Honors 1: Undergraduate Math Lab*

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Abstract

Lecture by Peter Sarnak; notes by Steven Miller.

1 Definitions

Let G be a connected, simple (no multiple bonds or edges) k-regular graph with adjacency matrix $A = (a_{v,w})$. Here

$$a_{v,w} = \begin{cases} 1 & \text{if } v \sim w \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Thus, $a_{v,w}$ is the number of paths of length one from v to w.
Let $A^2 = (a^{(2)}_{v,w})$, and here

$$a^{(2)}_{v,w} = \text{number of paths of length 2 from } v \text{ to } w. \tag{2}$$

Thus,

$$a^{(2)}_{v,w} = \sum_{v'} a_{v,v'} a_{v',w}. \tag{3}$$

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Similarly let $A^n = (a_{v,w}^{(n)})$, and here

$$a_{v,w}^{(n)} = \text{number of paths of length } n \text{ from } v \text{ to } w. \tag{4}$$

Recall

$$\text{Trace}(B) \;=\; Tr(B) \;=\; \sum_v b_{v,v}. \tag{5}$$

Given an adjacency matrix A, let D be the diagonal matrix of eigenvalues.

$$D = \begin{pmatrix} \lambda_0 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_{N-1} \end{pmatrix} \tag{6}$$

$A = Q^{-1}DQ$ for some orthogonal matrix Q. Thus, $A^n = Q^{-1}DQ$, and we find:

Lemma 1.1. *$Tr(A^n) = Tr(D^n)$.*

Lemma 1.2 (Trace Formula). *For any $n \geq 0$,*

$$\sum_{j=0}^{N-1} \lambda_j^n \;=\; \sum_{v \in V} a_{v,v}^{(n)}. \tag{7}$$

2 $\rho_k(2n)$ and $\lambda_{\max}$

To count walks of length n from v to v, it is clearly at least as many walks as there are on a k-regular infinite tree.

A tree is a homogeneous object: any vertex looks exactly the same as any other. There is no special vertex on a tree, though we will often name a vertex **the root**.

Definition 2.1. *$\rho_k(n)$ is the number of paths of length n from v to v, where v is* any *vertex of* the *k-regular tree.*

Remark 2.2. *$\rho_k(n) = 0$ for n odd.*

Remark 2.3. *The number of paths of length n from v to v on our graph G is at least the number of paths of length n from any vertex to itself on the infinite tree. Thus,*

$$\forall v \in V,\ a_{v,v}^{(n)} \ \geq \ \rho_k(n). \tag{8}$$

Remember that we've labeled the N eigenvalues by $\lambda_0 = k, \ldots, \lambda_{N-1}$.
In the trace formula, we find

$$\sum_{j=0}^{N-1} \lambda_j^{2n} \ \geq \ N\rho_k(2n). \tag{9}$$

Therefore

$$\frac{1}{N}\sum_{j=0}^{N-1} \lambda_j^{2n} \ \geq \ \rho_k(2n). \tag{10}$$

and as $\lambda_0 = k$

$$\frac{k^{2n}}{N} + \frac{1}{N}\sum_{j=1}^{N-1} \lambda_j^{2n} \ \geq \ \rho_k(2n). \tag{11}$$

Fix n and let $N \to \infty$.
Let $\lambda_{\max} = \max\Big(|\lambda_1|, |\lambda_{N-1}|\Big)$.
Thus, substituting into Equation 11 we find

$$\frac{k^{2n}}{N} + \lambda_{\max}^{2n} \geq \rho_k(2n) \tag{12}$$

in the limit as $N \to \infty$ (as we have λ_{max} a total of $N-1$ times, and we divide by N; in the limit, $\frac{N-1}{N} \to 1$.
As $N \to \infty$, we find

$$\begin{aligned} \lambda_{\max}^{2n} \ &\geq \ \rho_k(2n) \\ \text{or } \lambda_{\max} \ &\geq \ \Big(\rho_k(2n)\Big)^{\frac{1}{2n}}. \end{aligned} \tag{13}$$

Exercise 2.4. *Show*

1. $\rho_k(2n) \geq \frac{1}{m}\binom{2m-2}{m-1}k(k-1)^{m-1}$.

2. $\Big(\rho_k(2n)\Big)^{\frac{1}{2n}} \to 2\sqrt{k-1}$.

Using the above exercise, we now find that

$$\lambda_{\max} \geq 2\sqrt{k-1}. \tag{14}$$

3 Measure from the Eigenvalues

The trace formula told us that

$$\sum_{j=0}^{N-1} \lambda_j^n = \sum_{v\in V} a_{v,v}^{(n)}. \tag{15}$$

Definition 3.1 (girth). *The girth of a graph is the length of the shortest closed cycle that returns to the starting vertex without any backtracking.*

Assume that the girth of G_N tends to ∞ as $N \to \infty$.

Fix n, let N be very large. Then by assumption the girth is greater than say $2n+1$. Thus, $a_{v,v}^{(n)}$ cannot have any contribution from cycles without backtracking. Thus, locally, to calculate $a_{v,v}^{(n)}$, we look like a tree, and we find $a_{v,v}^{(n)} = \rho_k(n)$ for every $v \in V$. It is *essential* that we have fixed n.

Therefore, we now have (for fixed n under our assumption) that

$$\begin{aligned} \sum_{j=0}^{N-1} \lambda_j^n &= N\rho_k(n) \\ \frac{1}{N}\sum_{j=0}^{N-1} \lambda_j^n &= \rho_k(n). \end{aligned} \tag{16}$$

The left hand side looks like a Riemann sum.

Suppose the density of the eigenvalues of the 3-regular graph is $d\mu = f(x)dx$.

We have just shown, for polynomials $p_n(x) = x^n$, that

$$\frac{1}{N}\sum_{j=0}^{N-1} p_n(\lambda_j) \to \int_{-3}^{3} p_n(x)d\mu(x), \tag{17}$$

where the above converges to $\rho_k(n)$ (here $k = 3$).

Thus, we are looking for a density function $f(x)$ such that

$$\int_{-3}^{3} x^n f(x)dx = \rho_3(n), \ \ n \geq 0. \tag{18}$$

Is there such a function? Is it unique? What does it look like? This is the **Inverse Moment Problem**.

If there were two such functions, they would have to be equal by the **Weierstrass Approximation Theorem**, as their difference integrates to zero against any polynomial.

Exercise 3.2. *Compute the generating function*

$$F(z) = \sum_{n=0}^{\infty} \rho_3(n) z^n, \tag{19}$$

which is something like

$$\frac{1}{\sqrt{4(k-1)^2 - z^2}} \tag{20}$$

if $|z|$ is small (or maybe complex and outside $[-k, k]$).

$$\begin{aligned} \sum_{n=0}^{\infty} \rho_3(n) z^n &= \sum_{n=0}^{\infty} \left(\int_{-3}^{3} x^n f(x) dx \right) z^n \\ &= \int_{-3}^{3} f(x) \sum_{n=0}^{\infty} (zx)^n dx \\ &= \int_{-3}^{3} \frac{f(x)dx}{1 - zx} = F(z). \end{aligned} \tag{21}$$

If we let $z = \frac{1}{w}$ we find

$$\begin{aligned} F(w) &= \sum_{n=0}^{\infty} \rho_3(n) \frac{1}{w^n} \\ &= w \int_{-3}^{3} \frac{f(x)dx}{w - x}. \end{aligned} \tag{22}$$

Let $w \in C$ be such that $w \notin [-3,3]$. Letting $w \to a \in [-3,3]$, one gets a different value (a jump) if w approaches from above or below, and the jump is basically $f(a)$.

Look at $a+ib$ and $a-ib$, $b \to 0$.

4 Summary

The above is all based on the assumption that the girth was big. For the random graph, there are very few short closed cycles. Thus, when we use the trace formula, we now have

$$\frac{1}{N}\sum_{j=0}^{N-1}\lambda_j^n = \rho_k(n) + O\left(\frac{1}{N}\right). \tag{23}$$