NEWMAN'S CONJECTURE IN VARIOUS SETTINGS

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Abstract. De Bruijn and Newman introduced a deformation of the Riemann zeta function $\zeta(s)$, and found a real constant Λ which encodes the movement of the zeros of $\zeta(s)$ under the deformation. The Riemann hypothesis (RH) is equivalent to $\Lambda \leq 0$. Newman made the conjecture that $\Lambda \geq 0$ along with the remark that "the new conjecture is a quantitative version of the dictum that the Riemann hypothesis, if true, is only barely so."

Newman's conjecture is still unsolved, and previous work could only handle the Riemann zeta function and quadratic Dirichlet L-functions, obtaining lower bounds very close to zero (for example, for $\zeta(s)$ the bound is at least $-1.14541 \cdot 10^{-11}$, and for quadratic Dirichlet L-functions it is at least $-1.17 \cdot 10^{-7}$). We generalize the techniques to apply to automorphic L-functions as well as function field L-functions. We further determine the limit of these techniques by studying linear combinations of L-functions, proving that these methods are insufficient.

We explicitly determine the Newman constants in various function field settings, which has strong implications for Newman's quantitative version of RH. In particular, let $\mathcal{D} \in \mathbb{Z}[T]$ be a square-free polynomial of degree 3. Let D_p be the polynomial in $\mathbb{F}_p[T]$ obtained by reducing $\mathcal{D}$ modulo p. Then the Newman constant Λ_{D_p} equals $\log \frac{|a_p(\mathcal{D})|}{2\sqrt{p}}$; by Sato–Tate (if the curve is non-CM) there exists a sequence of primes such that $\lim_{n\to\infty} \Lambda_{D_{p_n}} = 0$. We end by discussing connections with random matrix theory.

Contents

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1. Introduction

1.1. Newman's conjecture for the Riemann zeta function.

Let

$$\xi(s) \;=\; \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{1.1}$$

be the completed Riemann zeta function, and let

$$\Xi(x) \;=\; \xi\left(\frac{1}{2}+ix\right). \tag{1.2}$$

Because of the functional equation $\xi(s) = \xi(1-s)$, we know that $x \in \mathbb{R}$ implies $\Xi(x) \in \mathbb{R}$. In general, we allow x to be complex.

As $\Xi(x)$ decays rapidly as $x \to \infty$ along the real line, we have

$$\Xi(x) \;=\; \int_{-\infty}^{\infty} \Phi(u)e^{iux}\,du \;=\; \int_0^{\infty} \Phi(u)(e^{iux}+e^{-iux})\,du, \tag{1.3}$$

where $\Phi(u) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \Xi(x)e^{-iux}\,dx = \frac{1}{2\pi}\int_0^{\infty} \Xi(x)(e^{iux}+e^{-iux})\,dx$ is the Fourier transform of $\Xi(x)$. In the 1920s, Pólya introduced a "time" parameter t to Ξ, given as follows:

$$\Xi_t(x) \;:=\; \int_0^{\infty} e^{tu^2}\Phi(u)(e^{iux}+e^{-iux})\,du. \tag{1.4}$$

We refer to the process beginning with (1.1) and ending with (1.4) as *Pólya's setup.* For each $t \in \mathbb{R}$, (1.4) gives us a function in x that is both $\mathbb{C} \to \mathbb{C}$ and $\mathbb{R} \to \mathbb{R}$. For $t = 0$, we recover our original function Ξ. The Riemann Hypothesis (RH) is equivalent to the assertion that Ξ_0 has only real zeros. Pólya hoped to show that the function Ξ_t has only real zeros for all $t \in \mathbb{R}$, so RH would follow.

De Bruijn and Newman proved the following results about $\Xi_t(x)$.

Lemma 1.1 (De Bruijn [dB])**.** *If Ξ_t has only real zeros, then so does $\Xi_{t'}$ for all $t' > t$.*

Lemma 1.2 (Newman [New])**.** *There exists some $t \in \mathbb{R}$ such that Ξ_t has a non-real zero.*

In particular, Newman's result shows that what Pólya had been trying to prove was actually false. However, by combining the two results of De Bruijn and Newman, we see the following.

Corollary 1.3. *There exists a constant $\Lambda \in \mathbb{R}$ (called the De Bruijn–Newman constant) such that*

(1) *if $t \geq \Lambda$ then Ξ_t has only real zeros, and*
(2) *if $t < \Lambda$ then Ξ_t has a non-real zero.*

The Riemann hypothesis is equivalent to $\Lambda \leq 0$. Newman made the following complementary conjecture.

Conjecture 1.4 (Newman's conjecture)**.** *Let Λ be the De Bruijn–Newman constant. Then $\Lambda \geq 0$.*

In Newman's own words: "The new conjecture is a quantitative version of the dictum that the Riemann hypothesis, if true, is only barely so."

Csordas, Smith and Varga [CSV] used the differential equations governing the motion of the zeros to show that "unusually" close pairs of zeros can give lower bounds on Λ. [SGD] builds on this method of Csordas et. al. and achieves the current best-known lower-bound: $\Lambda \geq -1.14541 \times 10^{-11}$.

A key step in the argument of [CSV] is the following.

Lemma 1.5 (Theorem 2.2 of [CSV])**.** *If $\Xi_{t_0}(x)$ has a zero x_0 of order at least 2, then $t_0 \leq \Lambda$.*

Remark 1.6. *Observe that if we set $F(x,t) = \Xi_t(x)$, then F satisfies $\partial_t F + \partial_{xx} F = 0$, the* backwards heat equation. *This PDE provides physical intuition for why Lemma 1.5 is true: as we* decrease t, *the graph of Ξ_t changes in accordance to the diffusion of heat. If Ξ_{t_0} has a double zero, these zeros are likely to "pop off" the real line as we further decrease t. See Appendix A for an example of this phenomenon.*

Remark 1.7. *It is conjectured that all the zeros of $\zeta(s)$ are simple. If this is false, then Lemma 1.5 implies that Newman's conjecture is true. However, if the zeros of $\zeta(s)$ are indeed all simple, we cannot make any conclusions of Newman's conjecture. This is discussed in Remark 3.13.*

1.2. **Structure of this paper.** In Section 2, we look at conditions needed for a generalized version of Newman's conjecture. Stopple [Sto] has shown that Pólya's setup also holds for quadratic Dirichlet L-functions. We show it is possible to state a version of Newman's conjecture for automorphic L-functions. However, as we only see the same behavior as before and the arguments are similar, we content ourselves with just describing how to extend the previous work here. We then quickly move on to rational function fields $\mathbb{F}_q(T)$, where new behavior emerges.

As in the number field case, each quadratic Dirichlet L-function $L(s, \chi_D)$ in the function field setting also gives rise to a constant Λ_D. This case, which we look at in Section 3 (the main section of the paper), exhibits *very* different behavior. First of all, RH is true, so we know $\Lambda_D \leq 0$. Second, the statement of Newman's conjecture is different.

Whereas Newman's conjecture in number fields is $\Lambda \geq 0$, it is not necessarily true that $\Lambda_D \geq 0$ in the function field setting. In fact, we can have $\Lambda_D = -\infty$. However, if we look at certain "families" $\mathcal{F}$ of L-functions (as discussed in Section 3.4), we have reason to believe that the supremum of Λ_D over these family is nonnegative.

Conjecture 1.8 (Newman's conjecture in the function fields setting)**.** *Let $\mathcal{F}$ be a family of L-functions over a function field. Then*

$$\sup_{D \in \mathcal{F}} \Lambda_D = 0. \tag{1.5}$$

For an example of a family, suppose we fix an elliptic curve $y^2 = \mathcal{D}(T)$ over $\mathbb{Q}$, and look at the polynomials $D_p \in \mathbb{F}_p[T]$ obtained by reducing $\mathcal{D}$ modulo p. Let $a_p(\mathcal{D})$ be the trace of Frobenius of the elliptic curve $y^2 = \mathcal{D}(T)$. In Section 3.5, we prove that Newman's conjecture is true for this family, and explicitly relate the Newman constant to $a_p(\mathcal{D})$.

Theorem 1.9 (Newman's conjecture for fixed $\mathcal{D}$, $\deg \mathcal{D} = 3$)**.** *Let $\mathcal{D} \in \mathbb{Z}[T]$ be a square-free polynomial of degree* 3*. Let D_p be the polynomial in $\mathbb{F}_p[T]$ obtained by reducing $\mathcal{D}$ modulo p. Then*

$$\Lambda_{D_p} \;=\; \log \frac{|a_p(\mathcal{D})|}{2\sqrt{p}}, \tag{1.6}$$

which implies $\sup_p \Lambda_{D_p} = 0$*.*

To show the supremum is zero, we use the recent proof of the Sato–Tate conjecture [BLGHT,CHT,HSBT,Tay]. This implies that the Newman conjecture for function fields is connected to deep results in number theory.

Next, we change our approach to Newman's conjecture in function fields and use results from random matrix theory to support our conjecture. By relating random matrix theory statistics to the distributions of the zeros of our functions $\Xi(x, \chi_D)$, we prove Newman's conjecture for a different family. For detailed statements, see Section 3.8.

Finally, in Appendix B we examine the results of some numerical computations. In particular, we observe that as we increase the degree, we find elements $D \in \mathbb{F}_3[T]$ such that the Newman constants Λ_D approach zero.

2. Conditions for a generalized Newman's conjecture

As the results and proofs in this section are similar to existing results in the literature, we content ourselves with quickly sketching the extensions to other automorphic forms.

2.1. Stopple's generalization of Newman's conjecture.

Stopple [Sto] showed that if D is a fundamental discriminant and $\chi_D(n)$ is the Kronecker symbol $(\frac{D}{n})$, then we can apply Pólya's setup for $\zeta(s)$ to the Dirichlet L-function $L(s, \chi_D)$. This gives us an analogue of (1.4):

$$\Xi_t(x, \chi_D) \;:=\; \int_0^\infty e^{tu^2} \Phi(u, \chi_D)(e^{iux} + e^{-iux})\, du. \tag{2.1}$$

Each D has its own De Bruijn–Newman constant Λ_D, and most of the techniques in [CSV] for attaining lower bounds on Λ carry over to Λ_D.

Conjecture 2.1 (Newman's conjecture for quadratic Dirichlet L-functions)**.** *Let $D \in \mathbb{Z}$ be a fundamental discriminant. Then $\Lambda_D \geq 0$.*

Stopple investigated a variation of this conjecture.

Conjecture 2.2 (Newman's conjecture for quadratic Dirichlet L-functions, weaker form)**.** *We have $\sup_D \Lambda_D \geq 0$, where the supremum is taken over all fundamental discriminants D.*

Note that Conjecture 2.1 implies Conjecture 2.2. Instead of looking for close pairs of zeros along the real line, Stopple looks for L-functions $L(s, \chi_D)$ with "unusually" low-lying zeros.[1] If an L-function has an unusually low-lying zero γ, then the zeros $\pm\gamma$ would then form a close pair.

Stopple found that for $D = -175990483$, we have $-1.13 \cdot 10^{-7} < \Lambda_D$.

2.2. **Sufficient conditions for generalization.** Let $L(s, f)$ be the L-function associated with some object f. In accordance with notation introduced earlier, let $\xi(s, f)$ be the completed L-function and let $\Xi(x, f) = \xi(\frac{1}{2} + ix, f)$.

If we try to define $\Xi_t(x, f)$ analogously, we need the following.

(1) $\Xi(\,\cdot\,, f)$ has to restrict to a $\mathbb{R} \to \mathbb{R}$ function, so that we can define the Fourier transform $\Phi(u, f)$, as in (1.3). It is sufficient to have the functional equation $\xi(s, f) = \xi(1 - s, f)$.
(2) $\Phi(u, f)$ has to have extremely rapid decay in order for the integral in (1.4) to converge for each $t \in \mathbb{R}$. It is sufficient to have $\Phi(u, f) = O(e^{-|u|^{2+\epsilon}})$ for some $\epsilon > 0$.

Usually, the rapid decay of $\Phi(u, f)$ can be seen because it has an infinite sum representation. For instance, in the case of the Riemann zeta function, we have

$$\Phi(u) \;=\; 2\sum_{n=1}^{\infty}(2n^4\pi^2 e^{9u/2} - 3n^2\pi e^{5u/2})e^{-n^2\pi e^{2u}}, \tag{2.2}$$

which shows that $\Phi(u)$ decays as quickly as a double exponential. See [Tit, (10.1.4)].

2.3. **Slight modifications of Pólya's setup and limitations.** The most stringent requirement is the even functional equation: $\xi(s, f) = \xi(1 - s, f)$. This is why Stopple did not investigate *all* Dirichlet L-functions – the complex L-functions do not have the proper type of symmetry.

In general, a completed L-function satisfies a functional equation of the form $\xi(s, f) = \epsilon_f \xi(1 - s, \overline{f})$, where $|\epsilon_f| = 1$. To obtain an even functional equation, we need f to be self-dual (i.e., $f = \overline{f}$) and we need the root number ϵ_f to be 1.

There are two straightforward attempts to "fix" a bad functional equation, but they both fail when we attempt to state Newman's conjecture for the L-function.

[1] Let the positive zeros of $\Xi(x, \chi_D)$ be denoted $\gamma_1, \gamma_2, \ldots$ with $0 < \gamma_1 < \gamma_2 < \cdots$. Then the zero γ_1 is "unusually" low-lying, in the sense given in [Sto, (15)], if

$$5\gamma_1^2 \sum_{|j| \geq 2} \left[\frac{1}{(-\gamma_1 - \gamma_j)^2} + \frac{1}{(\gamma_1 - \gamma_j)^2}\right] \;<\; 1.$$

Stopple calls such D "Low discriminants" ("Low" is a person's name).

(1) We replace $\xi(s,f)$ with $\tilde{\xi}(s,f) := |\xi(s,f)|$. Then $\tilde{\xi}(s,f) = \tilde{\xi}(1-s,f)$ for any L-function. However, $\tilde{\xi}(s,f)$ is no longer a smooth function in s. Thus, we lose the backwards heat equation and other desirable properties and the results of [CSV] do not carry over.
(2) We replace $\xi(s,f)$ with $\tilde{\xi}(s,f) := |\xi(s,f)|^2$, which is smooth in s. In this case we have an analogue of Lemma 1.5, but since every zero of $\tilde{\xi}(s,f)$ is doubled, the lemma would make Newman's conjecture for $\tilde{\xi}(s,f)$ trivially true.

If we have an L-function with odd functional equation $\xi(s,f) = -\xi(1-s,f)$, what we *can* do is define $\tilde{\xi}(s,f) = \frac{i}{s-1/2}\xi(s,f)$, which will then satisfy the conditions in Section 2.2.

Alternatively, we can consider products of different L-functions. For example, if we have two odd L-functions $\xi(s,f)$ and $\xi(s,g)$, then the product $\tilde{\xi}(s) := \xi(s,f)\xi(s,g)$ has the desired even symmetry. Thus there is a Pólya setup for $\tilde{\xi}$, and a corresponding constant Λ. In this case, it is the distribution of union of the zeros of each L-function that become relevant. (If the two L-functions share a zero then we have a double zero and Newman's conjecture is trivially true.)

Because of the lack of a proper functional equation, we cannot generalize Pólya's setup to (for example) the Hurwitz zeta function or linear combinations of L-functions.

2.4. **Automorphic L-functions.** One class of examples which can be analyzed with these methods is $H_k^+(N)$, the holomorphic cuspidal newforms of weight k and level N with even functional equation.

Lemma 2.3. *As in [ILS, Section 3], consider the Hecke L-function given by $L(s,f) = \sum_{n\geq 1}\lambda_f(n)n^{-s}$ for $f \in H_k^+(N)$, and let $\Xi(x,f)$ be the completed L-function evaluated at $s = \frac{1}{2}+ix$. Then we can follow Pólya's setup and introduce the analogous deformation $\Xi_t(x,f)$, so there is a De Bruijn–Newman constant Λ_f for each $f \in H_k^+(N)$.*

Proof. By definition of $H_k^+(N)$, the L-functions have even symmetry. Also, we have

$$\Xi(x,f) \ = \ \int_0^\infty \Phi(u)(e^{iux}+e^{-iux})\,du, \tag{2.3}$$

where

$$\Phi(u,f) \ = \ \left(\frac{2\pi}{\sqrt{N}}\right)^{(k-1)/2} \sum_{n\geq 1} \lambda_f(n) n^{(k-1)/2} \exp\left(-\frac{2\pi n}{\sqrt{N}}e^u + \frac{k}{2}u\right), \tag{2.4}$$

which shows that $\Phi(u)$ decays rapidly as $u \to \infty$. Thus, both conditions described in Section 2.2 are satisfied. □

Conjecture 2.4 (The generalized Newman's conjecture for $H_k^+(N)$)**.** *Let $f \in H_k^+(N)$. Then $\Lambda_f \geq 0$.*

In fact, most of the results in [CSV] and [Sto] on lower bounds of Λ_f carry over. However, while we are able to make a Newman's conjecture in the automorphic forms setting, we see only the same behavior as before. Thus in the next section we focus our attention on function field L-functions, where many new phenomena appear.

3. Newman's conjecture for function fields

In this section we explore Newman's conjecture for function fields. The situation is very different than the number field case due to the fact that the Riemann hypothesis is known. For example, we are able to find L-functions where the associated constant is $-\infty$! This result indicates that some care is needed in formulating the correct analogue. Briefly, we show that for certain families of L-functions then, in the limit, the constants converge to zero. A key ingredient in our analysis is the recent proof of the Sato-Tate conjecture for elliptic curves without complex multiplication. Interestingly, all that is needed for the proof is that for such an elliptic curve E there is a sequence of primes p_n such that the normalized coefficients of its L-function, $a_{p_n}(E)/2\sqrt{p_n}$, converge to 1; we are unaware of an elementary proof of this fact.

3.1. Background on function fields.

In the function fields setting, the appropriate analogue of $\mathbb{Z}$ is $\mathbb{F}_q[T]$, the coordinate ring of the affine line over $\mathbb{F}_q$. The background introduced here is given in more detail in [Rud, Section 2] or [AK, Section 3]. For a comprehensive text on number theory in function fields, see [Ros].

Definition 3.1. *Let q be an odd prime power and let $D \in \mathbb{F}_q[T]$. For this paper, we will say that (q, D) is a* good pair *if*

- *D is square-free and monic,*
- *$\deg D$ is odd,*
- *$\deg D \geq 3$.*

For (q, D) a good pair, let $\chi_D : \mathbb{F}_q[T] \to \{-1, 0, 1\}$ be the quadratic character modulo D. That is, $\chi_D(f) = (\frac{D}{f})$, where $(\frac{\cdot}{\cdot})$ is the Kronecker symbol.

Remark 3.2. *We assume q is odd because if a field has characteristic 2, then every element is a perfect square. We assume D is square-free and monic because this corresponds to the fundamental discriminants in the number field setting.*

We assume $\deg D$ is odd only for simplicity and ease of exposition. The case when $\deg D$ is even can be handled similarly with some modifications.

For (q, D) a good pair, we define the L-function

$$L(s, \chi_D) \ := \ \sum_{f \text{ monic}} \frac{\chi_D(f)}{|f|^s}. \tag{3.1}$$

By collecting terms, we can write

$$L(s, \chi_D) \ = \ \sum_{n=0}^{\infty} c_n (q^{-s})^n, \tag{3.2}$$

where

$$c_n \ = \ \sum_{\substack{f \text{ monic} \\ \deg f = n}} \chi_D(f). \tag{3.3}$$

It can be shown that $c_n = 0$ for all $n \geq \deg D$, so $L(s, \chi_D)$ is a polynomial in q^{-s} of at most degree $\deg D - 1$. In fact, the degree is exactly $\deg D - 1$.

Let $g = \frac{1}{2}(\deg D - 1)$; we use the letter g because the value of g is the genus of the hyperelliptic curve $y^2 = D(T)$. The completed L-function $\xi(s, \chi_D) := q^{gs} L(s, \chi_D)$

satisfies the functional equation $\xi(s,\chi_D) = \xi(1-s,\chi_D)$. Note that this satisfies the symmetry type we need, as discussed in Section 2.

Remark 3.3. *By the Riemann Hypothesis for curves over a finite field, proved by Andre Weil, we know that all the zeros of $L(s,\chi_D)$ lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. (A detailed exposition of Bombieri's proof is given in the appendix of [Ros].)*

Using the functional equation, we can write

$$\Xi(x,\chi_D) \;:=\; \Lambda\left(\frac{1}{2} + i\frac{x}{\log q}, \chi_D\right) \;=\; \Phi_0 + \sum_{n=1}^{g} \Phi_n \cdot (e^{inx} + e^{-inx}) \tag{3.4}$$

for some constants

$$\Phi_n \;=\; c_{g-n} q^{n/2} \;=\; c_{g+n} q^{-n/2}; \tag{3.5}$$

the two equivalent expressions for Φ_n are due to the symmetry of the completed L-function.

3.2. **Introducing the t parameter.** We observe that the right side of (3.4) gives the Fourier series of our completed L-function. We can introduce a new parameter as in (2.1), and find

$$\Xi_t(x,\chi_D) \;:=\; \Phi_0 + \sum_{n=1}^{g} \Phi_n e^{tn^2}(e^{inx} + e^{-inx}). \tag{3.6}$$

Remark 3.4. *What we call Φ_n here is the analogue of $\Phi(u)$ defined in the number field setting. In both cases, Φ is the Fourier transform of Ξ. The difference is that in the number field setting, $\Xi(x)$ is a function on $\mathbb{R}$ with rapid decay as $|x| \to \infty$, whereas here in the function field $\Xi(x)$ is now defined on the circle ($x \in [0, 2\pi]$). This is the reason that Φ is now a $\mathbb{Z} \to \mathbb{R}$ function.*

In order to guarantee the existence of a De Bruijn–Newman constant Λ_D, we need the following analogue of Lemma 1.1.

Lemma 3.5. *Suppose for some t that $\Xi_t(x,\chi_D)$ has only real zeros. Then for all $t' > t$, $\Xi_{t'}$ has only real zeros.*

Lemma 3.5 immediately follows from the following lemma.

Lemma 3.6 (Analogue of Theorem 13 in [dB])**.** *Suppose $F : \mathbb{Z} \to \mathbb{C}$ satisfies $\sum |F(n)| < \infty$, $F(n) = \overline{F(-n)}$ and $F(n) = O(e^{-|u|^{2+\epsilon}})$ for some $\epsilon > 0$. Also suppose that the roots of $\sum_{n=-\infty}^{\infty} F(n)e^{inx}$ satisfy $|\mathrm{Im}\, z| \le \Delta$ for some $\Delta \ge 0$. Then all the roots of $\sum_{n=-\infty}^{\infty} F(n)e^{tn^2}e^{inx}$ lie in the strip $|\mathrm{Im}\, z| \le \max(\Delta^2 - 2t, 0)^{1/2}$.*

Proof. The key idea is to take (3.6) in De Bruijn's paper, which is the trigonometric integral $f(z) = \int_{-\infty}^{\infty} F(t)e^{izt}dt$, and replace it with the trigonometric sum $f(z) = \sum_{n=-\infty}^{\infty} F(n)e^{inx}$. Then we note that the arguments to Theorems 11, 12, and 13 in De Bruijn's paper can be generalized to this situation. □

Proof of Lemma 3.5. Let

$$F(n) \;=\; \begin{cases} \Phi_{|n|} & \text{if } |n| \le g \\ 0 & \text{if } |n| > g, \end{cases} \tag{3.7}$$

and apply Lemma 3.6. □

Remark 3.7. *Lemma 3.5 can be phrased as "zeros on the real line remain on the real line." Lemma 3.6 gives us more than that. It also tells us that the zeros off the real line move towards the line, and furthermore provides a lower bound on the speed at which the zeros move.*

For instance, if we know all the zeros of $\Xi_{t_0}(x,\chi_D)$ lie in the strip $|\operatorname{Im} x| \le \Delta$, then we know all the zeros are real by the time $t = t_0 + \frac{1}{2}\Delta^2$. In the number field case, since we know the zeros of $\Xi(x)$ (for the Riemann zeta function) lie in the critical strip $|\operatorname{Im}(x)| \le \frac{1}{2}$, we know that $\Lambda \le \frac{1}{2}$.

By Lemma 3.5 and RH (see Remark 3.3), we know that for each good pair (q, D), there exists a constant $\Lambda_D \in [-\infty, 0]$ such that

(1) if $t \ge \Lambda$ then $\Xi_t(\,\cdot\,,\chi_D)$ has only real zeros, and
(2) if $t < \Lambda$ then $\Xi_t(\,\cdot\,,\chi_D)$ has a non-real zero.

Note that we have not eliminated the possibility of $\Lambda_D = -\infty$. However, it turns out that the analogue of Lemma 1.2 is false in the function field setting; that is, there are L-functions with the property that $\Xi_t(\,\cdot\,,\chi_D)$ has only real zeros for *all* t. (Remark 3.10 contains an example.)

There is a partial analogue, which holds for irreducible D. This at least assures us that $\Lambda_D \ne -\infty$ often.

Lemma 3.8. *Let (q, D) be a good pair and suppose D is irreducible. Then there exists some $t \in \mathbb{R}$ such that Ξ_t has a non-real zero.*

Proof. First we show that $\Phi_n \ne 0$ for all $0 \le n \le g$. Using (3.3) and (3.5), we have

$$\frac{\Phi_n}{q^{n/2}} \;=\; c_{g-n} \;=\; \sum_{\substack{f \text{ monic} \\ \deg f \,=\, g-n}} \chi_D(f). \tag{3.8}$$

Since q is odd, the number of terms in the sum is odd. Every f in the sum is relatively prime to D, since $g - n < 2g + 1 = \deg D$. Hence, every term in the sum is either $+1$ or -1. Thus c_{g-n} is odd, so $\Phi_n \ne 0$.

Using the fact that $\Phi_n \ne 0$, we can complete the proof of the lemma. For very negative t (i.e., as $t \to -\infty$), the main terms of $\Xi_t(x,\chi_D)$ are $\Phi_0 + \Phi_1 e^t(e^{ix} + e^{-ix})$. If x is a zero, we have

$$|\Phi_0/\Phi_1| e^{|t|} \;\approx\; |e^{ix} + e^{-ix}|. \tag{3.9}$$

As $t \to -\infty$, the left side goes to ∞ (since $\Phi_0 \ne 0$), so for some t, the left side exceeds 2, which means x cannot be real. □

Remark 3.9. *We can see from the proof of Lemma 3.8 that the conclusion of the lemma holds if at least two of the Fourier coefficients of $\Xi_t(x,\chi_D)$ are nonzero.*

Remark 3.10. *An example of an L-function with $\Lambda_D = -\infty$ is $D = T^3 + T \in \mathbb{F}_3[T]$. For this polynomial, $\Xi_t(x,\chi_D) = \sqrt{3}e^t \cos x$. As expected, D is not irreducible – we have $D = T(T^2 + 1)$.*

3.3. **The failure of Newman's conjecture for individual L-functions.** Using the results of the previous section, we know that for each L-function $L(s,\chi_D)$, there is a De Bruijn–Newman constant Λ_D.

At first, the "obvious" generalization of Newman's conjecture to this setting is that $\Lambda_D \geq 0$ for each D. However, this is false. Remark 3.10 provides an example with $\Lambda_D = -\infty$. Appendix A provides an example with $-\infty < \Lambda_D < 0$. In fact, for most (if not all) D, Λ_D will be strictly negative.

Lemma 3.11. *Let (q, D) be a good pair. If $\Xi_0(x, \chi_D)$ has only simple zeros, then $\Lambda_D < 0$.*

Proof. The two key ideas of this technical argument are to use the implicit function theorem, and to note that there are only finitely many zeros (which is very different than the number field cases). Write $F(x, t) = \Xi_t(x)$. Suppose γ is a simple zero of $\Xi_0(x, \chi_D)$. By the implicit function theorem, we can find a time interval $(-\epsilon, 0]$ and a function $y : (-\epsilon, 0] \to \mathbb{R}$ defined on this time interval such that $y(0) = \gamma$ and $F(y(s), s) = 0$ for all $s \in (-\epsilon, 0]$.

Because $\Phi_g \neq 0$, we know that $\Xi_0(x, \chi_D)$ has exactly $2g$ zeros (with multiplicity) in a period $0 \leq \mathrm{Re}(x) < 2\pi$. Suppose all these zeros are simple, so we can write them as $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_{2g} < 2\pi$. (We know the zeros of Ξ_0 are real because of Remark 3.3.)

For each zero, there is a time interval $(-\epsilon_n, 0]$ such that the zero γ_n stays real in this interval. Then all the zeros stay real inside the time interval $(-\epsilon, 0]$, where $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_{2g}\}$. Finally, since $\Xi_t(x, \chi_D)$ has exactly $2g$ zeros in $0 \leq \mathrm{Re}(x) < 2\pi$ for every t, we have accounted for all of them. □

Remark 3.12. *It is not known whether there exists a good pair (q, D) such that $\Xi_0(x, \chi_D)$ has a double zero.*

Remark 3.13. *There is no analogue of Lemma 3.11 in the number field setting. A crucial part of the argument is the periodicity of $\Xi_t(x, \chi_D)$. Thus, ϵ is the minimum of a finite set of positive numbers (as opposed to the infimum of an infinite set), which allows us to conclude that ϵ is strictly positive.*

3.4. **Newman's conjecture for families of L-functions.** Because of Lemma 3.11, we do not look at individual L-functions. Instead, following Stopple, we study families of L-functions.

Conjecture 3.14 (Newman's conjecture, fixed q)**.** *Keep q, the number of elements of the finite field, fixed. Then*

$$\sup_{(q,D)\ \text{good pair}} \Lambda_D \ \geq\ 0. \tag{3.10}$$

Conjecture 3.15 (Newman's conjecture, fixed g)**.** *Keep g, the genus, fixed. Then*

$$\sup_{\substack{\deg D\ =\ 2g+1 \\ (q,D)\ \text{good pair}}} \Lambda_D \ \geq\ 0. \tag{3.11}$$

Conjecture 3.16 (Newman's conjecture, fixed $\mathcal{D}$)**.** *Fix $\mathcal{D} \in \mathbb{Z}[T]$ square-free. For each prime p, let D_p be the polynomial in $\mathbb{F}_p[T]$ obtained by reducing $\mathcal{D}$ modulo p. Then*

$$\sup_{(p,D_p)\ \text{good pair}} \Lambda_{D_p} \ \geq\ 0, \tag{3.12}$$

Remark 3.17. *As RH has been proved in this setting in the conjectures above, we could replace the greater than or equal to* 0 *with equal to* 0*; we wrote it as above to remind the reader of the analogues of Newman's conjecture in the number field setting.*

More generally, let $\mathcal{F}$ be a set of polynomials D belonging to good pairs (q, D) (where q can vary). The corresponding family of L-functions is $\{L(s, \chi_D) : D \in \mathcal{F}\}$. (We often use $\mathcal{F}$ to refer to not only the family of polynomials but also the family of L-functions.) The Newman's conjecture for a family $\mathcal{F}$ is the statement that

$$\sup_{D \in \mathcal{F}} \Lambda_D \ \geq \ 0. \tag{3.13}$$

The families $\mathcal{F}$ corresponding to Conjecture 3.14, Conjecture 3.15 and Conjecture 3.16, respectively, are

- Fix q and let $\mathcal{F} = \{D : (q, D) \text{ is a good pair}\}$.
- Fix g and let $\mathcal{F} = \{D : (q, D) \text{ is a good pair}, \deg D = 2g + 1\}$.
- Fix $\mathcal{D} \in \mathbb{Z}[T]$ square-free and let $\mathcal{F} = \{D_p : (p, D_p) \text{ is a good pair}\}$.

3.5. **The case** $\deg \mathcal{D} = 3$ **and the Sato–Tate Conjecture.** We examine a special case of Conjecture 3.16 in which the fixed square-free polynomial $\mathcal{D} \in \mathbb{Z}[T]$ satisfies $\deg \mathcal{D} = 3$, so $g = 1$. In this section we prove this special case of Newman's conjecture.

For a fixed $\mathcal{D}$ of degree 3, the corresponding Ξ functions have the form

$$\Xi_t(x, \chi_{D_p}) \ = \ -a_p(\mathcal{D}) + 2\sqrt{q}e^t \cos x, \tag{3.14}$$

where

$$a_p(\mathcal{D}) \ = \sum_{\substack{f \in \mathbb{F}_p[T] \\ \deg f = 1 \\ f \text{ monic}}} \chi_{D_p}(f). \tag{3.15}$$

Note that $a_p(\mathcal{D})$ is the trace of Frobenius of the elliptic curve $y^2 = \mathcal{D}(T)$. In this setting, we get an explicit formula for Λ_{D_p}.

Lemma 3.18. *Let* $\mathcal{D} \in \mathbb{Z}[T]$ *be a square-free polynomial of degree* 3*. Let* D_p *be the polynomial in* $\mathbb{F}_p[T]$ *obtained by reducing* $\mathcal{D}$ *modulo* p*. Then*

$$\Lambda_{D_p} \ = \ \log \frac{|a_p(\mathcal{D})|}{2\sqrt{p}}. \tag{3.16}$$

Proof. Fix t and suppose x_0 is a zero of Ξ_t. Then

$$\cos x_0 \ = \ \frac{1}{e^t} \cdot \frac{2\sqrt{p}}{a_p(\mathcal{D})}. \tag{3.17}$$

If $e^t \geq \frac{|a_p(\mathcal{D})|}{2\sqrt{p}}$, then $-1 \leq \cos x_0 \leq 1$, so x_0 is real. On the other hand, if $e^t < \frac{|a(\mathcal{D})|}{2\sqrt{q}}$, then $|\cos x_0| > 1$, which implies x_0 is not real. □

In order to show that $\sup_p \Lambda_{D_p} = 0$, we need a sequence of primes $p_1, p_2, \ldots$ such that

$$\lim_{n \to \infty} \frac{a_{p_n}(\mathcal{D})}{2\sqrt{p_n}} \to 1. \tag{3.18}$$

Thus we need to investigate the distribution of $\frac{|a_p(\mathcal{D})|}{2\sqrt{p}}$ as p varies. Hasse showed in the 1930s that $-1 < \frac{a_p(\mathcal{D})}{2\sqrt{p}} < 1$ [Sil, Theorem V.1.1]. (In fact, this is a special case of RH for curves over a finite field.)

It is natural to let $\theta_p \in (0, \pi)$ satisfy

$$\cos\theta_p \;=\; \frac{a_p(D)}{2\sqrt{p}} \tag{3.19}$$

and to study the distribution of θ_p as p ranges.

If $\mathcal{D}$ has complex multiplication, then there is a spike at $\theta_p = \pi$ and otherwise a uniform distribution on $(0, \pi)$.

If $\mathcal{D}$ does not have complex multiplication, then the distribution is conjectured to satisfy the semi-circle distribution:

$$\lim_{N\to\infty} \frac{\#\{p \le N : \alpha < \theta_p < \beta\}}{\#\{p \le N\}} \;=\; \frac{2}{\pi}\int_\alpha^\beta \sin^2\theta\, d\theta. \tag{3.20}$$

This is a specific case of the Sato–Tate conjecture. Clozel, Harris, Shepherd-Barron and Taylor [CHT,HSBT,Tay] proved this for elliptic curves without complex multiplication, provided that there is at least one prime of multiplicative reduction; that assumption was recently removed by Barnet-Lamb, Geraghty, Harris and Taylor [BLGHT]. Thus, for any square-free $\mathcal{D} \in \mathbb{Z}[T]$, we can find a sequence of primes satisfying (3.18). We have therefore proved the following.

Theorem 3.19 (Newman's conjecture for fixed $\mathcal{D}$, $\deg\mathcal{D} = 3$)**.** *Let $\mathcal{D} \in \mathbb{Z}[T]$ be square-free with $\deg\mathcal{D} = 3$. Then $\sup_p \Lambda_{D_p} = 0$.*

Remark 3.20. *When $g \ge 2$, then Ξ_t contains multiple e^t terms and multiple $\cos nx$ terms, making it much harder to find the explicit expression of Λ_{D_p}.*

3.6. **Zeros of $\Xi_0(x, \chi_D)$.** In this section, we introduce notation for the zeros of $\Xi(x, \chi_D)$ and discuss basic properties, which will be used in the remainder of the paper.

Remark 3.21. *Because of (3.4), a zero γ of $\Xi(x, \chi_D)$ corresponds to a zero $\frac{1}{2} + \frac{i\gamma}{\log q}$ of $L(s, \chi_D)$.*

The following analogue of Lemma 1.5 gives us a lower bound on Λ_D via double zeros.

Lemma 3.22. *Let (q, D) be a good pair. Let $t_0 \in \mathbb{R}$. If $\Xi_{t_0}(x, \chi_D)$ has a zero x_0 of order at least 2, then $t_0 \le \Lambda_D$.*

Proof. If $F(x, t) = \Xi_t(x, \chi_D)$, then F satisfies the backwards heat equation: $\partial_t F + \partial_{xx} F = 0$. Using the observation, the lemma follows via the argument given in [Sto, Lemma, page 7]. □

Remark 3.23. *Because of Lemma 3.22, if $\Xi_0(x, \chi_D)$ has a double zero, then Newman's conjecture is true. For most of the remaining paper, we assume that all the zeros of Ξ_0 are simple.*

Let (q, D) be a good pair and assume the zeros of $\Xi_0(x, \chi_D)$ are simple. Because of evenness, this implies that Ξ_0 does not have a zero at $x = 0$. Let the positive zeros of $\Xi_0(x, \chi_D)$ be denoted $\gamma_1, \gamma_2, \ldots$, counted with multiplicity. We assume the zeros are ordered so that $0 < \gamma_1 < \gamma_2 < \cdots$.

By (3.4), we see that the first $2g$ zeros lie in the interval $(0, 2\pi)$, and the remaining zeros are repeated by periodicity. Thus, all the zeros of Ξ are given by

$$\{\gamma_j + 2\pi\ell : j \in \{1, 2, \ldots, 2g\}, \ell \in \mathbb{Z}\}. \tag{3.21}$$

Next, by evenness and periodicity, we know for $1 \le j \le g$, we have $\gamma_{2g+1-j} = -\gamma_j + 2\pi$. This implies that the first g zeros lie in $(0, \pi)$ and the next g zeros lie in $(\pi, 2\pi)$. Thus, all the zeros of Ξ_0 are given by

$$\{\epsilon\gamma_j + 2\pi\ell : \epsilon \in \{\pm 1\}, j \in \{1, 2, \ldots, g\}, \ell \in \mathbb{Z}\}. \tag{3.22}$$

In other words, once we compute the first g zeros of Ξ_0, we know the remaining zeros.

Remark 3.24. *The observations above still apply if $\Xi_0(x, \chi_D)$ does not have only simple zeros. The only technical detail we have to pay attention to is if Ξ_0 has a zero at $x = 0$. We know that Ξ_0 has a zero of even order, say $2n$. Then we must let $0 = \gamma_1 = \cdots = \gamma_n < \gamma_{n+1}$, so that $-\gamma_1, \ldots, -\gamma_n$ cover the remaining multiplicities.*

3.7. **Main result of Csordas et. al.** We have an analogue of the main result of [CSV] and [Sto], which can be used to give lower bounds on Λ.

Lemma 3.25. *Let (q, D) be a good pair and suppose the zeros of $\Xi_0(x, \chi_D)$ are simple. Let the positive zeros of $\Xi_0(z, \chi_D)$ be denoted $\gamma_1, \gamma_2, \ldots$ as described in Section 3.6. Define the quantity*

$$G = \sum_{j=2}^{\infty} \frac{2}{(\gamma_1 - \gamma_j)^2}. \tag{3.23}$$

Then if $5\gamma_1^2 G < 1$, we have

$$\Lambda_D > \frac{(1 - 5\gamma_1^2 G)^{4/5} - 1}{8G}. \tag{3.24}$$

Proof. This is a direct generalization of [Sto, Theorem 1]. In [Sto], the quantity G has the same form except the sum is over the zeros of a number field L-function.[2] The condition $(5\gamma_1^2 G < 1)$ is the same as in [Sto], and the conclusion (a lower bound on Λ_D) is the same. The method of proof uses differential equations governing the motion of the zero γ_1 as t changes to find a time $t < 0$ when γ_1 coalesces with $-\gamma_1$. □

3.8. **Low-lying zeros and connections with random matrix theory.** We now show that the condition $5\gamma_1^2 G$ in Lemma 3.25 does occur in certain families, using connections to random matrix theory. We begin by analyzing (3.23).

We assume the zeros of $\Xi_0(x, \chi_D)$ are simple, so by the discussion in Section 3.6, we can write the first g positive zeros as $0 < \gamma_1 < \cdots < \gamma_g < \pi$. Then all of the zeros of Ξ_0 are given by Equation (3.22). Using this fact, we can write (3.23) as

$$G = \sum_{\epsilon \in \{\pm 1\}} \sum_{j=1}^{g} {\sum_{\ell \in \mathbb{Z}}}' \frac{2}{[\gamma_1 - (\epsilon\gamma_j + 2\pi\ell)]^2}, \tag{3.25}$$

[2] In [Sto], the quantity analogous to (3.23) is actually called "$g(0)$." (See [Sto, (13)].) However, in this paper, we use g for the genus of the hyperelliptic curve defined by D.

where the prime mark ($'$) means we omit the two terms $(\epsilon, j, \ell) = (\pm 1, 1, 0)$. Using the identity $\sum_{n\in\mathbb{Z}}(n+\alpha)^{-2} = \pi\csc^2\pi\alpha$, after some algebraic manipulations, we obtain

$$G \;=\; \frac{1}{6} - \frac{1}{2\gamma_1^2} + \frac{1}{2}\csc^2\gamma_1 + \frac{1}{2}\sum_{\epsilon\in\{\pm1\}}\sum_{j=2}^{g}\csc^2\left(\frac{\gamma_1+\epsilon\gamma_j}{2}\right). \tag{3.26}$$

Observe that the sum on the right is now a finite sum.

With some work, we can determine sufficient conditions for $5\gamma_1^2 G$, which allows us to apply Lemma 3.25.

Lemma 3.26. *Let (q, D) be a good pair and let $\gamma_1, \ldots, \gamma_g$ be the zeros of $\Xi_0(x, \chi_D)$ in $[0, \pi]$. Assume the zeros are simple so that $0 < \gamma_1 < \cdots < \gamma_g < \pi$. Suppose the following conditions hold*

- $g \geq 13$,
- $\left(\frac{g}{\pi}\gamma_1\right)^2 \leq \frac{1}{500g}$,
- $\frac{1}{2} \leq \frac{g}{\pi}\gamma_2 \leq 2$.

Then $5\gamma_1^2 G < 1$ (where G is defined in (3.26)).

Before we present the proof, we make a few observations. The quantities $\tilde{\gamma}_j := \frac{g}{\pi}\gamma_j$ are rescalings of the zeros. Since $0 < \tilde{\gamma}_1 < \cdots < \tilde{\gamma}_g < g$, the normalized zeros $\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots$ on average have unit spacing on the positive real line. Thus the condition $\tilde{\gamma}_1^2 \leq \frac{1}{500g}$ says that the first zero is unusually small, while the condition $\frac{1}{2} \leq \tilde{\gamma}_2 \leq 2$ says that second zero is around where it is "expected" to be.

These conditions (along with $g \geq 13$) are very crude and the constants can easily be improved with some work. However, our focus is not on the optimum, but the fact that such a statement as the lemma exists.

Proof. This argument is technical, and relies on bounds for the function $\csc^2 x$. In particular, for $|x| \leq \frac{1}{2}$,

$$\csc^2 x \;\leq\; \frac{1.1}{x^2}, \tag{3.27}$$

$$\csc^2 x \;\leq\; \frac{1}{x^2} + 0.36. \tag{3.28}$$

Next, we take the expression (3.26) for G and break it into two parts by writing

$$G \;=\; \frac{1}{6} + S_I + S_{II}, \tag{3.29}$$

where

$$\begin{aligned} S_I \;&=\; \frac{1}{2}\csc^2\gamma_1 - \frac{1}{2\gamma_1^2} \\ S_{II} \;&=\; \frac{1}{2}\sum_{\epsilon\in\{\pm1\}}\sum_{j=2}^{g}\csc^2\left(\frac{\gamma_1+\epsilon\gamma_j}{2}\right). \end{aligned} \tag{3.30}$$

Using the bound (3.28) and our assumption on γ_1, we have

$$S_I \;\leq\; 0.18. \tag{3.31}$$

Next we bound S_{II}. The idea will be to bound the sum by the maximum term, i.e.,

$$S_{II} \;=\; \frac{1}{2}\cdot 2\cdot(g-1)\cdot \max_{\substack{\epsilon\in\{\pm1\}\\ 2\le j\le g}}\left\{\csc^2\left(\frac{\gamma_1+\epsilon\gamma_j}{2}\right)\right\}. \tag{3.32}$$

Notice that $\csc^2 x$ is large when x is near a multiple of π. The choice of $(\epsilon, j) \in \{\pm1\}\times\{2,\dots,g\}$ that minimizes the distance between $\frac{\gamma_1+\epsilon\gamma_j}{2}$ and 0 is $(\epsilon, j) = (-1, 2)$. That distance is

$$\left|0-\frac{\gamma_1-\gamma_2}{2}\right| \;=\; \frac{\gamma_2-\gamma_1}{2} \;\le\; \frac{\gamma_2}{2} \;\le\; \frac{\pi}{g} \;\le\; \frac{\pi}{13}. \tag{3.33}$$

The choice of (ϵ, j) that minimizes the distance between $\frac{\gamma_1+\epsilon\gamma_j}{2}$ and π is $(\epsilon, j) = (1, g)$. That distance is

$$\left|\pi-\frac{\gamma_1+\gamma_g}{2}\right| \;=\; \pi-\frac{\gamma_1+\gamma_g}{2} \;\ge\; \pi-\frac{1+\pi}{2} \;\ge\; \frac{\pi}{2}-\frac{1}{2}. \tag{3.34}$$

Note that it suffices to obtain a lower bound on the absolute value of the difference, as if it were large than it would be closer to a different multiple of π.

The choice of (ϵ, j) that minimizes the distance between $\frac{\gamma_1+\epsilon\gamma_j}{2}$ and $-\pi$ is $(\epsilon, j) = (-1, g)$. That distance is

$$\left|-\pi-\frac{\gamma_1-\gamma_g}{2}\right| \;=\; \pi-\frac{-\gamma_1+\gamma_g}{2} \;\ge\; \pi-\frac{0+\pi}{2} \;\ge\; \frac{\pi}{2}. \tag{3.35}$$

It follows that $\csc^2\left(\frac{\gamma_1+\epsilon\gamma_j}{2}\right)$ is maximized at $(\epsilon, j) = (-1, 2)$, so

$$S_{II} \;\le\; \frac{1}{2}\sum_{\epsilon\in\{\pm1\}}\sum_{j=2}^{g}\csc^2\left(\frac{\gamma_1-\gamma_2}{2}\right) \;\le\; g\csc^2\left(\frac{\gamma_1-\gamma_2}{2}\right). \tag{3.36}$$

As shown in (3.33), we have $\left|\frac{\gamma_1-\gamma_2}{2}\right| \le \frac{1}{2}$. Thus, combining (3.28) and (3.36) yields

$$S_{II} \;\le\; 1.1g\left(\frac{2}{\gamma_1-\gamma_2}\right)^2 \;=\; \frac{4.4g}{\gamma_2^2}\left(\frac{1}{1-\gamma_1/\gamma_2}\right)^2 \;\le\; \frac{18g}{\gamma_2^2}. \tag{3.37}$$

By combining (3.29), (3.31), and (3.37), we arrive at

$$5\gamma_1^2 G \;\le\; 1.8\gamma_1^2 + 90g\cdot\frac{\gamma_1^2}{\gamma_2^2}. \tag{3.38}$$

By using $\tilde{\gamma}_2 \ge \frac{1}{2}$, we have

$$90g\cdot\frac{\gamma_1^2}{\gamma_2^2} \;=\; 90g\cdot\frac{\tilde{\gamma}_1^2}{\tilde{\gamma}_2^2} \;\le\; 450g\tilde{\gamma}_1^2 \;\le\; 0.9, \tag{3.39}$$

where we use $\tilde{\gamma}_1^2 \le \frac{1}{500g}$ at the end. Then $1.8\tilde{\gamma}_1^2 \le 1.8\cdot\frac{1}{500\cdot 13} \le 0.0003$, so $5\gamma_1^2 G < 1$. □

Analogous to [CSV, (4.25)] and [Sto, (17)], the expression on the right hand side of (3.24) has the power series expansion

$$\frac{(1-5\gamma_1^2 G)^{4/5}-1}{8G} \;=\; -\frac{1}{2}\gamma_1^2\left(1+\frac{\gamma_1^2 G}{2}+O(\gamma_1^4 G^2)\right). \tag{3.40}$$

Thus the smaller the first zero γ_1 is, the better the lower bound on Λ given by Lemma 3.26 is.

We discuss an interpretation of the above. Since our Newman conjectures vary over families, we write $\tilde{\gamma}_j(D)$ and $g(D)$ to remind ourselves of dependence on D.

Corollary 3.27. *Let $\mathcal{F}$ be a family of polynomials D belonging to good pairs (q, D). Suppose there exists a sequence $D_1, D_2, \dots$ in $\mathcal{F}$ such that $\gamma_1(D_n) \to 0$ as $n \to \infty$ and for all n,*

- $g(D_n) \geq 13$
- $\tilde{\gamma}_1(D_n)^2 \leq \frac{1}{500g(D_n)}$
- $\frac{1}{2} \leq \tilde{\gamma}_2(D_n) \leq 2.$

Then $\Lambda_{D_n} \to 0$ as $n \to \infty$, so Newman's conjecture is true for the family $\mathcal{F}$.

The conditions above essentially say that there is a set of curves in our family where the first zero is unusually small and the second zero is on the order of its expected value. For many families with g and q tending to infinity, this is known due to work of Katz and Sarnak [KS2,KS1].

Appendix A. Example of $\Xi_t(x, \chi_D)$ in function fields and the role of the backwards heat equation.

For

$$D \;=\; T^5 + T^4 + T^3 + 2T + 2 \in \mathbb{F}_5[T], \tag{A.1}$$

we have

$$\Xi_t(x, \chi_D) \;=\; 10e^{4t}\cos 2x - 2\sqrt{5}e^t \cos x - 1. \tag{A.2}$$

For any t, we observe that $e^{2ix} \cdot \Xi_t(x, \chi_D)$ is a fourth degree polynomial in e^{ix}. Thus $\Xi_t(x, \chi_D)$ must have exactly four zeros with $\mathrm{Re}(x) \in [0, 2\pi)$.

Figure A.1 shows plots of $\Xi_t(x, \chi_D)$ for various times t. Observe that as we move backwards in time, the peaks get smaller. Because $\Xi_t(x, \chi_D)$ solves the backwards heat equation, the "flattening" of the function behaves like the diffusion of heat.

As we decrease t, the two zeros on the left move towards each other, until they coalesce at $t \approx -0.189$. If we keep going further back in time, these two zeros "pop off" the real line. For instance, at $t = -0.25$, the function has zeros at $x \approx \pm 0.152i$.

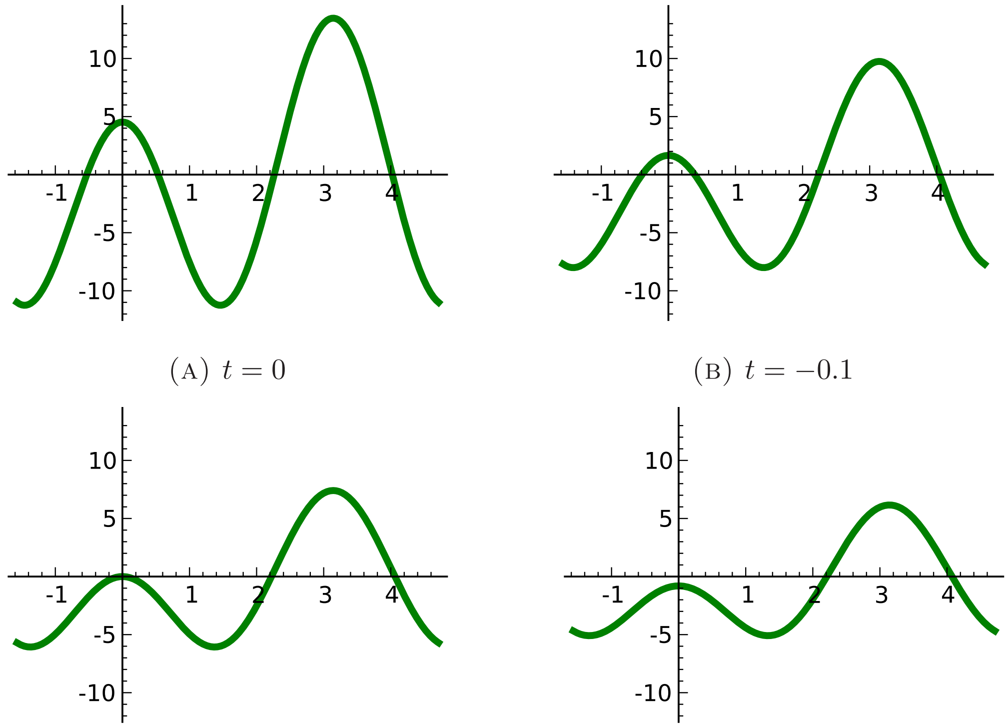
The time when the zeros coalesce ($t \approx -0.189$) is the De Bruijn–Newman constant Λ_D for this D. It is the largest real solution to $\Xi_t(0, \chi_D) = 0$. From (A.2), we see that Λ_D is the logarithm of the root of a fourth degree polynomial.

Appendix B. Numerical calculations

If $\Xi_t(\,\cdot\,, \chi_D)$ has a zero at $x = 0$, then it has a double zero there by evenness. Thus, by using Lemma 3.22, we see that a solution t to $\Xi_t(0, \chi_D) = 0$ is a lower bound for Λ_D. We have

$$\Xi_t(0, \chi_D) \;=\; \Phi_0 + 2\sum_{n=1}^{g} \Phi_n (e^t)^{n^2}, \tag{B.1}$$

which is a polynomial in e^t of degree g^2. As g increases, it becomes harder to find the exact roots of this polynomial, but we may still proceed numerically. This gives us a method to quickly find lower bounds of Λ_D for D.



(A) $t = 0$ (B) $t = -0.1$

(C) $t \approx -0.189$ (D) $t = -0.25$

FIGURE A.1. Plots of $\Xi_t(x, \chi_D)$ for different t.

For $q = 3$, this method produces the lower bounds given in Figure B.1 and Figure B.2.

g	$(c_0, \dots, c_g)$	lower bound on Λ_D
1	$(1, -3)$	$-1.44 \cdot 10^{-1}$
2	$(1, -3, 5)$	$-5.28 \cdot 10^{-2}$
3	$(1, -1, 1, -7)$	$-1.26 \cdot 10^{-2}$
4	$(1, -3, 9, -23, 39)$	$-1.05 \cdot 10^{-3}$
5	$(1, -3, 5, -3, -11, 27)$	$-1.23 \cdot 10^{-4}$
6	$(1, -1, 3, -7, 5, -13, 11)$	$-3.02 \cdot 10^{-5}$
7	$(1, 1, 5, 3, 1, -15, -51, -101)$	$-1.28 \cdot 10^{-5}$

FIGURE B.1. Lower bounds on Λ_D for certain $D \in \mathbb{F}_3[T]$. The values $c_0, \dots, c_g$ are the coefficients of the L-function as in (3.2) and (3.3).

The above supports the claim that

$$\lim_{g \to \infty} \sup_{\substack{D \in \mathbb{F}_3[T] \\ \deg D \leq 2g+1}} \Lambda_D = 0,$$

which supports Newman's conjecture for fixed q.

g	D
1	$T^3 + 2T + 1$
2	$T^5 + T^3 + T + 1$
3	$T^7 + 2T^5 + T^3 + 2T^2 + 2T + 2$
4	$T^9 + T^6 + T^4 + T^3 + T^2 + T + 1$
5	$T^{11} + 2T^9 + T^8 + 2T^7 + 2T^6 + 2T^5 + 2T^4 + T^2 + 2T + 1$
6	$T^{13} + 2T^{11} + T^{10} + 2T^7 + 2T^6 + 2T^4 + T^3 + 2T + 1$
7	$T^{15} + 2T^{14} + 2T^9 + T^8 + 2T^6 + T^3 + 2T^2 + T + 2$

Figure B.2. Polynomials in $\mathbb{F}_3[T]$ used in Figure B.1.

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