Honors 1: Undergraduate Math Lab*

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September 25^{th}, 2002

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Abstract

We review some basic definitions of Graph Theory, and prove a simple result about the size of the eigenvalues of adjacency graphs. Lecture by Peter Sarnak; notes by Steven Miller.

1 Definitions

Definition 1.1 *For a graph G, let $V(G)$ be the set of vertices and $E(G)$ the set of edges (an edge is a pair (v,w) with $v, w \in V$). We often just write V and E.*

Definition 1.2 (Connected Graph) *A graph G is connected if for any two vertices $v, w \in G$, there is a path of edges in E starting at v and ending at w.*

Definition 1.3 (Boundary) $\partial A = \{v \in V - A :$ *there is a* $w \in A$ *with* $(v,w) \in E\}$.

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Definition 1.4 (k-regular) *A graph G is k-regular if there are k edges coming out from each vertex.*

Definition 1.5 (Expander Graph) *Let G be a connected graph with n vertices. Let $A \subset V(G)$ be any subset of vertices with $|A| \leq \frac{n}{2}$ (ie, at most half of the vertices). We say G is a (n, c, k) expander if for any such A, $|\partial A| \geq c|A|$.*

Example 1.6 *Let G be a 2-regular graph with n vertices. For definateness, label the vertices $\{1, 2, \ldots, n\}$. Let the edges be $(1, 2)$, $(2, 3)$, $(3, 4), \ldots, (n, 1)$. Let A be the first half of the vertices: $A = \{1, 2, \ldots, \left[\frac{n}{2}\right]\}$. Then $|\partial A| = 2$, and G is not an $(n, c, 2)$ expander.*

2 Size of Eigenvalues

Consider a 3-regular graph with n vertices. We want the graph to have certain desirable connectivity properties.

Let $A = (a_{vw})$ is the adjacency matrix attached to the graph G, and $a_{vw} = 1$ if there is an edge from v to w and 0 otherwise. This is a very sparse matrix (only three non-zero entries in each row or column). Compute its eigenvalues (real numbers).

Lemma 2.1 *The eigenvalues $\lambda_i \in [-3, 3]$.*

Proof: Let $f : V \to \mathbb{R}$, define the action of the adjacency matrix A on f by

$$Af(v) = \sum_{(v,w) \in E} f(w). \tag{1}$$

As there are finitely many vertices (n, in fact), we can regard the function $f(v)$ as living in $\mathbb{R}^n$, with coordinates $\big(f(v_1), \ldots, f(v_n)\big)$.

Suppose $\forall w \in V$, $Af(w) = \lambda f(w)$, $f \neq 0$. Then λ is an eigenvalue (which must be real as A is real symmetric).

We use the Maximum Modulus Principle. As $f \neq 0$, let w_0 be such that $f(w_0)$ is the maximum value of $f(w)$ (not zero, and exists as we have finitely many vertices). Then

$$f(w_0) = \frac{1}{\lambda} \sum_{(w,w_0)} f(w). \tag{2}$$

If $\lambda > 3$, this cannot happen (we're assuming the graph is k-regular and connected). If $\lambda = 3$, then f is constant. Working with absolute values, we similarly obtain $\lambda > -3$.

Exercise 2.2 *Prove a k-regular graph G is connected if and only if $\lambda = k$ is a simple eigenvalue.*

Let λ_1 be the second largest eigenvalue; $\lambda = k$ is always an eigenvalue for a k-regular connected graph. How big can the gap be between λ_1 and k?