Honors 1: Undergraduate Math Lab*

Peter Sarnak†, Steven Miller‡, Alex Barnett§

October 16^{th}, 2002

Courant Institute of Mathematical Sciences
New York University
New York, NY

Abstract

More on Graphs; Arithmetic maps mod p. Lecture by Peter Sarnak; notes by Steven Miller. Appendix: Introduction to Fourier Series and $\{n\alpha\}$, by Steven Miller.

1 Kesten's Measure

For a k-regular graph, define

$$d\mu_k(t) = \begin{cases} \frac{c_k\sqrt{(k-1)-\frac{t^2}{4}}}{1-(\frac{t}{k})^2}dt & \text{if } |t| \leq 2\sqrt{k-1} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Fix a and b, and consider the N eigenvalues λ_j. Count

$$\frac{\#\{j : \lambda_j \in [a,b]\}}{N}. \tag{2}$$

Then

*Homepage: http://www.math.nyu.edu/∼millerj/
†E-mail: sarnak@math.princeton.edu
‡E-mail: millerj@cims.nyu.edu or sjmiller@math.princeton.edu
§E-mail: barnett@nmr.mgh.harvard.edu

Claim 1.1.

$$\lim_{N\to\infty} \frac{\#\{j : \lambda_j \in [a,b]\}}{N} \;=\; \mu_k\Big([a,b]\Big). \tag{3}$$

We have

$$\Big(\rho_k(2n)\Big)^{\frac{1}{2n}} \;\to\; 2\sqrt{k-1}. \tag{4}$$

2 Generating Functions on k-Regular Trees

2.1 $R(z)$

Fix k. The **generating function** $R(z)$ is

$$R(z) \;=\; \sum_{n=0}^{\infty} r_n z^n, \tag{5}$$

where

$$r_n \;=\; \frac{\rho(n)}{k^n} \;=\; \text{Probability that we return to } v \text{ in } n \text{ steps.} \tag{6}$$

For $|z| < 1$, the series expansion for $R(z)$ converges.

Let q_n be the probability of starting at v and ending at v for the first time (after n steps).

2.2 $Q(z)$

Define

$$Q(z) \;=\; \sum_{n=0}^{\infty} q_n z^n. \tag{7}$$

Exercise 2.1. *Prove*

$$R(z) \;=\; \frac{1}{1-Q(z)}. \tag{8}$$

2.3 $T(z)$

Define

$$T(z) \;=\; \sum_{n=0}^{\infty} t_n z^n, \qquad (9)$$

where for w adjacent to v, t_n is the probability of going from w to v in n-steps for the first time.

Further, let $t_{w,v}(n)$ be the probability of going from w to v in n-steps first time and $d(w,v) = m \geq 1$. Remember that $d(w,v)$ is the distance from w to v.

Exercise 2.2. *Prove*

1. $Q(z) = zT(z)$.
2. $\sum_{n=0}^{\infty} t_{w,v}(n) z^n = \Big(T(z)\Big)^n$.

Exercise 2.3. *Prove*

$$T(z) \;=\; \frac{z}{k} + \frac{k-1}{k} z T^2(z). \qquad (10)$$

Note that this explicitly gives us $T(z)$ by application of the quadratic formula:

$$T(z) \;=\; \frac{1 \pm \sqrt{1 - 4\left(\frac{k-1}{k} z\right)^2 \frac{z}{k}}}{2\frac{k-1}{k} z}. \qquad (11)$$

Now that we have $T(z)$ we have $Q(z)$, from which we get $R(z)$. $T(z)$ will have a square-root – it will be an algebraic function of z.

3 Recovering the Measure $f(x)$ from $R(z)$

We have

$$R(z) \;=\; \sum_{n=0}^{\infty} r_n z^n. \qquad (12)$$

As we know $T(z)$, we know $R(z)$, hence we know the numbers r_n.

Now,

$$r_n \;=\; \int_{\infty}^{\infty} x^n f(x)dx \;=\; \int_{-k}^{k} x^n f(x)dx. \tag{13}$$

How do we recover $f(x)$ given the numbers r_n? We've now normalized the eigenvalues to lie in $[-1,1]$.

$$\begin{aligned}
R(z) &= \sum_{n=0}^{\infty}\left(\int_{-1}^{1} x^n f(x)dx\right) \\
&= \int_{-1}^{1}\left(\sum_{n=0}^{\infty}(xz)^n\right)f(x)dx \\
&= \int_{-1}^{1}\frac{f(x)}{1-xz}dx \\
B(z) \;=\; \frac{1}{z}R\Big(\frac{1}{z}\Big) &= \int_{-1}^{1}\frac{f(x)}{z-x}dx.
\end{aligned} \tag{14}$$

Suppose we know the LHS. Can we recover $f(x)$? If $z \in [-1,1]$, the function will have a singularity. Thus, if $f(x)$ is a nice function, we do not expect to be able to make sense of the above relation if $z \in [-1,1]$. We will, however, consider z *close* to the interval $[-1,1]$.

Let $z = \xi + iy$, $\xi \in [-1,1]$, $y > 0$. Later we will take $z = \xi - iy$.

Look at

$$\begin{aligned}
B(\xi+iy) - B(\xi-iy) &= \int_{-1}^{1} f(x)\left[\frac{1}{\xi+iy-x} - \frac{1}{\xi-iy-x}\right]dx \\
&= 2i\int_{-1}^{1}\frac{yf(x)}{(\xi-x)^2+y^2}dx.
\end{aligned} \tag{15}$$

We will study the above as $y \to 0$.

3.1 Poisson Kernel

Recall $\xi \in [-1,1]$, $f(x)$ fixed, we are integrating $f(x)$ against the **Poisson Kernel**

$$\frac{y}{(\xi - x)^2 + y^2}. \tag{16}$$

As $y \to 0$, the above looks singular at $x = \xi$.

At $x = \xi$, the kernel has height $\frac{1}{y}$, which is quite large.

If $x = \xi + \epsilon$, then as $y \to 0$, the kernel goes to 0 very rapidly.

Basically, as $y \to 0$, the kernel becomes a higher, thinner spike centered at ξ.

Now

$$\begin{aligned}
\int_{\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2} dx &= \int_{-\infty}^{\infty} \frac{y}{t^2 + y^2} dt \\
&= \int_{-\infty}^{\infty} \frac{y^2}{y^2\eta^2 + y^2} d\eta, \text{ from } \frac{t}{y} = \eta \\
&= \int_{-\infty}^{\infty} \frac{1}{1 + \eta^2} d\eta \\
&= \pi. \qquad (17)
\end{aligned}$$

This is an **approximation to the identity**.

Thus,

$$B(\xi + iy) - B(\xi - iy) \;\to\; 2\pi i f(\xi). \tag{18}$$

3.2 Cauchy Integral Formula

If you have an analytic function $f(z)$ and γ is a curve enclosing z then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{z - \zeta} d\zeta \tag{19}$$

In our case above, we cannot apply Cauchy's Integral Formula, as our function $f(x)$ is not analytic. It is compactly supported, and no non-zero analytic function is compactly supported.

Call this permutation ϕ:

$$\phi : \mathbb{F}_p^* \to \mathbb{F}_p^*, \tag{20}$$

where ϕ^2 is the identity.

Question 3.1. *Does ϕ behave like a random permutation?*

4 Third Problem

4.1 Introduction

Let p be a large prime, and consider the map

$$x \mapsto x^{-1} \bmod p, \ x \neq 0. \tag{21}$$

This is a map from $\mathbb{F}_p \to \mathbb{F}_p$. The map is not completely random, as

$$\begin{aligned} 1 &\mapsto 1 \\ 2 &\mapsto \frac{p+1}{2} \\ p-1 &\mapsto p-1. \end{aligned} \tag{22}$$

The map which sends $x \to x^{-1}$ is a permutation of $\mathbb{F}_p^*$. It is not a completely arbitrary permutation, as it pairs x with x^{-1} (away from a few very special x's, such as $p-1$ and 1).

Thus, this permutation is a product of transpositions.

4.2 Character Sums

Let

$$1 \leq A \leq B \leq p, \ B - A \text{ large}. \tag{23}$$

Let

$$\overline{m} \text{ be the inverse of } m \bmod p \tag{24}$$

IE, $m\overline{m} \equiv 1 \bmod p$.
Let

$$e(z) \ = \ 2^{2\pi i z}. \tag{25}$$

For $\nu \in \mathbb{Z}/p\mathbb{Z}$, consider

$$S \ = \sum_{A \leq m \leq B} e\left(\frac{\overline{m}\nu}{p}\right). \tag{26}$$

These sums will measure how equidistributed or random the map $m \to \overline{m}$ is.

Exercise 4.1. *Prove the trivial bound for* $|S|$*:*

$$|S| \leq B - A. \tag{27}$$

Let $N = B - A + 1$. By the Central Limit Theorem, with high probability if we add N random numbers of modulus one we expect square-root cancellation. Thus, we expect (if the inverse map is random) that $|S| \approx \sqrt{N}$.

4.3 Completing the Square

Let $a, b \in \mathbb{Z}$, and consider the **Kloosterman Sum**

$$\mathrm{Kl}(a, b, p) \ = \ \sum_{\substack{x \mod p \\ x \neq 0}} e\left(\frac{ax + b\overline{x}}{p}\right). \tag{28}$$

How large can the Kloosterman Sum be? If $a = b = 0$, then trivially $\mathrm{Kl}(0, 0, p) = p - 1$.

If $b = 0$ and $a \neq 0$ (or, by symmetry, the other way around) then

$$\begin{aligned} \mathrm{Kl}(a, b, p) \ &= \ \sum_{\substack{x \mod p \\ x \neq 0}} e\left(\frac{ax}{p}\right) \\ &= \ \sum_{\substack{y \mod p \\ y \neq 0}} e\left(\frac{y}{p}\right) \\ &= \ \sum_{y=0}^{p-1} e\left(\frac{y}{p}\right) \ - \ 1 \ = \ -1. \end{aligned} \tag{29}$$

Exercise 4.2. *Prove* $\sum_{y=0}^{p-1} e\left(\frac{y}{p}\right) = 0$*. Hint: Let* T *be this sum. Then show* $e\left(\frac{1}{p}\right)T = T$*; thus* $T = 0$*.*

4.4 Weil's Bound

Let $a \not\equiv 0 \mod p$. Then

$$|\mathrm{Kl}(a, b, p)| \ \leq \ 2\sqrt{p}. \tag{30}$$

This is a very deep result.

How big is

$$\sum_{a \mod p} |\mathrm{Kl}(a,1,p)|^2 \tag{31}$$

If we believe Weil's bound, each term is of size at most $2\sqrt{p}$, we square, then sum p terms. Thus, we expect a size of at most $4p^2$. We will show on average that Weil's bound is correct.

$$\begin{aligned}
\sum_{a \mod p} |\mathrm{Kl}(a,1,p)|^2 &= \sum_{a(p)} \left| \sum_{\substack{x \mod p \\ x \neq 0}} e\Big(\frac{\overline{x}+ax}{p}\Big) \right| \\
&= \sum_{a(p)} \sum_{\substack{x_1,x_2 \mod p \\ x_1,x_2 \neq 0}} e\Big(\frac{\overline{x}_1 - \overline{x}_2 + a(x_1-x_2)}{p}\Big) \\
&= \sum_{\substack{x_1,x_2 \mod p \\ x_1,x_2 \neq 0}} e\Big(\frac{\overline{x}_1 - \overline{x}_2}{p}\Big) \sum_{a(p)} e\Big(\frac{a(x_1-x_2)}{p}\Big) \\
&= (p-1)p,
\end{aligned} \tag{32}$$

where the last line follows from the a-sum vanishes unless $x_1 = x_2$, which then collapses the sums. There are $p-1$ ways $x_1 = x_2$, and when this occurs, the a-sum gives p.

Exercise 4.3. *Consider*

$$\sum_{a(p)} |Kl(a,1,p)|^4. \tag{33}$$

Above there are p-terms, each term of size $(2\sqrt{p})^4 = 16p^2$. *Thus, show the sum is at most* $16p^3$. *You will find* cp^3 *for some* c *independent of* p.

By looking at one term, as every summand is positive, we find

$$|\mathrm{Kl}(a,1,p)|^4 \ \leq\ cp^3. \tag{34}$$

Thus, taking the fourth-root yields

$$|\mathrm{Kl}(a,1,p)| \ \leq\ c^{\frac{1}{4}} p^{\frac{3}{4}}. \tag{35}$$

4.5 Fourier Expansion of Sums

Define the indicator function

$$I(y) = \begin{cases} 1 & A \leq y \leq A+N \\ 0 & \text{otherwise} \end{cases} \tag{36}$$

Consider

$$\begin{aligned} S &= \sum_{A \leq x \leq A+N} e\Big(\frac{\nu\overline{x}}{p}\Big) \\ &= \sum_{x(p)} e\Big(\frac{\nu\overline{x}}{p}\Big) I(x). \end{aligned} \tag{37}$$

We want to write $I(x)$ in terms of its Fourier Coefficients

$$\widehat{I}(m) = \int_0^1 e(-mt)I(t)dt. \tag{38}$$

Then

$$I(y) = \sum_{m=-\infty}^{\infty} \widehat{I}(m) e\Big(\frac{my}{p}\Big). \tag{39}$$

4.6 Brief Review of Fourier Series

Consider the unit interval $[0,1]$. Define

$$\phi_m(x) = e(mx). \tag{40}$$

Then (if our function is sufficiently nice)

$$f(x) = \sum_{m \in \mathbb{Z}} \widehat{f}(m) e(mx), \tag{41}$$

where

$$\widehat{f}(m) = \int_0^1 f(x) e(-mx) dx. \tag{42}$$

5 Fourier Analysis and the Equi-Distribution of $\{n\alpha\}$

5.1 Inner Product of Functions

We define the exponential function by means of the series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \tag{43}$$

which converges everywhere. Given the Taylor series expansion of $\sin x$ and $\cos x$, we can verify the identity

$$e^{ix} = \cos x + i \sin x. \tag{44}$$

Exercise 5.1. *Prove e^x converges for all $x \in \mathbb{R}$ (even better, for all $x \in \mathbb{C}$. Show the series for e^x also equals*

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n, \tag{45}$$

which you may remember from compound interest problems.

Exercise 5.2. *Prove, using the series definition, that $e^{x+y} = e^x e^y$. Use this fact to calculate the derivative of e^x. If instead you try to differentiate the series directly, you must justify the derivative of the infinite sum is the infinite sum of the derivatives.*

Remember the definition of **inner or dot product**: for two vectors $\vec{v} = (v_1, \cdots, v_n)$, $\vec{w} = (w_1, \cdots, w_n)$, we take the *inner product* $\vec{v} \cdot \vec{w}$ (also denoted $\langle v, w \rangle$) to mean

$$\vec{v} \cdot \vec{w} \;=\; \langle v, w \rangle = \sum_i v_i \bar{w}_i. \tag{46}$$

Further, the length of a vector v is

$$|v| \;=\; \langle v, v \rangle. \tag{47}$$

We generalize this for functions. For definiteness, assume f and g are functions from $[0, 1]$ to $\mathbb{C}$. Divide the interval $[0, 1]$ into n equal pieces. Then we can represent the functions by

$$f(x) \quad \longleftrightarrow \quad \left(f(0), f\Big(\frac{1}{n}\Big), \dots, f\Big(\frac{n-1}{n}\Big)\right), \tag{48}$$

and similarly for g. Call these vectors f_n and g_n. As before, we consider

$$\langle f_n, g_n \rangle \;=\; \sum_{i=0}^{n-1} f\Big(\frac{i}{n}\Big) \cdot \bar{g}\Big(\frac{i}{n}\Big). \tag{49}$$

In general, as we continue to divide the interval ($n \to \infty$), the above sum diverges. For example, if f and g are identically 1, the above sum is n.

There is a natural rescaling: we multiply each term in the sum by $\frac{1}{n}$, the size of the sub-interval. Note for the constant function, the sum is now independent of n.

Thus, for good f and g we are led to

$$\langle f, g \rangle \;=\; \lim_{n\to\infty} \sum_{i=0}^{n-1} f\Big(\frac{i}{n}\Big) \cdot \bar{g}\Big(\frac{i}{n}\Big)\frac{1}{n} = \int_0^1 f(x)\overline{g(x)}dx. \tag{50}$$

The last result follows by Riemann Integration.

Definition 5.3. *We say two continuous functions on* $[0,1]$ *are orthogonal (or perpendicular) if their dot product equals zero.*

Exercise 5.4. *Prove* x^n *and* x^m *are not perpendicular on* $[0,1]$ *for* $n \neq m$.

We will see that the exponential function behaves very nicely under the inner product. Define

$$e_n(x) = e^{2\pi i n x} \quad \text{for } n \in \mathbb{Z}. \tag{51}$$

Then a straightforward calculation shows

$$\langle e_n(x), e_m(x) \rangle = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{otherwise.} \end{cases} \tag{52}$$

Thus $e_0(x), e_1(x), e_2(x), \cdots$ are an **orthogonal set** of functions, which means they are pairwise perpendicular. As each function has length 1, we say the functions $e_n(x)$ are an **orthonormal set** of functions.

Exercise 5.5. *Prove* $\langle e_n(x), e_m(x) \rangle$ *is* 1 *if* $n = m$ *and* 0 *otherwise.*

5.2 Fourier Series and $\{n\alpha\}$

5.2.1 Fourier Series

Let f be continuous and periodic on $\mathbb{R}$ with period one. Define the n**th Fourier coefficient** $\hat{f}(n)$ of f to be

$$\hat{f}(n) = a_n = \langle f(x), e_n(x)\rangle = \int_0^1 f(x)e^{-2\pi i n x}dx. \tag{53}$$

Returning to the intuition of $\mathbb{R}^m$, we can think of the $e_n(x)$'s as an infinite set of perpendicular directions. The above is simply the projection of f in the direction of $e_n(x)$.

Exercise 5.6. *Show*

$$\langle f(x) - \hat{f}(n)e_n(x), e_n(x)\rangle = 0. \tag{54}$$

This agrees with our intuition, namely, that if you remove the projection in a certain direction, what is left is perpendicular to that direction.

The N^{th} **partial Fourier series** of f is

$$s_N(x) = \sum_{n=-N}^{N} \hat{f}(n)e_n(x). \tag{55}$$

Exercise 5.7. *Prove*

1. $\langle f(x) - s_N(x), e_n(x)\rangle = 0$ *if* $|n| \leq N$.
2. $|\hat{f}(n)| \leq \int_0^1 |f(x)|dx$.
3. *If* $\langle f, f\rangle < \infty$, *then* $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \leq \langle f, f\rangle$.
4. *If* $\langle f, f\rangle < \infty$, *then* $\lim_{|n|\to\infty} \hat{f}(n) = 0$.

As $\langle f(x) - s_N(x), e_n(x)\rangle = 0$ if $|n| \leq N$, we might think that we just have to let N go to infinity to obtain a series s_∞ such that

$$\langle f(x) - s_\infty(x), e_n(x)\rangle = 0. \tag{56}$$

Assume that for a periodic function $g(x)$ to be orthogonal to $e_n(x)$ for every n it must be zero for every x. Then $f(x) - s_\infty(x) = 0$, and hence

$f = s_\infty$. Voilá – an expression for f as a sum of exponentials! Be careful, however. We have just glossed over the two central issues – completeness and, even worse, convergence. We will now see a way of avoiding some of our problems.

5.2.2 Weighted partial sums

Define

$$\begin{aligned} D_N(x) &= \sum_{n=-N}^{N} e_n(x) = \frac{\sin((2N+1)\pi x)}{\sin \pi x}, \\ F_N(x) &= \frac{\sin^2(N\pi x)}{N \sin^2 \pi x} = \frac{1}{N}\sum_{n=0}^{N-1} D_n(x). \end{aligned} \tag{57}$$

Here F stands for Féjer, D for Dirichlet. In general, functions which we are interested in taking their inner product against f are called **kernels**; thus, the Dirichlet kernel, the Féjer kernel, etc.

Note that, no matter what N is, $F_N(x)$ is positive for all x.

We say that a sequence $f_1(x), f_2(x), f_3(x), \ldots$ of functions is an **approximation to the identity** if

1. $f_N(x) \geq 0$ for all x and every N;
2. $\int_0^1 f_N(x)dx = 1$;
3. $\lim_{N\to\infty} \int_\delta^{1-\delta} f_N(x)dx = 0$ if $0 < \delta < \frac{1}{2}$.

Theorem 5.8. *The Féjer kernels $F_1(x), F_2(x), F_3(x), \ldots$ are an approximation to the identity.*

Proof: The first property is immediate. The second follows from the observation that $F_N(x)$ can be written as

$$F_N(x) = e_0(x) + \frac{N-1}{N}\Big(e_{-1}(x) + e_1(x)\Big) + \cdots, \tag{58}$$

and all integrals are zero but the first, which is 1.

To prove the third property, note that $F_N(x) \leq \frac{1}{N \sin^2 \pi\delta}$ for $\delta \leq x \leq 1-\delta$. $\square$

Let f be a continuous, periodic function on $\mathbb{R}$ with period one. Thus, we can consider f as a function on just $[0,1]$, with $f(0) = f(1)$. Define

$$T_N(x) = \int_0^1 f(y) F_N(x-y) dy. \tag{59}$$

Recall the following definition and theorem:

Definition 5.9 (Uniform Continuity). *A continuous function is uniformly continuous if given an $\epsilon > 0$, there exists a $\delta > 0$ such that $|x-y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. Note that the same δ works for all points.*

Theorem 5.10. *Any continuous function on a closed, compact interval is uniformly continuous.*

Exercise 5.11. *Show x^n is uniformly continuous on $[a,b]$ for $-\infty < a < b < \infty$.*

Theorem 5.12. *Given $\epsilon > 0$, there is an N such that*

$$|f(x) - T_N(x)| \leq \epsilon \tag{60}$$

for every $x \in [0,1]$.

Proof. For any positive N,

$$\begin{aligned}
T_N(x) - f(x) &= \int_0^1 f(x-y) F_N(y) dy - f(x) \cdot 1 \\
&= \int_0^1 f(x-y) F_N(y) dy - \int_0^1 f(x) F_n(y) dy \text{ (property 2 of } F_N) \\
&= \int_0^\delta \Big(f(x-y) - f(x) \Big) F_N(y) dy \\
&+ \int_\delta^{1-\delta} \Big(f(x-y) - f(x) \Big) F_N(y) dy \\
&+ \int_{1-\delta}^1 \Big(f(x-y) - f(x) \Big) F_N(y) dy.
\end{aligned} \tag{61}$$

Let $\delta \in (0, 1/2)$. Then, using the fact that the $F_N(x)$'s are an approximation to the identity, we find

$$\left|\int_{\delta}^{1-\delta}\Big(f(x-y)-f(x)\Big)F_N(y)dy\right| \le 2\max|f(x)|\cdot\int_{\delta}^{1-\delta}F_N(y)dy. \tag{62}$$

Since

$$\lim_{N\to\infty}\int_{\delta}^{1-\delta}F_N(y)dy = 0, \tag{63}$$

we obtain

$$\lim_{N\to\infty}\int_{\delta}^{1-\delta}(f(x-y)-f(x))F_N(y)dy = 0. \tag{64}$$

Thus, by choosing N large enough (where large depends on δ), we can insure that this piece is at most $\frac{\epsilon}{3}$.

It remains to estimate what happens near zero. Since f is continuous and $[0,1]$ is compact, f is uniformly continuous. Thus, we can choose δ small enough that $|f(x-y)-f(x)| < \frac{\epsilon}{3}$ for any x and any positive $y < \delta$. Then

$$\left|\int_{0}^{\delta}\Big(f(x-y)-f(x)\Big)F_N(y)dy\right| \le \int_0^{\delta}\frac{\epsilon}{3}F_N(y)dy \le \frac{\epsilon}{3}\int_0^1 F_N(y)dy \le \frac{\epsilon}{3}. \tag{65}$$

Similarly

$$\left|\int_{1-\delta}^{1}\Big(f(x-y)-f(x)\Big)F_N(y)dy\right| \le \frac{\epsilon}{3}. \tag{66}$$

Therefore

$$|T_N(x)-f(x)| \le \epsilon \tag{67}$$

for all N sufficiently large. □

Definition 5.13 (Trigonometric Polynomials). *Any finite linear combination of the functions $e_n(x)$ is called a trigonometric polynomial.*

From Theorem 5.12 we immediately get the Stone-Weierstrass theorem:

Theorem 5.14 (Stone-Weierstrass). *Any continuous period function can be uniformly approximated by trigonometric polynomials.*

5.3 Equidistribution

We say that a sequence $\{x_n\}$, $x_n \in [0,1]$ is *equidistributed* if

$$\lim_{N\to\infty} \frac{1}{2N+1} \#\{n : |n| \leq N, x_n \in (a,b)\} = b - a \tag{68}$$

for all $(a,b) \subset [0,1]$.

Theorem 5.15 (Weyl). *Let α be an irrational number in $[0,1]$. Let $x_n = \{n\alpha\}$, where $\{y\}$ denotes the fractional part of y. Then the sequence $\{x_n\}$ is equidistributed.*

Proof. We will estimate $\frac{1}{2N+1}\sum_{-N}^{N} \chi_{(a,b)}(x_n)$ as $N \to \infty$, where $\chi_{(a,b)}$ is the function taking the value 0 outside (a,b) and 1 inside (a,b). We call $\chi_{(a,b)}$ the **characteristic function** of the interval (a,b).

Thus, we must show

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \chi_{(a,b)}(x_n) = b - a. \tag{69}$$

Consider $e_k(x) = e^{2\pi ikx}$. Since $x_n = \{n\alpha\} = n\alpha - [n\alpha]$ and $e_k(x) = e_k(x+m)$ for every integer m,

$$e_k(x_n) = e^{2\pi ikn\alpha}. \tag{70}$$

Hence

$$\begin{aligned}
\frac{1}{2N+1} \sum_{n=-N}^{N} e_k(x_n) &= \frac{1}{2N+1} \sum_{n=-N}^{N} e_k(n\alpha) \\
&= \frac{1}{2N+1} \sum_{n=-N}^{N} (e^{2\pi ik\alpha})^n \\
&= \begin{cases} 1 & \text{if } k = 0 \\ \frac{1}{2N+1}\frac{e_k(-N\alpha) - e_k((N+1)\alpha)}{1 - e_k(\alpha)} & \text{if } k > 0. \end{cases}
\end{aligned} \tag{71}$$

Now for a fixed irrational α, $|1 - e_k(\alpha)| > 0$. Therefore if $k \neq 0$:

$$\lim_{N\to\infty} \frac{1}{2N+1} \frac{e_k(-N\alpha) - e_k((N+1)\alpha)}{1 - e_k(\alpha)} = 0. \tag{72}$$

Let $P(x) = \sum_k a_k e_k(x)$ be a finite sum (ie, $P(x)$ is a trigonometric polynomial). By possibly adding some zero coefficients, we can write $P(x)$ as a sum over a symmetric range: $P(x) = \sum_{k=-K}^{K} a_k a_k(x)$.

Exercise 5.16. *Show* $\int_0^1 P(x)dx = a_0$.

By the above arguments, we have shown that for any (finite) trigonometric polynomial $P(x)$:

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} P(x_n) \ \to \ a_0 \ = \ \int_0^1 P(x)dx. \tag{73}$$

Consider two approximations to the characteristic function $\chi_{(a,b)}$:

1. f_{1m}: $f_{1m}(x) = 1$ if $a + \frac{1}{m} \leq x \leq b - \frac{1}{m}$, drops linearly to 0 at a and b, and is zero elsewhere.

2. f_{2m}: $f_{1m}(x) = 1$ if $a \leq x \leq b$, drops linearly to 0 at $a - \frac{1}{m}$ and $b + \frac{1}{m}$, and is zero elsewhere.

Note there are trivial modifications if $a = 0$ or $b = 1$. Clearly

$$f_{1m}(x) \ \leq \ \chi_{(a,b)}(x) \ \leq \ f_{2m}(x). \tag{74}$$

Therefore

$$\frac{1}{2N+1} \sum_{n=-N}^{N} f_{1m}(x_n) \ \leq \ \frac{1}{2N+1} \sum_{n=-N}^{N} \chi_{(a,b)}(x_n) \ \leq \ \frac{1}{2N+1} \sum_{n=-N}^{N} f_{2m}(x_n). \tag{75}$$

By Theorem 5.12, for each m, given $\epsilon > 0$ we can find trigonometric polynomials $P_{1m}(x)$ and $P_{2m}(x)$ such that $|P_{1m}(x) - f_{1m}(x)| < \epsilon$ and $|P_{2m}(x) - f_{2m}(x)| < \epsilon$.

As f_{1m} and f_{2m} are continuous functions, we can replace

$$\frac{1}{2N+1} \sum_{n=-N}^{N} f_{im}(x_n) \ \ \text{with} \ \ \frac{1}{2N+1} \sum_{n=-N}^{N} P_{im}(x_n) \tag{76}$$

at a cost of at most ϵ.

As $N \to \infty$,

$$\frac{1}{2N+1}\sum_{n=-N}^{N} P_{im}(x_n) \ \rightarrow \ \int_0^1 P_{im}(x)dx. \tag{77}$$

But $\int_0^1 P_{1m}(x)dx = (b-a)-\frac{1}{m}$ and $\int_0^1 P_{2m}(x)dx = (b-a)+\frac{1}{m}$. Therefore, given m and ϵ, we can choose N large enough so that

$$(b-a)-\frac{1}{m}-\epsilon \ \leq \ \frac{1}{2N+1}\sum_{n=-N}^{N} \chi_{(a,b)}(x_n) \ \leq \ (b-a)+\frac{1}{m}+\epsilon. \tag{78}$$

Letting m tend to ∞ and ϵ tend to 0, we see $\frac{1}{2N+1}\sum_{n=-N}^{N} \chi_{(a,b)}(x_n) \rightarrow b-a$. $\square$

Exercise 5.17. *Rigorously do the necessary book-keeping to prove the previous theorem.*

Exercise 5.18. *Prove*

1. *If $\alpha \in \mathbb{Q}$, then $\{n\alpha\}$ is periodic.*

2. *If $\alpha \notin \mathbb{Q}$, then no two $\{n\alpha\}$ are equal.*