Calculating the Level Density a la Katz-Sarnak

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Abstract

These are an expanded set of notes from [Mil1]. Below we calculate the 1 and 2-Level Densities for the Classical Compact Groups, using the determinant expansions from Katz-Sarnak.

1. 1- and 2-Level Density Kernels for the Classical Compact Groups

By [KS1], the m-level densities for the classical compact groups are

$$\begin{aligned}
W_{m,\epsilon}(x) &= \mathbf{det}\Big(K_\epsilon(x_i,x_j)\Big)_{i,j\leqslant m}\\
W_{m,O^+}(x) &= \mathbf{det}(K_1(x_i,x_j))_{i,j\leqslant m}\\
W_{m,O^-}(x) &= \mathbf{det}(K_{-1}(x_i,x_j))_{i,j\leqslant m} + \sum_{k=1}^{m}\delta(x_k)\mathbf{det}(K_{-1}(x_i,x_j))_{i,j\neq k}\\
&= (W_{m,O^-})_1(x) + (W_{m,O^-})_2(x)\\
W_{m,O}(x) &= \frac{1}{2}W_{m,O^+}(x) + \frac{1}{2}W_{m,O^-}(x)\\
W_{m,U}(x) &= \mathbf{det}(K_0(x_i,x_j))_{i,j\leqslant m}\\
W_{m,Sp}(x) &= \mathbf{det}(K_{-1}(x_i,x_j))_{i,j\leqslant m}
\end{aligned} \tag{1.1}$$

where $K(y) = \frac{\sin \pi y}{\pi y}$, $K_\epsilon(x,y) = K(x-y) + \epsilon K(x+y)$ for $\epsilon = 0, \pm 1$, O^+ denotes the group $SO(\text{even})$ and O^- the group $SO(\text{odd})$.

In all applications below, there are enormous simplifications as all functions are even (we do not need to worry about signs in the Fourier transform or the inversion formulas).

Definition 1.1 $I(x)$. *$I(x)$ will denote the characteristic function of* $[-1,1]$.

1.1 Needed Fourier Transforms

Let δ be the Dirac Delta functional: $\int f(x)\delta(x) = f(0)$. Let $I(x) = \chi_{[-1,1]}(x)$ be the characteristic function of the unit interval.

Lemma 1.2. $\widehat{1} = \delta$

Proof. This is proved in the theory of distributions. Formally, using duality, one can argue $\int f \cdot 1 = \widehat{f}(0) = \int \widehat{f} \cdot \delta$. □

Lemma 1.3. $\widehat{\chi_{[-\frac{1}{2},\frac{1}{2}]}}(u) = K(u)$

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Proof.

$$\begin{aligned}\int_{-\infty}^{\infty} \chi_{[-\frac{1}{2},\frac{1}{2}]}(x)e^{2\pi ixu}dx &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \Big[\cos(2\pi xu) + i\sin(2\pi xu)\Big]dx \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(2\pi xu)dx \\ &= \frac{\sin \pi u}{\pi u} \end{aligned} \tag{1.2}$$

□

Lemma 1.4. $\widehat{K(2x)}(u) = \frac{1}{2}I(u)$.

Proof.

$$\begin{aligned}\widehat{K(2x)}(u) &= \int_{-\infty}^{\infty} K(2x)e^{2\pi ixu}dx \\ &= \int_{-\infty}^{\infty} K(2x)e^{2\pi i2x(\frac{1}{2}u)}\frac{2dx}{2} \\ &= \frac{1}{2}\int_{-\infty}^{\infty} K(t)e^{2\pi it(\frac{1}{2}u)}dt \\ &= \frac{1}{2}\chi_{[-\frac{1}{2},\frac{1}{2}]}\left(\frac{1}{2}u\right) = \frac{1}{2}I(u). \end{aligned} \tag{1.3}$$

□

Lemma 1.5. $\widehat{K^2}(u) = (1-|u|)I(u)$.

Proof. We use duality for even functions: $\int f(x)g(x)dx = \int \widehat{f}(y)\widehat{g}(y)dy$. See [La2], pages $242-243$. Let $K_u(t) = K(t)e^{2\pi iut}$. Then $\widehat{K_u}(y) = \widehat{K}(y+u)$, and recall $\widehat{K}(y) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(y)$. As K is even, the arguments below are justified.

$$\begin{aligned}\int_{-\infty}^{\infty} K^2(t)e^{2\pi iut}dt &= \int_{-\infty}^{\infty} \Big(K(t)\Big)\Big(K(t)e^{2\pi iut}\Big)dt \\ &= \int_{-\infty}^{\infty} K(t)K_u(t)dt \\ &= \int_{-\infty}^{\infty} \chi_{[-\frac{1}{2},\frac{1}{2}]}(y)\chi_{[-\frac{1}{2},\frac{1}{2}]}(y+u)dy \end{aligned} \tag{1.4}$$

$\chi_{[-\frac{1}{2},\frac{1}{2}]}(y)\chi_{[-\frac{1}{2},\frac{1}{2}]}(y+u)$ is one on the intersection of $\{-\frac{1}{2} \leqslant y \leqslant \frac{1}{2}\}$ and $\{-\frac{1}{2} \leqslant y+u \leqslant \frac{1}{2}\}$ and zero elsewhere. If $|u| > 1$, $\chi_{[-\frac{1}{2},\frac{1}{2}]}(y)\chi_{[-\frac{1}{2},\frac{1}{2}]}(y+u) = 0$, and the integral vanishes. If $u \in [0,1]$, the intersection is $-\frac{1}{2} \leqslant y \leqslant \frac{1}{2} - u$, and integrating over y gives $1-u$. If $u \in [-1,0]$, it is one on the intersection of $\{-\frac{1}{2} \leqslant y \leqslant \frac{1}{2}\}$ and $\{-\frac{1}{2} \leqslant y - |u| \leqslant \frac{1}{2}\}$. We get $-\frac{1}{2} + |u| \leqslant y \leqslant \frac{1}{2}$, and integrating over y gives $1-|u|$. Therefore the Fourier Transform of K^2 is $(1-|u|)I(u)$. □

1.2 1-Level Densities

For $|u_1| \leqslant 1$, $\frac{1}{2}I(u_1) = -\frac{1}{2}I(u_1) + 1$.

$$\begin{aligned}W_{1,O^+}(x_1) &= \mathbf{det}\Big(K_1(x_i,x_j)\Big)_{i,j\leqslant 1} \\ &= K_1(x_1,x_1) = 1 + K(2x_1)\end{aligned}$$

$$\begin{aligned} &= 1(x_1) + K(2x_1) \\ \widehat{W_{1,O^+}}(u_1) &= \delta(u_1) + \frac{1}{2}I(u_1). \end{aligned} \tag{1.5}$$

$$\begin{aligned} W_{1,O^-}(x_1) &= \mathbf{det}\Big(K_{-1}(x_i, x_j)\Big)_{i,j\leqslant 1} \\ &\quad + \sum_{k=1}^{1} \delta(x_k)\mathbf{det}\Big(K_{-1}(x_i, x_j)\Big)_{i,j\neq 1} \\ &= K_{-1}(x_1, x_1) + \delta(x_1) \\ &= 1 - K(2x_1) + \delta(x_1) \\ &= 1(x_1) - K(2x_1) + \delta(x_1) \\ \widehat{W_{1,O^-}}(u_1) &= \delta(u_1) - \frac{1}{2}I(u_1) + 1(u_1). \end{aligned} \tag{1.6}$$

$$\begin{aligned} W_{1,Sp}(x_1) &= \mathbf{det}\Big(K_{-1}(x_i, x_j)\Big) \\ &= K_{-1}(x_1, x_1) \\ &= 1(x_1) - K(2x_1) \\ \widehat{W_{1,Sp}}(u_1) &= \delta(u_1) - \frac{1}{2}I(u_1). \end{aligned} \tag{1.7}$$

$$\begin{aligned} W_{1,U}(x_1) &= \mathbf{det}\Big(K_0(x_i, x_j)\Big) \\ &= K_0(x_1, x_1) = 1(x_1) \\ \widehat{W_{1,U}}(u_1) &= \delta(u_1). \end{aligned} \tag{1.8}$$

We have shown

Theorem 1.6 1-Level Densities.

$$\begin{aligned} \widehat{W_{1,O^+}}(u) &= \delta(u) + \frac{1}{2}I(u) \\ \widehat{W_{1,O}}(u) &= \delta(u) + \frac{1}{2} \\ \widehat{W_{1,O^-}}(u) &= \delta(u) - \frac{1}{2}I(u) + 1 \\ \widehat{W_{1,Sp}}(u) &= \delta(u) - \frac{1}{2}I(u) \\ \widehat{W_{1,U}}(u) &= \delta(u). \end{aligned} \tag{1.9}$$

For functions whose Fourier Transforms are supported in $[-1, 1]$, the three orthogonal densities are indistinguishable, though they are distinguishable from U and Sp. To detect differences between the orthogonal groups using the 1-level density, one needs to work with functions whose Fourier Transforms are supported beyond $[-1, 1]$.

1.3 Preliminaries for the 2-Level Densities

$$\begin{aligned} W_{2,\epsilon}(x) &= \mathbf{det}\Big(K_\epsilon(x_i, x_j)\Big)_{i,j\leqslant 2} \\ &= K_\epsilon(x_1, x_1)K_\epsilon(x_2, x_2) - K_\epsilon(x_1, x_2)K_\epsilon(x_2, x_1) \end{aligned}$$

$$\begin{aligned} &= \Big[1+\epsilon K(2x_1)\Big]\Big[1+\epsilon K(2x_2)\Big] - \\ &\quad \Big[K(x_1-x_2)+\epsilon K(x_1+x_2)\Big]\Big[K(x_2-x_1)+\epsilon K(x_1+x_2)\Big] \\ &= W_{2,\epsilon,a}(x) - W_{2,\epsilon,b}(x). \end{aligned} \tag{1.10}$$

We now calculate $\widehat{W_{2,\epsilon,a}}(u)$.

$$\begin{aligned} W_{2,\epsilon,a}(x) &= \Big[1+\epsilon K(2x_1)\Big]\Big[1+\epsilon K(2x_2)\Big] \\ &= 1+\epsilon K(2x_1)+\epsilon K(2x_2)+\epsilon^2 K(2x_1)K(2x_2) \\ &= 1(x_1)1(x_2)+\epsilon K(2x_1)1(x_2)+\epsilon 1(x_1)K(2x_2)+\epsilon^2 K(2x_1)K(2x_2) \\ \widehat{W_{2,\epsilon,a}}(u) &= \widehat{1(x_1)}\widehat{1(x_2)}+\epsilon\widehat{K(2x_1)}\widehat{1(x_2)}+\epsilon\widehat{1(x_1)}\widehat{K(2x_2)}+\widehat{K(2x_1)}\widehat{K(2x_2)} \\ &= \delta(u_1)\delta(u_2)+\frac{\epsilon}{2}I(u_1)\delta(u_2)+\frac{\epsilon}{2}\delta(u_1)I(u_2)+\frac{\epsilon^2}{4}I(u_1)I(u_2) \\ &\quad \text{where } \ I(u) == \ \chi_{[-1,1]}(u) \end{aligned} \tag{1.11}$$

It is straightforward to calculate the Fourier Transforms of the above, as each function is even and of the form $g_1(x_1)g_2(x_2)$. We also use the fact that $\widehat{g(2x)} = \frac{1}{2}\widehat{g}(\frac{x}{2})$.

We now calculate $\widehat{W_{2,\epsilon,b}}(u)$. Note that K is even, so $K(x_i - x_j) = K(x_j - x_i)$.

$$\begin{aligned} W_{2,\epsilon,b}(x) &= \Big[K(x_1-x_2)+\epsilon K(x_1+x_2)\Big]\Big[K(x_2-x_1)+\epsilon K(x_1+x_2)\Big] \\ &= K^2(x_1-x_2)+\epsilon^2 K^2(x_1+x_2)+2\epsilon K(x_1-x_2)K(x_1+x_2) \\ \widehat{W_{2,\epsilon,b}}(u) &= \widehat{T_-}(u_1,u_2)+\epsilon^2\widehat{T_+}(u_1,u_2)+2\epsilon\widehat{T_3}(u_1,u_2) \end{aligned} \tag{1.12}$$

Let $\eta = \pm 1$. Then

$$\widehat{T_\eta}(u_1,u_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K^2(x_1+\eta x_2)e^{2\pi i (u_1,u_2)\cdot(x_1,x_2)}dx_1dx_2. \tag{1.13}$$

Change variables: $t_1 = x_1 + \eta x_2, t_2 = x_2$. Then $x_1 = t_1 - \eta t_2, x_2 = t_2$, and the Jacobian is $+1$.

$$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -\eta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}. \tag{1.14}$$

Hence $dx_1dx_2 = dt_1dt_2$, and $(u_1,u_2)\cdot(x_1,x_2) = u_1t_1 + (-\eta u_1 + u_2)t_2$. Hence

$$\begin{aligned} \widehat{T_\eta}(u_1,u_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K^2(t_1)e^{2\pi i(u_1t_1+(-\eta u_1+u_2)t_2)}dt_1dt_2 \\ &= \int_{-\infty}^{\infty} K^2(t_1)e^{2\pi i u_1t_1}dt_1 \int_{-\infty}^{\infty} 1(t_2)e^{2\pi i(-\eta u_1+u_2)t_2}dt_2 \\ &= \int_{-\infty}^{\infty} K^2(t_1)e^{2\pi i u_1t_1}dt_1 \cdot \delta(-\eta u_1+u_2) \end{aligned} \tag{1.15}$$

We have previously shown the Fourier Transform of $K^2(x_1)$ is $(1-|u_1|)I(u_1)$. We therefore find

$$T_\eta(u_1,u_2) = \delta(-\eta u_1+u_2)\cdot(1-|u_1|)\cdot I(u_1). \tag{1.16}$$

We calculate $\widehat{T_3}(u_1, u_2)$, the Fourier Transform of $K(x_1 - x_2)K(x_1 + x_2)$. Change variables: $t_1 = x_1 - x_2$, $t_2 = x_1 + x_2$. Therefore $x_1 = \frac{1}{2}t_1 + \frac{1}{2}t_2$, $x_2 = -\frac{1}{2}t_1 + \frac{1}{2}t_2$. The Jacobian is the absolute value of the determinant of the transformation, which is $\frac{1}{2}$. In the exponential we have $u_1x_1 + u_2x_2$, which becomes $\frac{1}{2}(u_1 - u_2)t_1 + \frac{1}{2}(u_1 + u_2)t_2$.

$$\begin{aligned}
\widehat{T_3}(u_1, u_2) &= \int\int K(x_1 - x_2)K(x_1 + x_2)e^{2\pi i(u_1x_1+u_2x_2)}dx_1dx_2 \\
&= \int\int K(t_1)K(t_2)e^{2\pi i(\frac{1}{2}(u_1-u_2)t_1+\frac{1}{2}(u_1+u_2)t_2)}\frac{dt_1dt_2}{2} \\
&= \frac{1}{2}\int K(t_1)e^{2\pi i\frac{1}{2}(u_1-u_2)t_1}dt_1 \int K(t_2)e^{2\pi i\frac{1}{2}(u_1+u_2)t_2}dt_2 \\
&= \frac{1}{2}\chi_{[-\frac{1}{2},\frac{1}{2}]}(\frac{u_1 - u_2}{2})\chi_{[-\frac{1}{2},\frac{1}{2}]}(\frac{u_1 + u_2}{2}) \\
&= \frac{1}{2}I(u_1 - u_2)I(u_1 + u_2), \qquad (1.17)
\end{aligned}$$

where I is the characteristic function of $[-1, 1]$. If $|u_1| + |u_2| > 1$, the above vanishes; I is symmetric, and either $u_1 - u_2$ or $u_1 + u_2$ is $\pm(|u_1| + |u_2|)$. If $|u_1| + |u_2| \leqslant 1$, the above is 1. Hence

$$\widehat{T_3}(u_1, u_2) = \frac{1}{2}I(|u_1| + |u_2|). \qquad (1.18)$$

Collecting the pieces we obtain

$$\begin{aligned}
\widehat{W_{2,\epsilon}}(u) &= \widehat{T_-}(u_1, u_2) + \epsilon^2\widehat{T_+}(u_1, u_2) + 2\epsilon\widehat{T_3}(u_1, u_2) \\
&= \delta(u_1 + u_2) \cdot (1 - |u_1|)I(u_1) + \epsilon^2\delta(-u_1 + u_2) \cdot (1 - |u_1|)I(u_1) + \epsilon I(|u_1| + |u_2|). \qquad (1.19)
\end{aligned}$$

We have proved

Lemma 1.7 Expansion for $\widehat{W_{2,\epsilon}}(u)$. *Let* $K(y) = \frac{\sin \pi y}{\pi y}$, $K_\epsilon(x, y) = K(x - y) + \epsilon K(x + y)$, $\epsilon = \pm 1$, *and* $W_{2,\epsilon}(x) = \boldsymbol{det}(K_\epsilon(x_i, x_j))$. *We have*

$$\begin{aligned}
\widehat{W_{2,\epsilon}}(u) &= \widehat{W_{2,\epsilon,a}}(u) - \widehat{W_{2,\epsilon,b}}(u) \\
&= \delta(u_1)\delta(u_2) + \frac{\epsilon}{2}I(u_1)\delta(u_2) + \frac{\epsilon}{2}\delta(u_1)I(u_2) + \frac{\epsilon^2}{4}I(u_1)I(u_2) + \\
&\quad \big[\delta(u_1 + u_2) + \epsilon^2\delta(-u_1 + u_2)\big] \cdot (|u_1| - 1)I(u_1) - \epsilon I(|u_1| + |u_2|). \qquad (1.20)
\end{aligned}$$

By duality, $\int\int f_1(x_1)f_2(x_2)W_{2,\epsilon}(x)dx_1dx_2 = \int\int \widehat{f_1}(u_1)\widehat{f_2}(u_2)\widehat{W_{2,\epsilon}}(u)du_1du_2$. Note (since f_i is even)

$$\int\int \widehat{f_1}(u_1)\widehat{f_2}(u_2)\delta(\pm u_1 + u_2) \cdot (|u_1| - 1)I(|u_1|)du_1du_2 = \int_{-1}^{1}(|u| - 1)\widehat{f_1}(u)\widehat{f_2}(u)du. \qquad (1.21)$$

We simplify $\int \widehat{f_1}(u)\widehat{f_2}(u)du$. **We assume the support of each $\widehat{f_i}$ is at most 1; this will still allow us to deal with non-trivial regions, as we can have $|u_1| + |u_2| > 1$.** By duality (for even functions), as $\widehat{f_i}$ is supported in $(-1, 1)$,

$$\int_{-1}^{1}\widehat{f_1}(u)\widehat{f_2}(u)du = \int f_1(x)f_2(x)dx$$

$$= \int (f_1 f_2)(x) dx$$
$$= \widehat{f_1 f_2}(0). \tag{1.22}$$

Therefore, for even functions of the form $f_1(x_1)f_2(x_2)$ whose Fourier Transforms are supported in $|u_i| \leqslant 1$,

Lemma 1.8.

$$\begin{aligned}\int\int f_1(x_1)f_2(x_2)W_{2,\epsilon}(x)dx &= \widehat{f_1}(0)\widehat{f_2}(0) + \frac{\epsilon}{2}f_1(0)\widehat{f_2}(0) + \frac{\epsilon}{2}\widehat{f_1}(0)f_2(0) + \frac{\epsilon^2}{4}f_1(0)f_2(0) \\ &\quad + (1+\epsilon^2)\int_{-1}^{1}(|u|-1)\widehat{f_1}(u)\widehat{f_2}(u)du - \epsilon\int\int \widehat{f_1}(u_1)\widehat{f_2}(u_2)I(|u_1|+|u_2|)du_1du_2 \\ &= \left[\widehat{f_1}(0) + \frac{\epsilon}{2}f_1(0)\right]\left[\widehat{f_2}(0) + \frac{\epsilon}{2}f_2(0)\right] + \\ &\quad (1+\epsilon^2)\int_{-1}^{1}|u|\widehat{f_1}(u)\widehat{f_2}(u)du - (1+\epsilon^2)\widehat{f_1f_2}(0) \\ &\quad -\epsilon\int\int \widehat{f_1}(u_1)\widehat{f_2}(u_2)I(|u_1|+|u_2|)du_1du_2. \end{aligned} \tag{1.23}$$

We could replace $\widehat{f_1f_2}(0)$ by leaving the -1 in the $|u|$-integral:

$$\begin{aligned}\int\int f_1(x_1)f_2(x_2)W_{2,\epsilon}(x)dx &= \left[\widehat{f_1}(0) + \frac{\epsilon}{2}f_1(0)\right]\left[\widehat{f_2}(0) + \frac{\epsilon}{2}f_2(0)\right] + \\ &\quad (1+\epsilon^2)\int_{-1}^{1}(|u|-1)\widehat{f_1}(u)\widehat{f_2}(u)du - \epsilon\int\int \widehat{f_1}(u_1)\widehat{f_2}(u_2)I(|u_1|+|u_2|)du_1du_2. \end{aligned} \tag{1.24}$$

For $SO(even)$, setting $\epsilon = 1$ above yields the 2-level expansion, valid for any support.

1.4 2-Level Densities

We calculate the pieces needed to evaluate the densities (Equation 1.1). We calculate $\sum_{k=1}^{2}\delta(x_k)\mathbf{det}(K_{-1}(x_i,x_j))_{i,j\neq k}$; we've already calculated $W_{2,\epsilon}(x) = \mathbf{det}(K_\epsilon(x_i,x_j))$.

$$\begin{aligned}(W_{2,O^-})_2(x) &= \sum_{k=1}^{2}\delta(x_k)\mathbf{det}(K_{-1}(x_i,x_j))_{i,j\neq k} \\ &= \delta(x_1)K_{-1}(x_2,x_2) + \delta(x_2)K_{-1}(x_1,x_1) \\ &= \delta(x_1)\Big(1 - K(2x_2)\Big) + \delta(x_2)\Big(1 - K(2x_1)\Big) \\ &= \delta(x_1) + \delta(x_2) - \delta(x_1)K(2x_2) - \delta(x_2)K(2x_1) \\ &= \delta(x_1)1(x_2) + 1(x_1)\delta(x_2) - \delta(x_1)K(2x_2) - K(2x_1)\delta(x_2) \\ (\widehat{W_{2,O^-}})_2(u) &= 1(u_1)\delta(u_2) + \delta(u_1)1(u_2) - \frac{1}{2}1(u_1)I(u_2) - \frac{1}{2}I(u_1)1(u_2). \end{aligned} \tag{1.25}$$

We determine the effect of $(\widehat{W_{2,O^-}})_2(u)$ on $\widehat{f_1}(u_1)\widehat{f_2}(u_2)$ when $\text{supp}(f_i) < 1$.

$$\int\int \widehat{f_1}(u_1)\widehat{f_2}(u_2)(\widehat{W_{2,O^-}})_2(u) = \int\int \widehat{f_1}(u_1)\widehat{f_2}(u_2)1(u_1)\delta(u_2) + \int\int \widehat{f_1}(u_1)\widehat{f_2}(u_2)\delta(u_1)1(u_2)$$

$$\begin{aligned}
&-\frac{1}{2}\int\int \widehat{f_1}(u_1)\widehat{f_2}(u_2)1(u_1)I(u_2)\\
&-\frac{1}{2}\int\int \widehat{f_1}(u_1)\widehat{f_2}(u_2)I(u_1)1(u_2)\\
&= f_1(0)\widehat{f_2}(0)+\widehat{f_1}(0)f_2(0)-\frac{1}{2}f_1(0)f_2(0)-\frac{1}{2}f_1(0)f_2(0)\\
&= f_1(0)\widehat{f_2}(0)+\widehat{f_1}(0)f_2(0)-f_1(0)f_2(0). \qquad (1.26)
\end{aligned}$$

Theorem 1.9 $\mathcal{G} = SO(\text{even})$, O, or $SO(\text{odd})$. *Let O^ϵ represent SO(even) for $\epsilon = 1$ and SO(odd) for $\epsilon = -1$. For even functions supported with supp$(f_i) < 1$,*

$$\begin{aligned}
\int\int \widehat{f_1}(u_1)\widehat{f_2}(u_2)\widehat{W_{2,\mathcal{O}^\epsilon}}(u)du_1du_2 &= \left[\widehat{f_1}(0)+\frac{1}{2}f_1(0)\right]\left[\widehat{f_2}(0)+\frac{1}{2}f_2(0)\right]\\
&+2\int_{-1}^{1}(|u|-1)\widehat{f_1}(u)\widehat{f_2}(u)du-\epsilon\int\int \widehat{f_1}(u_1)\widehat{f_2}(u_2)I(|u_1|+|u_2|)du_1du_2\\
&-\frac{1-\epsilon}{2}f_1(0)f_2(0). \qquad (1.27)
\end{aligned}$$

Remark 1.10. *The term $\epsilon\int \widehat{f_1}(u_1)\widehat{f_2}(u_2)I(|u_1|+|u_2|)du_1du_2$ arises from $\int\int f_1(x_1)f_2(x_2)K(x_1-x_2)K(x_1+x_2)dx_1dx_2$.*

Remark 1.11. *Remember $I(|u_1|+|u_2|) = I(u_1-u_2)I(u_1+u_2)$.*

For arbitrarily small support, the three 2-level densities differ. One increases by a factor of $\frac{1}{2}f_1(0)f_2(0)$ moving from $\widehat{W_{2,O^+}}$ to $\widehat{W_{2,O}}$ to $\widehat{W_{2,O^-}}$ when each $\widehat{f_i}$ is supported in $(-\frac{1}{2},\frac{1}{2})$.

Remark 1.12. *For comparison purposes, we record the 2-level moment for SO(even). We assume $\widehat{\phi_1} = \widehat{\phi_2} = \widehat{\phi}$ is supported in $(-1,1)$. Thus, there are no combinatorial terms in the 2-level density arising from odd elements (as everything is even). We have to* add back *twice the 1-level density with test function ϕ^2 (this was because we were summing over $j_1 \neq \pm j_2$); thus, we add back $2D_{1,SO(even)}(\phi^2)$.*

If supp$(\widehat{\phi}) = \sigma \in (\frac{1}{2},1)$, then supp$(\widehat{\phi^2}) > 1$. Thus, as $\sigma < 1$,

$$\begin{aligned}
D_{1,SO(even)}(\phi^2) &= \widehat{\phi^2}(0)+\frac{1}{2}\int\widehat{\phi^2}(u)I(u)du\\
&= \int\widehat{\phi}(u)^2I(u)du+\frac{1}{2}\int\widehat{\phi^2}(u)I(u)du. \qquad (1.28)
\end{aligned}$$

We subtract twice this, as well as the mean, $\left[\widehat{\phi}(0)+\frac{1}{2}\phi(0)\right]^2$. We are left with the 2-level moment:

$$2\int|u|\widehat{\phi}(u)^2I(u)du-\int\int\widehat{\phi}(u)^2I(|u_1|+|u_2|)du_1du_2+\int\widehat{\phi^2}(u)I(u)du. \qquad (1.29)$$

I believe the above is the centered second moment, assuming $\widehat{\phi}$ is supported in $(-1,1)$. For support in $(-\frac{1}{2},\frac{1}{2})$, the last two terms add to zero. Further, the middle term comes from the $2K(x_1-x_2)K(x_1+x_2)$ term; in your notation, this was $2S(x_1-x_2)S(x_1+x_2)$.

To determine the density for Sp, we use Lemma 1.8 with $\epsilon = -1$. Rewriting the result in a similar form as the orthogonal densities yields

Theorem 1.13 $\mathcal{G} = Sp$. *For even functions supported with supp*$(f_i) < 1$,

$$\int\int \widehat{f_1}(u_1)\widehat{f_2}(u_2)\widehat{W_{2,Sp}}(u)du_1du_2 = \left[\widehat{f_1}(0) - \frac{1}{2}f_1(0)\right]\left[\widehat{f_2}(0) - \frac{1}{2}f_2(0)\right] + 2\int_{-1}^{1}(|u|-1)\widehat{f_1}(u)\widehat{f_2}(u)du + \int \widehat{f_1}(u_1)\widehat{f_2}(u_2)I(|u_1|+|u_2|)du_1du_2. \tag{1.30}$$

To calculate $W_{2,U}(x)$, we need to take the determinant of

$$\begin{pmatrix} 1 & \frac{\sin\pi(x_1-x_2)}{\pi(x_1-x_2)} \\ \frac{\sin\pi(x_1-x_2)}{\pi(x_1-x_2)} & 1 \end{pmatrix} \tag{1.31}$$

Thus we need the Fourier Transform of $1 - \left(\frac{\sin\pi(x_1-x_2)}{\pi(x_1-x_2)}\right)^2$. We find

Theorem 1.14 $\mathcal{G} = U$.

$$\widehat{W_{2,U}}(u) = \delta(u_1)\delta(u_2) - \delta(u_1+u_2)\cdot(1-|u_1|)I(u_1). \tag{1.32}$$

Thus

$$\int\int \widehat{f_1}(u_1)\widehat{f_2}(u_2)\widehat{W_{2,U}}du_1du_2 = \widehat{f_1}(0)\widehat{f_2}(0) + \int_{-1}^{1}(|u|-1)\widehat{f_1}(u)\widehat{f_2}(u)du. \tag{1.33}$$

Thus, for test functions with arbitrarily small support, the 2-level densities for the classical compact groups are mutually distinguishable.

Appendix A. Fourier Transform Simplifications

Let $I(u)$ be the characteristic function of $[-1,1]$. If $K(x) = \frac{\sin\pi x}{\pi x}$, note $\widehat{K(2x)}(u) = \frac{1}{2}I(u)$. All functions below will be even Schwartz functions whose Fourier Transforms have finite support. As all functions below are even, we can use $\widehat{\widehat{f}}(u) = f(u)$, $\int \widehat{f}(u)\widehat{g}(u)dt = \int f(x)g(x)dx$.

Lemma A.1.

$$\int_{u_1} I(|u_1|+|u_2|)e^{-2\pi i u_1 x_1}du_1 = \frac{\sin(2\pi(1-|u_2|)x_1)}{\pi x_1}. \tag{A.34}$$

Proof. This follows immediately from integrating. □

Lemma A.2.

$$\int_{u_2}\int_{u_1} \widehat{\phi}(u_2)\widehat{\phi}(u_1)I(|u_1|+|u_2|)du_1du_2 = \int_u \widehat{\phi^2}(u)I(u)du. \tag{A.35}$$

Proof. We have calculated the Fourier Transform of $I(|u_1|+|u_2|)$ with respect to u_1 above. Then

$$\int_{u_2}\int_{u_1} \widehat{\phi}(u_2)\widehat{\phi}(u_1)I(|u_1|+|u_2|)du_1du_2 = \int_{u_2}\widehat{\phi}(u_2)\int_{x_1}\phi(x_1)\frac{\sin(2\pi(1-|u_2|)x_1)}{\pi x_1}dx_1du_2. \tag{A.36}$$

When we expand $\sin(2\pi(1-|u_2|)x_1)$, as $\widehat{\phi}$ is even, we need only keep the even term in u_2, $\sin(2\pi x_1)\cos(2\pi|u_2|x_1)$. **This is wrong – it is $|u_2|$ which appears, not u_2; thus, we don't get**

a cancellation from oddness. As $\widehat{\phi}$ is even, we may replace $\cos(2\pi|u_2|x_1)$ with $e^{-2\pi i u_2 x_1}$. **What follows is just one of the terms, from the sin cos.** Thus,

$$\begin{aligned}\int_{u_2}\int_{u_1}\widehat{\phi}(u_2)\widehat{\phi}(u_1)I(|u_1|+|u_2|)du_1du_2 &= \int_{u_2}\widehat{\phi}(u_2)\int_{x_1}\phi(x_1)\frac{\sin(2\pi x_1)}{\pi x_1}e^{-2\pi i u_2 x_1}dx_1du_2\\ &= 2\int_{x_1}\phi(x_1)\frac{\sin(2\pi x_1)}{2\pi x_1}\int_{u_2}\widehat{\phi}(u_2)e^{-2\pi i u_2 x_1}du_2dx_1\\ &= 2\int_{x_1}\phi(x_1)K(2x)\cdot\phi(x_1)dx_1\\ &= 2\int_{x_1}\phi(x_1)^2K(2x)dx_1 \;=\; \int_u\widehat{\phi^2}(u)I(u)du. \quad (A.37)\end{aligned}$$

□

Lemma A.3. *$\widehat{\Psi_2}(x)=\phi(x)^2$ and $\widehat{\Psi_3}(x)=\phi(x)^2$.*

Proof. The Fourier Transform converts convolution to multiplication:

$$\begin{aligned}\Psi_2(u) &= \int_w\widehat{\phi}(w)\widehat{\phi}(u-w)dw\\ \widehat{\Psi_2}(x) &= \int_u\int_w\widehat{\phi}(w)\widehat{\phi}(u-w)e^{-2\pi i x u}dwdu\\ &= \int_w\widehat{\phi}(w)e^{-2\pi i w x}dw\int_u\widehat{\phi}(u)e^{-2\pi i x u}du \;=\; \phi(u)^2. \quad (A.38)\end{aligned}$$

Similarly $\widehat{\Psi_3}(x)=\widehat{\Psi_2}(x)\phi(x)=\phi(x)^3$. □

Lemma A.4. *$\widehat{\phi^2}(0)=\int\widehat{\phi}(u)^2I(u)du$.*

Proof. As $\mathrm{supp}(\phi)\subset(-1,1)$,

$$\widehat{\phi^2}(0) = \int_x\phi(x)^2dx = \int_x\phi(x)\phi(x)dx = \int_u\widehat{\phi}(u)\widehat{\phi}(u)du = \int_u\widehat{\phi}(u)^2I(u)du. \quad (A.39)$$

□

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