as each x_n is equally likely to be the first to the *right* of x_1 (where, of course, $.001$ is just a little to the right of $.999$), and we have $N - 1$ choices left for x_n.

The probability that $x_n \in \left[\frac{t}{N}, \frac{t+\Delta t}{N}\right]$ is $\frac{\Delta t}{N}$.

For the remaining $N-2$ of the x_ns, each must be further than $\frac{t+\Delta t}{N}$ from x_n. Thus, they must *all* lie in an interval (or possibly two intervals if we wrap around) of length $1-\frac{t+\Delta t}{N}$. The probability that they all lie in this region is $\left(1-\frac{t+\Delta t}{N}\right)^{N-2}$.

Thus, if $x_1 = y_l$, we want to calculate the probability that $||y_{l+1} - y_l|| \in \left[\frac{t}{N}, \frac{t+\Delta t}{N}\right]$. This is

$$\begin{aligned} \text{Prob}\left(||y_{l+1} - y_l|| \in \left[\frac{t}{N}, \frac{t+\Delta t}{N}\right]\right) &= \binom{N-1}{1} \cdot \frac{\Delta t}{N} \cdot \left(1 - \frac{t+\Delta t}{N}\right)^{N-2} \\ &= \left(1-\frac{1}{N}\right) \cdot \left(1 - \frac{t+\Delta t}{N}\right)^{N-2} \Delta t. \end{aligned} \tag{2.12}$$

For N enormous and Δt small,

$$\begin{aligned} \left(1-\frac{1}{N}\right) &\approx 1 \\ \left(1 - \frac{t+\Delta t}{N}\right)^{N-2} &\approx e^{-(t+\Delta t)} \approx e^{-t}. \end{aligned} \tag{2.13}$$

Thus

$$\text{Prob}\left(||y_{l+1} - y_l|| \in \left[\frac{t}{N}, \frac{t+\Delta t}{N}\right]\right) \to e^{-t}\Delta t. \tag{2.14}$$

Remark 2.4.2. *The above argument is infinitesimally wrong. Once we've located y_{l+1}, the remaining x_ns do not need to be more than $\frac{t+\Delta t}{N}$ units to the right of $x_1 = y_l$; they only need to be further to the right than y_{l+1}. As the incremental gain in probabilities for the locations of the remaining x_ns is of order Δt, these contributions will not influence the large N, small Δt limits. Thus, we ignore these effects.*

To rigorously derive the limiting behavior of the nearest neighbor spacings using the above arguments, one would integrate over x_m ranging from $\frac{t}{N}$ to $\frac{t+\Delta t}{N}$, and the remaining events x_n would be in the a segment of length $1-x_m$. As

$$\left|\left(1-x_m\right) - \left(1 - \frac{t+\Delta t}{N}\right)\right| \leq \frac{\Delta t}{N}, \tag{2.15}$$

this will lead to corrections of higher order in Δt, hence negligible.

We can rigorously avoid this by instead considering the following:

1. Calculate the probability that all the other x_ns are at least $\frac{t}{N}$ units to the right of x_1. This is

$$p_t \ = \ \left(1 - \frac{t}{N}\right)^{N-1} \ \to \ e^{-t}. \tag{2.16}$$

2. Calculate the probability that all the other x_ns are at least $\frac{t+\Delta t}{N}$ units to the right of x_1. This is

$$p_{t+\Delta t} \ = \ \left(1 - \frac{t+\Delta t}{N}\right)^{N-1} \ \to \ e^{-(t+\Delta t)}. \tag{2.17}$$

3. The probability that no x_ns are within $\frac{t}{N}$ units to the right of x_1 but at least one x_n is between $\frac{t}{N}$ and $\frac{t+\Delta t}{N}$ units to the right is $p_{t+\Delta t} - p_t$:

$$\begin{aligned} p_t - p_{t+\Delta t} \ &\to \ e^{-t} - e^{-(t+\Delta t)} \\ &= \ e^{-t}\left(1 - e^{-\Delta t}\right) \\ &= \ e^{-t}\left(1 - 1 + \Delta t + O\left((\Delta t)^2\right)\right) \\ &\to \ e^{-t}\Delta t. \end{aligned} \tag{2.18}$$

Definition 2.4.3 (Unfolding Spacings). *If $y_{l+1} - y_l \in \left[\frac{t}{N}, \frac{t+\Delta t}{N}\right]$, then $N(y_{l+1} - y_l) \in [t, t+\Delta t]$. The new spacings $z_{l+1} - z_l$ have unit mean spacing. Thus, while we expect the average spacing between adjacent y_ns to be $\frac{1}{N}$ units, we expect the average spacing between adjacent z_ns to be 1 unit.*

2.4.2 k^{th} Neighbor Spacings

Similarly, one can easily analyze the distribution of the k^{th} neighbor spacings when each x_n is chosen independently from the uniform distribution on $[0, 1)$.

Again, consider $x_1 = y_l$. Now we want to calculate the probability that y_{l+k} is between $\frac{t}{N}$ and $\frac{t+\Delta t}{N}$ units to the *right* of y_l.

Therefore, we need exactly $k-1$ of the x_ns to lie between 0 and $\frac{t}{N}$ units to the right of x_1, exactly one x_n (which will be y_{l+k}) to lie between $\frac{t}{N}$ and $\frac{t+\Delta t}{N}$ units to the right of x_1, and the remaining x_ns to lie at least $\frac{t+\Delta t}{N}$ units to the right of y_{l+k}.

Remark 2.4.4. *We face the same problem discussed in Remark 2.4.2; a similar argument will show that ignoring these affects will not alter the limiting behavior. Therefore, we will make these simplifications.*

There are $\binom{N-1}{k-1}$ ways to choose the x_ns that are at most $\frac{t}{N}$ units to the right of x_1; there is then $\binom{(N-1)-(k-1)}{1}$ ways to choose the x_n between $\frac{t}{N}$ and $\frac{t+\Delta t}{N}$ units to the right of x_1.

Thus,

$$\begin{aligned}
&\text{Prob}\left(||y_{l+k} - y_l|| \in \left[\frac{t}{N}, \frac{t+\Delta t}{N}\right]\right) = \\
= &\binom{N-1}{k-1}\left(\frac{t}{N}\right)^{k-1} \cdot \binom{(N-1)-(k-1)}{1}\frac{\Delta t}{N} \cdot \left(1 - \frac{t+\Delta t}{N}\right)^{N-(k+1)} \\
= &\frac{(N-1)\cdots(N-1-(k-2))}{N^{k-1}}\frac{(N-1)-(k-1)}{N}\frac{t^{k-1}}{(k-1)!}\left(1-\frac{t+\Delta t}{N}\right)^{N-(k+1)}\Delta t \\
\to &\frac{t^{k-1}}{(k-1)!}e^{-t}\Delta t. \qquad (2.19)
\end{aligned}$$

Again, one way to avoid the complications is to integrate over x_m ranging from $\frac{t}{N}$ to $\frac{t+\Delta t}{N}$.

Or, similar to before, we can proceed more rigorously as follows:

1. Calculate the probability that exactly $k-1$ of the other x_ns are at most $\frac{t}{N}$ units to the right of x_1, and the remaining $(N-1)-(k-1)$ of the x_ns are at least $\frac{t}{N}$ units to the right of x_1. As there are $\binom{N-1}{k-1}$ ways to choose $k-1$ of the x_ns to be at most $\frac{t}{N}$ units to the right of x_1, this probability is

$$\begin{aligned}
p_t &= \binom{N-1}{k-1}\left(\frac{t}{N}\right)^{k-1}\left(1-\frac{t}{N}\right)^{(N-1)-(k-1)} \\
&\to \frac{N^{k-1}}{(k-1)!}\frac{t^{k-1}}{N^{k-1}}e^{-t} \\
&\to \frac{t^{k-1}}{k-1!}e^{-t}. \qquad (2.20)
\end{aligned}$$

2. Calculate the probability that exactly $k-1$ of the other x_ns are at most $\frac{t}{N}$ units to the right of x_1, and the remaining $(N-1)-(k-1)$ of the x_ns are at least $\frac{t+\Delta t}{N}$ units to the right of x_1. Similar to the above, this gives

$$\begin{aligned} p_t &= \binom{N-1}{k-1}\left(\frac{t}{N}\right)^{k-1}\left(1-\frac{t+\Delta t}{N}\right)^{(N-1)-(k-1)} \\ &\to \frac{N^{k-1}}{(k-1)!}\frac{t^{k-1}}{N^{k-1}}e^{-(t+\Delta t)} \\ &\to \frac{t^{k-1}}{(k-1)!}e^{-(t+\Delta t)}. \end{aligned} \tag{2.21}$$

3. The probability that exactly $k-1$ of the x_ns are within $\frac{t}{N}$ units to the right of x_1 and at least one x_n is between $\frac{t}{N}$ and $\frac{t+\Delta t}{N}$ units to the right is $p_{t+\Delta t} - p_t$:

$$p_t - p_{t+\Delta t} \ \to\ \frac{t^{k-1}}{(k-1)!}e^{-t} - \frac{t^{k-1}}{(k-1)!}e^{-(t+\Delta t)} \ \to\ \frac{t^{k-1}}{(k-1)!}e^{-t}\Delta t. \tag{2.22}$$

Note that when $k=1$, we recover the nearest neighbor spacings.

2.5 Induced Probability Measures

We have proven the following:

Theorem 2.5.1. *Consider N independent random variables x_n chosen from the uniform distribution on the wrapped unit interval $[0,1)$. For fixed N, arrange the x_ns in increase order, labeled $y_1 \le y_2 \le \cdots \le y_N$.*

Form the induced probability measure $\mu_{N,1}$ from the nearest neighbor spacings. Then as $N \to \infty$ we have

$$\mu_{N,1}(t)dt \ =\ \frac{1}{N}\sum_{n=1}^{N}\delta\Big(t - N(y_n - y_{n-1})\Big)dt \ \to\ e^{-t}dt. \tag{2.23}$$

Equivalently, using $z_n = Ny_n$:

$$\mu_{N,1}(t)dt \ =\ \frac{1}{N}\sum_{n=1}^{N}\delta\Big(t - (z_n - z_{n-1})\Big)dt \ \to\ e^{-t}dt. \tag{2.24}$$

More generally, form the probability measure from the k^{th} nearest neighbor spacings. Then as $N \to \infty$ we have

$$\mu_{N,k}(t)dt \;=\; \frac{1}{N}\sum_{n=1}^{N}\delta\Big(t - N(y_n - y_{n-k})\Big)dt \;\to\; \frac{t^{k-1}}{(k-1)!}e^{-t}dt. \tag{2.25}$$

Equivalently, using $z_n = Ny_n$:

$$\mu_{N,k}(t)dt \;=\; \frac{1}{N}\sum_{n=1}^{N}\delta\Big(t - (z_n - z_{n-k})\Big)dt \;\to\; \frac{t^{k-1}}{(k-1)!}e^{-t}dt. \tag{2.26}$$

Definition 2.5.2 (Poissonian Behavior). *We say a sequence of points x_n has Poissonian Behavior if in the limit as $N \to \infty$ the induced probability measures $\mu_{N,k}(t)dt$ converge to $\frac{t^{k-1}}{(k-1)!}e^{-t}dt$.*

Exercise 2.5.3. *Let $\alpha \in \mathbb{Q}$, and define $\alpha_n = \{n^m\alpha\}$ for some positive integer m. Show the sequence of points α_n does not have Poissonian Behavior.*

Exercise 2.5.4. *Let $\alpha \notin \mathbb{Q}$, and define $\alpha_n = \{n\alpha\}$. Show the sequence of points α_n does not have Poissonian Behavior. Hint: for each N, show the nearest neighbor spacings take on at most three distinct values (the three values depend on N). As only three values are ever assumed for a fixed N, $\mu_{N,1}(t)dt$ cannot converge to $e^{-t}dt$.*

2.6 Non-Poissonian Behavior

Conjecture 2.6.1. *With probability one (with respect to Lebesgue Measure, see Definition* **??***), if $\alpha \notin \mathbb{Q}$, if $\alpha_n = \{n^2\alpha\}$ then the sequence of points α_n is Poissonian.*

There are constructions which show certain irrationals give rise to non-Poissonian behavior.

Theorem 2.6.2. *Let $\alpha \in \mathbb{Q}$ such that $\left|\alpha - \frac{p_n}{q_n}\right| < \frac{a_n}{q_n^3}$ holds infinitely often, with $a_n \to 0$. Then there exist integers $N_j \to \infty$ such that $\mu_{N_j,1}(t)$ does not converge to $e^{-t}dt$.*

As $a_n \to 0$, eventually $a_n < \frac{1}{10}$ for all n large. Let $N_n = q_n$, where $\frac{p_n}{q_n}$ is a good rational approximation to α:

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{a_n}{q_n^3}. \tag{2.27}$$

Remember that all subtractions are performed on the wrapped unit interval. Thus, $||.999 - .001|| = .002$.

We look at $\alpha_k = \{k^2\alpha\}$, $1 \le k \le N_n = q_n$. Let the β_ks be the α_ks arranged in increasing order, and let the γ_ks be the numbers $\{k^2\frac{p_n}{q_n}\}$ arranged in increasing order:

$$\begin{aligned} \beta_1 &\le \beta_2 \le \cdots \le \beta_N \\ \gamma_1 &\le \gamma_2 \le \cdots \le \gamma_N. \end{aligned} \tag{2.28}$$

2.6.1 Preliminaries

Lemma 2.6.3. *If $\beta_l = \alpha_k = \{k^2\alpha\}$, then $\gamma_l = \{k^2\frac{p_n}{q_n}\}$. Thus, the same permutation orders both the α_ks and the γ_ks.*

Proof. Multiplying both sides of Equation 2.27 by $k^2 \le q_n^2$ yields

$$\left|k^2\alpha - k^2\frac{p_n}{q_n}\right| \;<\; k^2\frac{a_n}{q_n^2} \;\le\; \frac{a_n}{q_n} \;<\; \frac{1}{2q_n}. \tag{2.29}$$

Thus, $k^2\alpha$ and $k^2\frac{p_n}{q_n}$ differ by at most $\frac{1}{2q_n}$. Therefore

$$\left|\left|\left\{k^2\alpha\right\} - \left\{k^2\frac{p_n}{q_n}\right\}\right|\right| \;<\; \frac{1}{2q_n}. \tag{2.30}$$

As the numbers $\{m^2\frac{p_n}{q_n}\}$ all have denominators of size at most $\frac{1}{q_n}$, we see that $\{k^2\frac{p_n}{q_n}\}$ is the closest of the $\{m^2\frac{p_n}{q_n}\}$ to $\{k^2\alpha\}$.

This implies that if $\beta_l = \{k^2\alpha\}$, then $\gamma_l = \{k^2\frac{p_n}{q_n}\}$, completing the proof. □

Exercise 2.6.4. *Prove the ordering is as claimed. Hint: about each $\beta_l = \{k^2\alpha\}$, the closest number of the form $\{c^2\frac{p_n}{q_n}\}$ is $\{k^2\frac{p_n}{q_n}\}$.*

2.6.2 Proof of Theorem 2.6.2

Exercise 2.6.5. *Assume* $||a-b||, ||c-d|| < \frac{1}{10}$. *Show*

$$||(a-b)-(c-d)|| \ < \ ||a-b|| + ||c-d||. \tag{2.31}$$

Proof of Theorem 2.6.2: We have shown

$$||\beta_l - \gamma_l|| \ < \ \frac{a_n}{q_n}. \tag{2.32}$$

Thus, as $N_n = q_n$:

$$\Big|\Big|N_n(\beta_l - \gamma_l)\Big|\Big| \ < \ a_n, \tag{2.33}$$

and the same result holds with l replaced by $l-1$.

By Exercise 2.6.5,

$$\Big|\Big|N_n(\beta_l - \gamma_l) - N_n(\beta_{l-1} - \gamma_{l-1})\Big|\Big| \ < \ 2a_n. \tag{2.34}$$

Rearranging gives

$$\Big|\Big|N_n(\beta_l - \beta_{l-1}) - N_n(\gamma_l - \gamma_{l-1})\Big|\Big| \ < \ 2a_n. \tag{2.35}$$

As $a_n \to 0$, this implies the difference between $\Big|\Big|N_n(\beta_l - \beta_{l-1})\Big|\Big|$ and $\Big|\Big|N_n(\gamma_l - \gamma_{l-1})\Big|\Big|$ goes to zero.

The above distance calculations were done mod 1. The actual differences will differ by an integer. Thus,

$$\mu^{\alpha}_{N_n,1}(t)dt \ = \ \frac{1}{N_n}\sum_{l=1}^{N_n} \delta\Big(t - N_n(\beta_l - \beta_{l-1})\Big) \tag{2.36}$$

and

$$\mu^{\frac{p_n}{q_n}}_{N_n,1}(t)dt \ = \ \frac{1}{N_n}\sum_{l=1}^{N_n} \delta\Big(t - N_n(\gamma_l - \gamma_{l-1})\Big) \tag{2.37}$$

are extremely close to one another; each point mass from the difference between adjacent β_ls is either within a_n units of a point mass from the difference between adjacent γ_ls, or is within a_n units of a point mass an integer number of units from a point mass from the difference between adjacent γ_ls. Further, $a_n \to 0$.

Note, however, that if $\gamma_l = \{k^2 \frac{p_n}{q_n}\}$, then

$$N_n \gamma_l \ = \ q_n \left\{ k^2 \frac{p_n}{q_n} \right\} \ \in \ \mathbb{N}. \tag{2.38}$$

Thus, the induced probability measure $\mu_{N_n,1}^{\frac{p_n}{q_n}}(t)dt$ formed from the γ_ls is supported on the integers! Thus, it is impossible for $\mu_{N_n,1}^{\frac{p_n}{q_n}}(t)dt$ to converge to $e^{-t}dt$.

As $\mu_{N_n,1}^{\alpha}(t)dt$, modulo some possible integer shifts, is arbitrarily close to $\mu_{N_n,1}^{\frac{p_n}{q_n}}(t)dt$, the sequence $\{k^2\alpha\}$ is *not* Poissonian along the subsequence of Ns given by N_n, where $N_n = q_n$, q_n is a denominator in a good rational approximation to α. □

2.6.3 Measure of $\alpha \notin \mathbb{Q}$ with Non-Poissonian Behavior along a sequence N_n

What is the (Lebesgue) measure of $\alpha \notin \mathbb{Q}$ such that there are infinitely many n with

$$\left|\alpha - \frac{p_n}{q_n}\right| \ < \ \frac{a_n}{q_n}, \ a_n \to 0. \tag{2.39}$$

If the above holds, then for any constant $k(\alpha)$, for n large (large depends on both α and $k(\alpha)$) we have

$$\left|\alpha - \frac{p_n}{q_n}\right| \ < \ \frac{k(\alpha)}{q_n^{2+\epsilon}}. \tag{2.40}$$

By Theorem **??**, this set has (Lebesgue) measure or size 0. Thus, almost no irrational numbers satisfy the conditions of Theorem 2.6.2, where *almost no* is relative to the (Lebesgue) measure.

Exercise 2.6.6. *In a topological sense, how many algebraic numbers satisfy the conditions of Theorem 2.6.2? How many transcendental numbers satisfy the conditions?*

Exercise 2.6.7. *Let α satisfy the conditions of Theorem 2.6.2. Consider the sequence N_n, where $N_n = q_n$, q_n the denominator of a good approximation to α. We know the induced probability measures $\mu_{N_n,1}^{\frac{p_n}{q_n}}(t)dt$ and $\mu_{N_n,1}^{\alpha}(t)dt$ do not converge to $e^{-t}dt$. Do these measures converge to anything?*

Remark 2.6.8. *In* The Distribution of Spacings Between the Fractional Parts of $\{n^2\alpha\}$ *(Z. Rudnick, P. Sarnak, A. Zaharescu), it is shown that for most* α *satisfying the conditions of Theorem 2.6.2, there* is *a sequence* N_j *along which* $\mu^{\alpha}_{N_n,1}(t)dt$ does *converge to* $e^{-t}dt$.