Honors 1: Undergraduate Math Lab*

Peter Sarnak†, Steven Miller‡, Alex Barnett§

November 13th and 20th, 2002

Courant Institute of Mathematical Sciences
New York University
New York, NY

Abstract

Lecture by Peter Sarnak, notes by Steven Miller.

Contents

1 Increasing Length Subsequences **1**

2 Tracy-Widom **2**

1 Increasing Length Subsequences

Consider the set of permutations S_n on n numbers. Let $\sigma \in S_n$ be a random permutation, and let L_σ be the length of the longest increasing sub-sequence. What is the expected value of L_σ?

Conjecture 1.1 (Ulam).

$$E[L_\sigma] \ \sim \ 2\sqrt{n}. \tag{1}$$

*Homepage: `http://www.math.nyu.edu/~millerj/`

†E-mail: `sarnak@math.princeton.edu`

‡E-mail: `millerj@cims.nyu.edu` or `sjmiller@math.princeton.edu`

§E-mail: `barnett@nmr.mgh.harvard.edu`

Proved by several people (Schepp, Vircheck (?), ...).

At Bell Labs, many people (including Odlyzko) investigated. Monte-Carlo simulations for variance (beginning 1993). Dividing variance by $n^{\frac{1}{3}}$ was good. Looking at the expectation of

$$\frac{L_\sigma - 2\sqrt{n}}{n^{\frac{1}{6}}} \tag{2}$$

and investigated whether or not it went to a limit. Noticed this distribution is negative (shifted to the left). Prefers to be *less* than $2\sqrt{n}$.

2 Tracy-Widom

On $\mathbb{R}^n$, we have

$$P_N(x)dx \;=\; e^{-\sum_{j=1}^N x_j^2} \prod_{j<k} |x_j - x_i|^\beta \, dx_1 \cdots dx_n, \quad \beta \in \{1, 2, 4\}. \tag{3}$$

Will use the weight $e^{-x^2}dx$, which will give rise to Hermite polynomials.

Definition 2.1 **($F_{N,\beta}(s)$).** *$F_{N,\beta}(s)$ is the probability that there is no $x \in [s, \infty)$.*

We can write this as a determinant (see the papers by Mehta and Mehta-Gaudin). We will have again

$$K_N(x, y) \;=\; \sum_{0 \le j \le N-1} \phi_j(x)\overline{\phi}_j(y). \tag{4}$$

Remember the semi-circle rule, that (with some normalization) the eigenvalues lie in $[-2\sqrt{N}, 2\sqrt{N}]$.

What is the expected value of the largest eigenvalue? We know most are in $[-2\sqrt{N}, 2\sqrt{N}]$. We haven't discussed whether or not there are outliers. With probability one, we can show that there will be no such outliers.

We have N eigenvalues. Near 0 is called *the bulk*. As there are N numbers, we expect eigenvalues in an interval of size $\frac{1}{N}$ near the origin.

What about eigenvalues near the edges, $\pm 2\sqrt{N}$? Let $s \in (-\infty, \infty)$. Look at

$$F_N\left(2\sqrt{N} + \frac{s}{N^{\frac{1}{6}}}\right). \tag{5}$$

This is the scaling limit. Say $s = 0$. this would give us what is happening at the origin.

Theorem 2.2 (Tracy-Widom).

$$\lim_{N\to\infty} F_N\left(2\sqrt{N} + \frac{s}{N^{\frac{1}{6}}}\right) \;=\; F_\beta(s), \quad F_\beta(s) = \frac{dF_\beta(s)}{ds}. \tag{6}$$

Here the $N^{\frac{1}{6}}$ arises from the particular problem we're interested in. Here, we are looking at eigenvalues from random matrices.