NEW BEHAVIOR IN LEGAL DECOMPOSITIONS ARISING FROM NON-POSITIVE LINEAR RECURRENCES

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Abstract. Zeckendorf's theorem states every positive integer has a unique decomposition as a sum of non-adjacent Fibonacci numbers. This result has been generalized to many sequences $\{a_n\}$ arising from an integer positive linear recurrence, each of which has a corresponding notion of a legal decomposition. Previous work proved the number of summands in decompositions of $m \in [a_n, a_{n+1})$ becomes normally distributed as $n \to \infty$, and the individual gap measures associated to each m converge to geometric random variables, when the leading coefficient in the recurrence is positive. We explore what happens when this assumption is removed in two special sequences. In one we regain all previous results, including unique decomposition; in the other the number of legal decompositions exponentially grows and the natural choice for the legal decomposition (the greedy algorithm) only works approximately 92.6% of the time (though a slight modification always works). We find a connection between the two sequences, which explains why the distribution of the number of summands and gaps between summands behave the same in the two examples. In the course of our investigations we found a new perspective on dealing with roots of polynomials associated to the characteristic polynomials. This allows us to remove the need for the detailed technical analysis of their properties which greatly complicated the proofs of many earlier results in the subject, as well as handle new cases beyond the reach of existing techniques.

1. Introduction

Previous work on Positive Linear Recurrence Sequences (PLRS) generalized Zeckendorf's theorem, which states that every positive integer can be uniquely written as a sum of nonconsecutive Fibonacci numbers. Papers such as [28, 29, 10, 11] showed that the decompositions of positive integers as sums of elements from a PLRS are unique and that the average number of summands displays Gaussian behavior; see also [9, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 30, 31, 32], and see [1, 10, 11] for other types of decomposition laws. Subsequent papers [2, 5] included proofs of the exponential decay in the gaps between summands. These papers hinge on technical arguments depending on the leading term of the recurrence relation defining the sequence being non-zero.

We have two goals in the work below: (1) we explore the behavior of some special integer sequences satisfying recurrences with leading term zero, and (2) we develop a new combinatorial method to bypass the technical arguments on polynomials associated to the recurrence relation which complicated arguments in previous work. The first is particularly interesting

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as all of the previous results are not applicable, and we have to develop new methods. While the two sequences we focus on may seem unrelated, knowledge of the first yields many results for the second (and thus explains why we study these two together).

Our first infinite two-parameter family of sequences are called the (s, b)-Generacci sequences. They were introduced in [6], where we showed that the $(1, 2)$-Generacci sequence, also referred to as the Kentucky sequence, has similar behavior to those displayed by a PLRS even though it is not a PLRS (the $(1, 1)$ case is the Fibonacci numbers, hence the name). This included the Gaussian behavior for the number of summands and the exponential decay in gaps between summands [6, Theorems 1.5 and 1.6]. In [7], we further expanded the study of the (s, b)-Generacci sequences and proved that these sequences lead to unique decompositions of all positive integers. In this paper, we introduce new methods which lead to proofs of Gaussian behavior in the number of summands, both for this sequence and others in the literature, which allow us to avoid complications involving roots of polynomials. This is very much in contrast to the very technical arguments presented in [28] for Positive Linear Recurrences. In addition, we provide an analogous result on the geometric decay in the distribution of gaps between bins (the arguments for gaps between summands is similar but involves uninteresting additional book-keeping, and hence we omit them here).

The other sequence of interest is called the Fibonacci Quilt sequence. This sequence arises naturally from a 2-dimensional construction of a log-cabin style quilt. The Fibonacci Quilt sequence, like the (s, b)-Generacci sequences, satisfies a recurrence with leading term zero, however in [7] we showed that the legal decompositions arising from this sequence have drastically different behavior than that of the (s, b)-Generacci sequence, with the major difference being that the decompositions arising from the Fibonacci Quilt sequence are not unique. In fact, we showed that the number of legal decompositions of a positive integer grows exponentially as the integer increases. Another surprising result is that among all of these decompositions, the decomposition arising from the greedy algorithm is a legal decomposition (approximately) 93% of the time. In [7], we defined a modified greedy algorithm, called the *Greedy-6 algorithm*, and showed that the decomposition arising from this algorithm always terminates in a legal decomposition. Moreover, we showed that the Greedy-6 algorithm results in a legal decomposition with minimal number of summands. Interestingly, while there is markedly different behavior between these two sequences in terms of uniqueness of decompositions, they exhibit similar behavior in terms of the number of summands and gaps between summands. In particular, for the Greedy-6 decomposition we obtain Gaussian behavior for the number of summands and geometric decay for the average and individual gap measures almost immediately by noticing a connection between the Fibonacci Quilt and $(4, 1)$-Generacci sequences.

Below we describe the sequences in greater detail and then state our main results. In the companion paper [7] we have collected many of the basic properties of the sequences we study; we repeat the statements here so this paper may be read independently of [7]. As many of the calculations follow analogously to similar computations in the literature, we only provide the details for the new arguments; the more standard proofs are available in the expanded arXiv version of this paper, [8].

1.1. (s, b)-Generacci Sequences and the Fibonacci Quilt Sequence.

1.1.1. (s, b)-*Generacci Sequences.*

We begin by restating the definition and some computational results for the (s, b)-Generacci sequences. The proofs of these results appeared in [7], and follow from straightforward algebra applied to the definitions.

Briefly, the sequence is defined as follows. We have a collection of bins $\mathcal{B}_j$, each containing b numbers. We construct a sequence $\{a_n\}$ such that each positive integer has a decomposition as a sum of elements such that (1) we take at most one element in a bin, and (2) if we take an element in bin $\mathcal{B}_j$, then we do not take any elements in any of the s bins preceding $\mathcal{B}_j$ nor the s bins succeeding $\mathcal{B}_j$. We formalize the above in the following two definitions.

Definition 1.1 ((s,b)-Generacci legal decomposition)**.** *For fixed integers $s, b \geq 1$, let an increasing sequence of positive integers $\{a_i\}_{i=1}^{\infty}$ and a family of subsequences*

$$\mathcal{B}_n = \{a_{b(n-1)+1}, \dots, a_{bn}\}$$

be given (we call these subsequences bins*). We declare a decomposition of an integer $m = a_{\ell_1} + a_{\ell_2} + \cdots + a_{\ell_k}$ where $a_{\ell_i} > a_{\ell_{i+1}}$ to be an* (s,b)-Generacci legal decomposition *provided $\{a_{\ell_i}, a_{\ell_{i+1}}\} \not\subset \mathcal{B}_{j-s} \cup \mathcal{B}_{j-s+1} \cup \cdots \cup \mathcal{B}_j$ for all i, j, with the convention that $\mathcal{B}_j = \emptyset$ for $j \leq 0$.*

Definition 1.2 ((s,b)-Generacci sequence)**.** *For fixed integers $s, b \geq 1$, an increasing sequence of positive integers $\{a_i\}_{i=1}^{\infty}$ is the* (s,b)-Generacci sequence *if every a_i for $i \geq 1$ is the smallest positive integer that does not have an (s,b)-Generacci legal decomposition using the elements $\{a_1, \dots, a_{i-1}\}$.*

We recall that Zeckendorf's theorem gave an equivalent definition of the Fibonacci numbers as the unique sequence which allows one to write all positive integers as a sum of nonconsecutive elements in the sequence. Note this holds provided we define the Fibonacci numbers beginning with $1, 2, 3, \dots$. It is then clear that the $(1,1)$-Generacci sequence is the Fibonacci sequence. However, other interesting sequences are also (s,b)-Generacci sequences. For example, Narayana's cow sequence is the $(2,1)$-Generacci sequence and the Kentucky sequence (studied at length by the authors in [6]) is the $(1,2)$-Generacci sequence.

Theorem 1.3 (Recurrence Relation and Explicit Formula)**.** *For $n > (s+1)b+1$, the n^{th} term of the (s,b)-Generacci sequence satisfies*

$$a_n \;=\; a_{n-b} + b a_{n-(s+1)b} \;=\; c_1 \lambda_1^n \left[1 + O\left((\lambda_2/\lambda_1)^n\right)\right], \tag{1.1}$$

where λ_1 is the largest root of $x^{(s+1)b} - x^{sb} - b = 0$, and c_1 and λ_2 are constants with $\lambda_1 > 1$, $c_1 > 0$ and $|\lambda_2| < \lambda_1$.

The proof of the recurrence follows from standard arguments involving the construction of the (s,b)-sequence (see, e.g., [7, Theorem 1.3]). The proof of the main term and error bound follows from a generalized Binet formula (see, e.g., [2, Theorem A.1]) and we provide a proof in §2.1 of [7]. There is a slight complication in that the leading coefficient of the recurrence is zero; we surmount this by passing to a related recurrence where the leading coefficient is positive and thus the standard arguments apply.

1.1.2. *Fibonacci Quilt Sequence.*

We state the definition and some computational results for the Fibonacci Quilt sequence; the proofs follow immediately by straightforward algebra (see [7]). Unlike many other works in the subject, here we use the more common convention for the Fibonacci numbers that $F_0 = 1$, $F_1 = 1$ (and of course still taking $F_{n+1} = F_n + F_{n-1}$). With this notation an interesting property of the Fibonacci numbers is that they can be used to tile the plane by squares (see Figure 1).

We have a different notion of legality based on the spiral and motivated by the Zeckendorf rule for the Fibonacci numbers involving the use of non-adjacent terms. We create a sequence

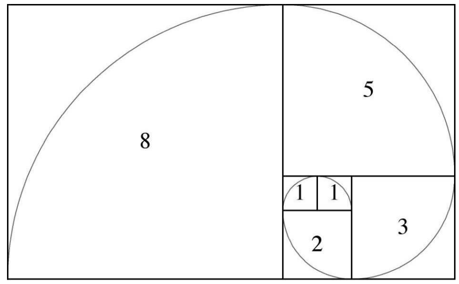

FIGURE 1. The (start of the) Fibonacci Spiral.

of integers by placing the integers of the sequence in the squares of the spiral (in the order the spiral is created) using the rule that we must be able to decompose every positive integer as a sum of elements in the sequence provided the squares they lie in do not share part of a side.

Definition 1.4 (FQ-legal decomposition). *Let an increasing sequence of positive integers $\{q_i\}_{i=1}^{\infty}$ be given. We declare a decomposition of an integer*

$$m \;=\; q_{\ell_1} + q_{\ell_2} + \cdots + q_{\ell_t} \tag{1.2}$$

(where $q_{\ell_i} > q_{\ell_{i+1}}$) to be an FQ-legal decomposition if for all i, j, $|\ell_i - \ell_j| \neq 0, 1, 3, 4$ and $\{1, 3\} \not\subset \{\ell_1, \ell_2, \ldots \ell_t\}$.

We compress the Fibonacci spiral so that the n^{th} square is replaced with a rectangle of thickness 1 (this allows us to display more of the pattern in the same space); we call this the Fibonacci Quilt (see Figure 3). The adjacency of the squares in the Fibonacci spiral is identical to the adjacency of the rectangles in the Fibonacci Quilt. (The latter figure is known in the quilting community as the log cabin quilt pattern, and we adopt the name Fibonacci Quilt sequence from this connection.) The definition above states that we cannot use two terms if the rectangles they are placed in share part of an edge. We see that $q_n + q_{n-1}$ is not legal but $q_n + q_{n-2}$ is legal for $n \geq 4$. For small n, the starting pattern of the quilt forbids decompositions that contain $q_3 + q_1$.

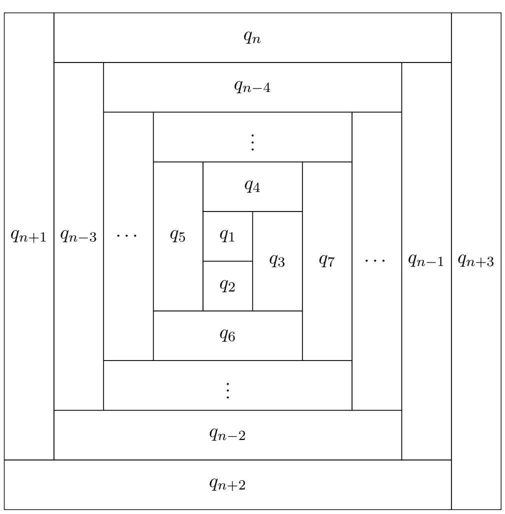

FIGURE 2. Log Cabin Quilt Pattern

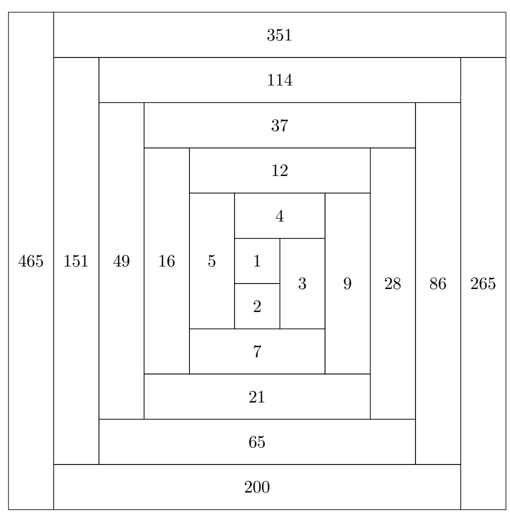

FIGURE 3. Fibonacci Quilt Sequence

The discussion above motivates the following definition of the Fibonacci Quilt Sequence.

Definition 1.5 (Fibonacci Quilt Sequence). *The Fibonacci Quilt Sequence* $\{q_i\}_{i=1}^{\infty}$ *has* $q_1 = 1$ *and every* q_i $(i \geq 2)$ *is the smallest positive integer that does not have an FQ-legal decomposition using the elements* $\{q_1, \dots, q_{i-1}\}$.

We display the first few terms of this sequence in Figure 3: $\{1, 2, 3, 4, 5, 7, 9, 12, \dots\}$.

Theorem 1.6 (Recurrence Relations). *Let* q_n *denote the* n^{th} *term in the Fibonacci Quilt Sequence. Then (1) for* $n \geq 6$, $q_{n+1} = q_n + q_{n-4}$, *(2) for* $n \geq 5$, $q_{n+1} = q_{n-1} + q_{n-2}$, *and (3) we have*

$$q_n \;=\; \alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \alpha_3 \overline{\lambda_2}^n, \tag{1.3}$$

where $\alpha_1 \approx 1.26724$,

$$\lambda_1 \;=\; \frac{1}{3}\left(\frac{27}{2} - \frac{3\sqrt{69}}{2}\right)^{1/3} + \frac{\left(\frac{1}{2}\left(9+\sqrt{69}\right)\right)^{1/3}}{3^{2/3}} \;\approx\; 1.32472 \tag{1.4}$$

and $\lambda_2 \approx -0.662359 - 0.56228i$ *(which has absolute value approximately 0.8688).*

The above result appeared in [7, Theorem 1.6 and Proposition 2.4], and follows from a straightforward constructive proof using induction.

1.2. **Results.** Both the (s, b)-Generacci sequences and the Fibonacci quilt sequence satisfy recurrence relations with leading term zero. They display drastically different behavior in some respects, but also have very similar behavior for other problems (which allows us to deduce results for the Fibonacci Quilt sequence from results for the $(4, 1)$-Generacci sequence). We begin by stating results related to the decompositions arising from these sequences, many of which are proved in the companion paper [7]. We then state new results on Gaussian behavior in the number of summands, and exponential decay in the gap measures between summands.

1.2.1. *Decompositions.* The (s, b)-Generacci legal decompositions are unique ([7, Theorem 1.9]) whereas FQ-legal decompositions are not. The average number of FQ-legal decompositions grows exponentially [7, Theorem 1.11].

Let m be a positive integer and let $d_{\rm FQ}(m)$ denote the number of FQ-legal decompositions of m. Let $d_{\rm FQ;ave}(n)$ denote the average number of FQ-legal decompositions of integers in $I_n := [0, q_{n+1})$. Hence

$$d_{\rm FQ;ave}(n) \;:=\; \frac{1}{q_{n+1}} \sum_{m=0}^{q_{n+1}-1} d_{\rm FQ}(m). \tag{1.5}$$

Theorem 1.7 (Growth Rate of Average Number of Decompositions). *There exist computable constants* $\lambda \approx 1.05459$ *and* $C_2 > C_1 > 0$ *such that for all* n *sufficiently large,*

$$C_1 \lambda^n \;\leq\; d_{\rm FQ;ave}(n) \;\leq\; C_2 \lambda^n. \tag{1.6}$$

Thus the average number of FQ-legal decompositions of integers in $[0, q_{n+1})$ *tends to infinity exponentially fast.*

The proof of Theorem 1.7 (found in [7]) derived recurrence relations and an explicit formula for the number of FQ-legal decompositions.

In many decomposition schemes including the (s, b)-Generacci case, there is a unique legal representation which can be found through a greedy algorithm. For the Fibonacci Quilt, not only does uniqueness often fail, but frequently the greedy algorithm does not terminate in a FQ-legal decomposition. For example, if we try to decompose $6 \in [q_5, q_6)$, the greedy algorithm would start with the largest summand possible, $q_5 = 5$. Unfortunately at this point we would need to take $q_1 = 1$ as our next term, but we cannot as q_1 and q_5 share a side. The only decomposition of 6 bypasses q_5 and uses q_4, writing it as $q_4 + q_2$. In [7, Theorem 1.13], we determined how often the greedy algorithm yields a legal decomposition.

Theorem 1.8. *There is a constant $\rho \in (0,1)$ such that, as $n \to \infty$, the percentage of positive integers in $[1, q_n)$ where the greedy algorithm terminates in a Fibonacci Quilt legal decomposition converges to ρ. This constant is approximately 0.92627.*

The proof of Theorem 1.8 (found in [7]) used a recurrence for h_n which denotes the number of positive integers between 1 and $q_{n+1} - 1$ where the greedy algorithm successfully terminates in a legal decomposition. The result then follows from the recurrence and the use of a generalized Binet formula.

Even though Theorem 1.8 shows that the greedy algorithm does not always terminate in a FQ-legal decomposition, a simple modification *does* always terminate in a FQ-legal decomposition. The Greedy-6 Algorithm (defined in Definition 1.9) is identical to the greedy algorithm with the caveat that if the greedy algorithm yields a decomposition including q_1 and q_5 (which sum to 6) we exchange them with the summands q_2 and q_4 (also summing to 6).

Definition 1.9. *(Greedy-6 Algorithm) Decompose m into sums of FQ-numbers as follows.*

- *If there is an n with $m = q_n$ then we are done.*
- *If $m = 6$, then we decompose m as $q_4 + q_2$ and we are done.*
- *If $m \geq q_6$ and $m \neq q_n$ for all $n \geq 1$, then we write $m = q_{\ell_1} + x$ where $q_{\ell_1} < m < q_{\ell_1+1}$ and $x > 0$. We then iterate the process with $m := x$.*

We denote the decomposition of m that results from the Greedy-6 Algorithm by $\mathcal{G}(m)$.

Theorem 1.10. *For all $m > 0$, the Greedy-6 Algorithm results in a FQ-legal decomposition. Moreover, if $\mathcal{G}(m) = q_{\ell_1} + q_{\ell_2} + \cdots + q_{\ell_{t-1}} + q_{\ell_t}$ with $q_{\ell_1} > q_{\ell_2} > \cdots > q_{\ell_t}$, then the decomposition satisfies exactly one of the following conditions:*

(1) *$\ell_i - \ell_{i+1} \geq 5$ for all i or*
(2) *$\ell_i - \ell_{i+1} \geq 5$ for $i \leq t-3$ and $\ell_{t-2} \geq 10$, $\ell_{t-1} = 4$, $\ell_t = 2$.*

Further, if $m = q_{\ell_1} + q_{\ell_2} + \cdots + q_{\ell_{t-1}} + q_{\ell_t}$ with $q_{\ell_1} > q_{\ell_2} > \cdots > q_{\ell_t}$ denotes a decomposition of m where either

(1) *$\ell_i - \ell_{i+1} \geq 5$ for all i or*
(2) *$\ell_i - \ell_{i+1} \geq 5$ for $i \leq t-3$ and $\ell_{t-2} \geq 10$, $\ell_{t-1} = 4$, $\ell_t = 2$,*

then $q_{\ell_1} + q_{\ell_2} + \cdots + q_{\ell_{t-1}} + q_{\ell_t} = \mathcal{G}(m)$. That is, the decomposition of m is the Greedy-6 decomposition.

The proof is straightforward; see [7, Theorem 1.15].

Let $\mathcal{D}(m)$ be a given decomposition of m as a sum of Fibonacci Quilt numbers (not necessarily legal):

$$m \;=\; c_1 q_1 + c_2 q_2 + \cdots + c_n q_n, \quad c_i \in \{0, 1, 2, \dots\}. \tag{1.7}$$

We define the number of summands by

$$\#\text{summands}(\mathcal{D}(m)) \;:=\; c_1 + c_2 + \cdots + c_n. \tag{1.8}$$

We can now state our final result for the Fibonacci Quilt sequence and the number of summands in FQ-legal decompositions; the proof is again standard and given in [7, Theorem 1.16].

Theorem 1.11. *If $\mathcal{D}(m)$ is any decomposition of m as a sum of Fibonacci Quilt numbers, then*

$$\#\text{summands}(\mathcal{G}(m)) \;\leq\; \#\text{summands}(\mathcal{D}(m)). \tag{1.9}$$

1.2.2. *Gaussian Distribution of the Number of Summands.* One of our main theorems regarding the (s,b)-Generacci sequences states that the number of summands in the (s,b)-Generacci legal decompositions of the positive integers follow a Gaussian distribution. We reiterate that previous results for Positive Linear Recurrences do not apply since the (s,b)-Generacci sequences are not Positive Linear Recurrences. Moreover, previous proofs of Gaussian behavior were very technical and relied heavily on knowledge of roots of polynomials. In this paper, some of the ideas we use are similar to those employed when studying Positive Linear Recurrence sequences but there is a major difference. We present a new technique that allows us to bypass all of the technical assumptions required in the other papers in their proofs of Gaussianity; see also [3, 24] for two different approaches (the first using Markov chains, the second using two dimensional recurrences) which also successfully avoid these complications. In §2 we give a proof to this main result, and then show it is applicable to the two sequences of this paper.

Theorem 1.12 (Gaussian Behavior of Summands for (s,b)-Generacci)**.** *Let the random variable Y_n denote the number of summands in the (unique) (s,b)-Generacci legal decomposition of an integer chosen uniformly at random from $[a_{(n-1)b+1}, a_{nb+1})$. Normalize Y_n to $Y_n' = (Y_n - \mu_n)/\sigma_n$, where μ_n and σ_n are the mean and variance of Y_n respectively, which satisfy*

$$\mu_n \ = \ An + B + o(1), \qquad \sigma_n^2 \ = \ Cn + D + o(1), \tag{1.10}$$

for some positive constants A, B, C, D. Then Y_n' converges in distribution to the standard normal distribution as $n \to \infty$.

Remark 1.13. *Using the methods of* [4]*, these results can trivially be extended to hold for an integer chosen uniformly at random from $[1, a_{nb+1})$ by trivially combining the results for intervals of the form $[a_{\ell b+1}, a_{(\ell+1)b+1})$.*

By specializing the above to the $(4,1)$-Generacci sequence we immediately obtain the same result for the Greedy-6 decompositions of the Fibonacci Quilt.

Theorem 1.14 (Gaussian Behavior of Summands for Greedy-6 FQ-Legal Decompositions)**.** *Let the random variable Y_n denote the number of summands in the (unique) Greedy-6 FQ-legal decomposition of an integer chosen uniformly at random from $[q_n, q_{n+1})$.*[1] *Normalize Y_n to $Y_n' = (Y_n - \mu_n)/\sigma_n$, where μ_n and σ_n are the mean and variance of Y_n respectively, which satisfy*

$$\mu_n \ = \ \widetilde{A}n + \widetilde{B} + o(1), \qquad \sigma_n^2 \ = \ \widetilde{C}n + \widetilde{D} + o(1), \tag{1.11}$$

for some positive constants $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}$. Then Y_n' converges in distribution to the standard normal distribution as $n \to \infty$.

1.2.3. *Gaps between Summands.* The following results concern the behavior of gaps between bins for (s,b)-Generacci sequences. For $m \in [a_{(n-1)b+1}, a_{nb+1})$, the legal decomposition

$$m \ = \ a_{\ell_1} + a_{\ell_2} + \cdots + a_{\ell_k} \quad \text{with} \quad \ell_1 \ > \ \ell_2 \ > \ \cdots \ > \ \ell_k, \tag{1.12}$$

where $a_{\ell_i} \in \mathcal{B}_{\lceil \frac{\ell_i}{b} \rceil}$ for all $1 \le i \le k$, we define the set of bin gaps as follows:

$$\text{BGaps}(m) \ := \ \left\{ \left\lceil \frac{\ell_1}{b} \right\rceil - \left\lceil \frac{\ell_2}{b} \right\rceil, \left\lceil \frac{\ell_2}{b} \right\rceil - \left\lceil \frac{\ell_3}{b} \right\rceil, \dots, \left\lceil \frac{\ell_{k-1}}{b} \right\rceil - \left\lceil \frac{\ell_k}{b} \right\rceil \right\}. \tag{1.13}$$

[1]Using the methods of [4], these results can be extended to hold almost surely for sufficiently large subinterval of $[q_n, q_{n+1})$.

Notice we do not include the wait to the first bin, $\left\lceil \frac{\ell_1}{b} \right\rceil - 0$, as a bin gap. We could include this if we wish; one additional bin gap will not affect the limiting behavior. We study the gaps between bins, and not between individual summands, because each bin contains at most one summand, and it is natural to view each bin as either 'on' or 'off'. At the cost of more involved formulas we could deduce similar results about gaps between summands.

In the theorem below we consider all the bin gaps in (s, b)-Generacci legal decompositions of all $m \in [a_{(n-1)b+1}, a_{nb+1})$. We let $P_n(g)$ be the fraction of all these bin gaps that are of length g (i.e., the probability of a bin gap of length g among (s, b)-Generacci legal decompositions of $m \in [a_{(n-1)b+1}, a_{nb+1})$). For example, when considering the $(4, 9)$-Generacci sequence notice $m = a_3 + a_{53} + a_{99} + a_{171} + a_{279}$ with $a_3 \in \mathcal{B}_1$, $a_{53} \in \mathcal{B}_6$, $a_{99} \in \mathcal{B}_{11}$, $a_{171} \in \mathcal{B}_{19}$ and $a_{279} \in \mathcal{B}_{31}$, contributes two bin gaps of length 5, one bin gap of length 8, and one bin gap of length 12.

Theorem 1.15 (Average Bin Gap Measure for the (s, b)-Generacci Sequences)**.** *For $P_n(g)$ as above, the limit $P(g) := \lim_{n\to\infty} P_n(g)$ exists. For $g < (s + 1)$, $P(g) = 0$, and*

$$P(g) \;=\; b(\lambda_1^b)^{-g} \quad (g \geq s + 1), \tag{1.14}$$

where λ_1 is the largest root of $x^{(s+1)b} - x^{sb} - b = 0$.

The proof of Theorem 1.15 is given in §3.1.

We obtain similar results for the individual spacing gap bin measure. We can use the result from [12] by showing certain combinatorial conditions are met. We quickly review the needed notation from that paper, then state the result.

Given a sequence $\{b_n\}$ and a decomposition rule that leads to unique decomposition, fix constants c_1, d_1, c_2, d_2 such that $I_n := [b_{c_1 n+d_1}, b_{c_2 n+d_2})$ is a well-defined interval for all $n > 0$. Below $\delta(x - a)$ denotes the Dirac delta functional, assigning a mass of 1 to $x = a$ and 0 otherwise.

- *Spacing gap measure:* The spacing gap measure of a $z \in I_n$ with $k(z)$ summands is

$$\nu_{z,n}(x) \;:=\; \frac{1}{k(z) - 1} \sum_{j=2}^{k(z)} \delta(x - (\ell_j - \ell_{j-1})). \tag{1.15}$$

- *Average spacing gap measure:* The total number of gaps for all $z \in I_n$ is

$$N_{\text{gaps}}(n) \;:=\; \sum_{z=b_{c_1 n+d_1}}^{b_{c_2 n+d_2}-1} (k(z) - 1). \tag{1.16}$$

  The average spacing gap measure for all $z \in I_n$ is

$$\begin{aligned} \nu_n(x) \quad &:= \quad \frac{1}{N_{\text{gaps}}(n)} \sum_{z=b_{c_1 n+d_1}}^{b_{c_2 n+d_2}-1} \sum_{j=2}^{k(z)} \delta\left(x - (\ell_j - \ell_{j-1})\right) \\ &= \quad \frac{1}{N_{\text{gaps}}(n)} \sum_{z=b_{c_1 n+d_1}}^{b_{c_2 n+d_2}-1} (k(z) - 1)\, \nu_{z,n}(x). \end{aligned} \tag{1.17}$$

  Letting $P_n(g)$ denote the probability of a gap of length g among all gaps from the decompositions of all $m \in I_n$, we have

$$\nu_n(x) \;=\; \sum_{g=0}^{c_2 n+d_2-1} P_n(g)\delta(x - g). \tag{1.18}$$

- *Limiting average spacing gap measure, limiting gap probabilities:* If the limits exist, let

$$\nu(x) \;=\; \lim_{n\to\infty} \nu_n(x), \qquad P(g) \;=\; \lim_{n\to\infty} P_n(g). \tag{1.19}$$

Although this notation was originally defined for gaps between summands, by taking the ℓ_i to represent the gaps between bins, this notation is applicable to our sequences.

Theorem 1.16 (Spacing Bin Gap Measure for (s, b)-Generacci sequences)**.** *Let $\{a_n\}$ denote the (s, b)-Generacci sequence, then for $z \in I_n := [a_{b(n-1)+1}, a_{bn+1})$, the spacing bin gap measures $\nu_{z,n}(x)$ converge almost surely in distribution to the limiting bin gap measure $\nu(x)$.*

As $\nu(x) = P(x)$, the spacing bin gap measure converges in distribution to geometric decay behavior.

The same ideas which gave us Gaussian behavior for the Fibonacci Quilt Greedy-6 decomposition from the Gaussian behavior for the $(4, 1)$-Generacci sequence also, with trivial tweaking, yield similar results on the average and spacing gap measures. We consider all $m \in I_n := [q_n, q_{n+1})$, i.e., those m with a Greedy-6 decomposition beginning with q_n. We let $P_n(g)$ be the fraction of all gaps from all $m \in I_n$ that are of length g.

Theorem 1.17 (Average and Spacing Gap Measures for the Greedy-6 Decomposition)**.** *Let $\{q_n\}$ denote the Fibonacci Quilt sequence, $P_n(g)$ as above, and consider $m \in I_n := [q_n, q_{n+1})$. The limit $P(g) := \lim_{n\to\infty} P_n(g)$ exists and agrees with the $(4, 1)$-Generacci limit, and the spacing gap measures $\nu_{z,n}(x)$ from the Greedy-6 decomposition converge almost surely in distribution to the limiting gap measure from the (4,1)-Generacci sequence.*

1.2.4. *New behavior for Fibonacci quilt sequence: $k_{\min}$ vs $k_{\max}$.* We do not have unique decompositions with the Fibonacci Quilt sequence. By Theorem 1.11, we know that the Greedy-6 algorithm results in a legal decomposition with a minimal number of summands. Here we investigate the range of the number of summands in any FQ-legal decomposition.

Definition 1.18. *We define $k_{\min}(m)$ (resp. $k_{\max}(m)$) to be the smallest (resp. largest) number of summands in any FQ-legal decomposition of m.*

The following result gives a lower bound for the growth of $k_{\max}(m) - k_{\min}(m)$ which holds for almost all $m \in [q_n, q_{n+1})$ as $n \to \infty$. In particular, we almost always have $k_{\max}(m) \neq k_{\min}(m)$. The proof is given in §5.

Theorem 1.19. *There is a $C_{\rm FQ} > 0$ such that, as $n \to \infty$, we have*

$$k_{\max}(m) - k_{\min}(m) \geq C_{\rm FQ} \log(n)$$

for almost all $m \in [q_n, q_{n+1})$.

2. Gaussian Behavior of Number of Summands

The following sections provide the pieces needed to prove Theorems 1.12 and 1.14. We introduce a new method that allows us to bypass many of the technical obstructions that arise when using standard techniques to handle the determination of the mean and variance in the number of summands. Using this approach we not only can reprove existing results, but also handle cases such as the (s, b)-Generacci and the Fibonacci Quilt sequences of this paper.

2.1. **Proof of Positivity of Linear Terms.** The idea of this section is to reprove and generalize many of the technical results from [28] without doing the involved analysis that is needed in order to derive properties of roots of certain polynomials in several variables. In many other papers the methods from [28] can be used without too much trouble, as there are explicit formulas available for all the polynomials which arise; however, there are many situations where this is not the case. These difficulties greatly lengthened that paper (and restricted the reach of other works) and resulted in several technical appendices on the behavior of the roots. We avoid these calculations by adopting a more combinatorial view.

Letting $\{a_n\}$ be any sequence of interest, we prove that the mean and the variance in the number of summands of $m \in [a_n, a_{n+1})$ diverge linearly with n. Standard generating function arguments show that the first grows like $Cn+d+o(1)$ and the second like $C'n+d'+o(1)$, where the constants are values of roots of certain associated polynomials (and their derivatives). The difficulty in the subsequent analysis of the Gaussianity of the number of summands is that C or C' could vanish. Briefly, the idea behind our combinatorial approach below is that if C were to vanish, we would count incorrectly and not have the right number of decompositions. The proof for C (the mean) is very straightforward; the proof for C' (the variance) is more involved, though it essentially reduces to a good approach to counting and then careful book-keeping.

In the arguments below we use a_n to denote the n^{th} term of the sequence; we use this and not G_n to emphasize the generality of the results (i.e., the results below are true for more than just PLRS).

2.2. **The Mean.** We introduce some terminology to help us prove results in great generality. Given a length L, a *segment* of summands in a generalized Zeckendorf decomposition starting at index i are the summands taken from $\{a_i, a_{i+1}, \dots, a_{i+L-1}\}$; note that for some decomposition rules we may choose a summand with multiplicity. If we write the expansion for $m \in [a_n, a_{n+1})$ we get

$$m \;=\; a_{r_1} + a_{r_2} + \cdots + a_{r_{k(m)}} \tag{2.1}$$

with $a_{r_1} \geq a_{r_2} \geq \cdots \geq a_{r_{k(m)}}$, where frequently $r_1 = n$. We denote the number of summands of m as $k(m)$, while the number of summands in the segment of length L starting at i is just the number of indices r_j with $i \leq r_j < i + L$.

Definition 2.1. *We say the legal decomposition* acts over a fixed distance *if there is some finite number f such that two segments of a legal decomposition do not interact if they are separated by at least f consecutive summands that are not chosen. This means that whatever summands we have (or do not have) in one segment does not affect our choices in the other, and for the entire decomposition to be legal each of these two segments must be legal.*

Note that the sequences we study in this paper both act over a fixed distance. For the (s,b)-Generacci sequence we can take $f = sb+1$ and for the Fibonacci Quilt sequence we can take $f = 5$. It is also the case that Positive Linear Recurrence relations, which come with a notion of a legal decomposition, act over a fixed distance (we can take f to be at least the length of the recurrence).

The next theorem states that for many generalized Zeckendorf decompositions, μ_n, the average number of summands of integers in $[a_n, a_{n+1})$, is a linear function in n with positive slope, up to an $o(1)$ term which vanishes in the limit.

Theorem 2.2. *Consider an increasing sequence $\{a_n\}$ which gives rise to unique legal decompositions of the positive integers such that*

- *the rule for the legal decomposition acts over a fixed distance,*

- *the average number of summands used for* $m \in [a_n, a_{n+1})$ *is* $\mu_n = Cn + d + o(1)$*, and*
- *given any constant* $A > 0$ *there is a length* L *and a probability* $p = p(A, L) > 0$ *that is less than or equal to the proportion of legal ways to choose summands in any segment of length* L *that have at least* A *summands, regardless of the choices of summands outside the segment.*

Then $C > 0$*.*

Remark 2.3. *Both* (s, b)*-Generacci Sequences and PLRS sequences satisfy all three conditions. To see that* (s, b)*-Generacci Sequences satisfy the third condition, given* A *if we take* $L \geq Asb$ *then there is at least one legal way to choose* A *summands from a segment of length* L*. Hence* $p(A, L) > 0$*.*

Proof of Theorem 2.2. Assume the claim is false and hence $C = 0$. We show that at least half of the integers have decompositions with at least twice the average number of summands, which contradicts the average number of summands.

For all n sufficiently large, as $C = 0$ we have $\mu_n \leq 2d$. We choose A to be much larger than $2d$, say $A = 1000(2d + 1)$. Let L be large relative to the fixed distance of the decomposition rule (for example, 100 times). For simplicity we assume n is a multiple of L so we may split decompositions up into n/L segments of length L, though of course this is not essential and we could just ignore the last segment. We also assume L is large enough so that the third condition holds, namely there is a constant $p(A, L) > 0$ such that in any segment of length L the probability we choose fewer than A summands is at most $1 - p(A, L) < 1$.

We claim that as $n \to \infty$, with probability 1 a decomposition has at least A summands. To see this, we can bound the probability that it has fewer summands by noting that if that were true, it must have fewer than A summands in each of the n/L segments of length L. Thus

$$\text{Prob}(m \in [a_n, a_{n+1}) \text{ has less than } A \text{ summands})) \ \leq \ (1 - p(A, L))^{n/L}. \tag{2.2}$$

Thus the probability an $m \in [a_n, a_{n+1})$ has at least A summands tends to 1 as desired:

$$\text{Prob}(m \in [a_n, a_{n+1}) \text{ has at least } A \text{ summands})) \ \geq \ 1 - (1 - p(A, L))^{n/L}. \tag{2.3}$$

As $p(A, L) > 0$ is independent of n, by taking n sufficiently large at least half of the m in the interval have at least A summands. If we assume all of these have exactly A summands and the rest have 0 then we see that the average number of summands is at least $A/2$, or $500(2d + 1)$. As this is far greater than $2d$ we have a contradiction. □

2.3. **The Variance.** We first define additional terminology (especially another notion of legal decompositions) that will help us state our result in great generality.

Definition 2.4. *A* ***block*** *is a nonempty finite sequence of nonnegative integers. The* ***size*** *of a block is the sum of the integers in the sequence, while the* ***length*** *of a block is the number of integers in the sequence.*

A ***block-batch****,* $\mathcal{S}$*, is a finite set of blocks with the following characteristics:*

(i) *If two blocks have the same size, then they have the same length,*
(ii) $\mathcal{S}$ *contains a block of size 0, whose length is minimal among all blocks in* $\mathcal{S}$*, and*
(iii) $\mathcal{S}$ *contains at least one block of size 1.*

Property (i) allows us to define a ***length function****:* $l(t)$ *is the length of all blocks with size* t*.*

Definition 2.5. ***(Definition of*** $(\mathcal{S}, \mathcal{T})$***-legal decompositions)*** *Consider a strictly increasing sequence of positive integers* $\{a_j\}_{j=1}^{\infty}$*. Let* $\mathcal{S}$ *be a given block-batch and* $\mathcal{T}$ *be a given finite set of blocks. Let* $\mathcal{L}_{\mathcal{T}}$ *be the maximum length of all blocks in* $\mathcal{T}$ *(*$\mathcal{L}_{\mathcal{T}} = 0$ *if* $\mathcal{T}$ *is empty). A* *decomposition of a positive integer* $\omega \in \mathbb{Z}$, $\omega = \sum_{i=1}^{m} c_i a_{m+1-i}$, *is* $(\mathcal{S}, \mathcal{T})$**-legal** *if the coefficient sequence* $\{c_i\}_{i=1}^{m}$ *has* $c_1 > 0$, *the other* $c_i \geq 0$, *and one of the following two conditions holds:*

- *Condition 1: We have* $m \leq \mathcal{L}_{\mathcal{T}}$ *and the sequence* $\{c_i\}_{i=1}^{m}$ *is a block in* $\mathcal{T}$.
- *Condition 2: There exists* $s \geq 1$ *such that the sequence* $\{c_i\}_{i=1}^{s}$ *is in block-batch* $\mathcal{S}$ *and* $\{b_i\}_{i=1}^{m-s}$ *(with* $b_i = c_{s+i}$*) is* $(\mathcal{S}, \mathcal{T})$*-legal or empty.*

We observe the following key properties.

(1) If a $(\mathcal{S}, \mathcal{T})$-legal decomposition contains a $\mathcal{T}$ type block, then it must be the last block. So any $(\mathcal{S}, \mathcal{T})$-legal decomposition contains at most one $\mathcal{T}$ type block.
(2) An $(\mathcal{S}, \mathcal{T})$-legal decomposition will stay $(\mathcal{S}, \mathcal{T})$-legal if an $\mathcal{S}$ type block is added or removed and indices are shifted accordingly. Only whole blocks can be added and removed. Moreover added blocks cannot be inserted in the middle of existing blocks.

Remark 2.6. *The usual legal decomposition rules for* (s, b)*-Generacci Sequences and Positive Linear Recurrence Sequences can be viewed as* $(\mathcal{S}, \mathcal{T})$*-legal decompositions. See* [8, Appendix D] *for examples showing how decompositions using several well-known sequences can be viewed as* $(\mathcal{S}, \mathcal{T})$*-legal decompositions.*

Let Ω_n be the set of all $(\mathcal{S}, \mathcal{T})$-legal decompositions of integers in $[a_n, a_{n+1})$. Take an $(\mathcal{S}, \mathcal{T})$-legal decomposition $\omega \in \Omega_n$ and define the number of summands in the decomposition: $Y_n(\omega) = \sum_{i=1}^{m} c_i$. We will define several other random variables that will assist in our study of Y_n. When $n > \mathcal{L}_{\mathcal{S}} + \mathcal{L}_{\mathcal{T}}$ (with $\mathcal{L}_{\mathcal{S}}$ the length of the longest block in $\mathcal{S}$), there are at least two $\mathcal{S}$ type blocks in each decomposition. We define the random variable Z_n by setting $Z_n(\omega)$ equal to the size of the last $\mathcal{S}$ type block of $\omega \in \Omega_n$. Similarly, we define the random variable L_n by setting $L_n(\omega)$ equal to the length of the last $\mathcal{S}$ type block of $\omega \in \Omega_n$.

Theorem 2.7. *Consider a strictly increasing sequence of positive integers* $\{a_n\}_{i=1}^{\infty}$ *with* $a_{i+1} - a_i \geq a_{j+1} - a_j$ *for all* $i \geq j$ *and* $a_{i+1} - a_i > 1$ *for all* $i > \mathcal{L}_{\mathcal{T}} + 1$, *block-batch* $\mathcal{S}$, *and set of blocks* $\mathcal{T}$ *such that all positive integers have unique* $(\mathcal{S}, \mathcal{T})$*-legal decompositions. If* $\mathbb{E}[Y_n] = Cn + d + f(n)$ *with* $C > 0$ *and* $f(n) = o(1)$, *and if* $\mathrm{Var}[Y_n] = C'n + d' + o(1)$, *then we can explicitly find* $\kappa > 0$, *such that* $\mathrm{Var}[Y_n] \geq \kappa n$ *for all* $n \geq \mathcal{L}_{\mathcal{T}} + 2$. *In other words,* $C' > 0$.

We assume the hypotheses of this theorem hold in all lemmas and corollaries below. Note that (s, b)-Generacci and PLRS Sequences satisfy these hypotheses.

We need additional notation. Let $\mathcal{Z}_{\mathcal{S}}$ be the maximum size of all blocks in $\mathcal{S}$. For all $0 \leq t \leq \mathcal{Z}_{\mathcal{S}}$, define $\mathcal{B}_t$ to be the subset of blocks in $\mathcal{S}$ whose size is t. For $\mathfrak{b} \in \mathcal{B}_t$, we define $\Upsilon_{n,\mathfrak{b}} = \{\omega \in \Omega_n \mid \text{the last } \mathcal{S} \text{ type block is } \mathfrak{b}\}$,

Lemma 2.8. *Let* $n > \mathcal{L}_{\mathcal{S}} + \mathcal{L}_{\mathcal{T}}$. *Define* $\phi_{t,\mathfrak{b}}(\omega)$ *to be the decomposition that results from removing the last* $\mathcal{S}$ *type block of* ω *and shifting indices appropriately. Then* $\phi_{t,\mathfrak{b}}$ *is a bijection between* $\Upsilon_{n,\mathfrak{b}}$ *and* $\Omega_{n-l(t)}$.

The proof follows by straightforward counting; see [8, Appendix D].

Corollary 2.9. *If* $\mathbb{E}[Y_n] = Cn + d + f(n)$ *then*

$$\mathbb{E}[Y_n | Z_n = t] \;=\; C(n - l(t)) + d + f(n - l(t)) + t, \tag{2.4}$$

$$\mathbb{E}[Y_n^2 | Z_n = t] \;=\; \mathbb{E}[Y_{n-l(t)}^2] + 2t[C(n - l(t)) + d + f(n - l(t))] + t^2, \tag{2.5}$$

and when $K_n := Z_n(\omega) + f(n - L_n(\omega)) - CL_n(\omega)$ *the*

$$\mathbb{E}[K_n] \;=\; f(n) \tag{2.6}$$

The proof relies upon the bijection between $\Upsilon_{n,\mathfrak{b}}$ and $\Omega_{n-l(t)}$ which allows us to conclude $\mathbb{E}[Y_n|\text{the last } \mathcal{S} \text{ type block is } \mathfrak{b}] = \mathbb{E}[Y_{n-l(t)}+t]$. The final form of the equations are a result of straightforward algebraic manipulation and rules of probability. The complete proof can be found in [8, Appendix D].

Lemma 2.10. *Assume that all integers in Ω_n have unique $(\mathcal{S},\mathcal{T})$-legal decompositions with respect to the sequence $\{a_n\}$. Then for $n > \mathcal{L}_\mathcal{S}+\mathcal{L}_\mathcal{T}$*

$$\mathbb{P}[Z_n=t] \;=\; |\mathcal{B}_t|\frac{a_{n-l(t)+1}-a_{n-l(t)}}{a_{n+1}-a_n}. \tag{2.7}$$

Proof. We have

$$\mathbb{P}[Z_n=t] \;=\; \sum_{\mathfrak{b}\in\mathcal{B}_t}\frac{|\Upsilon_{n,\mathfrak{b}}|}{|\Omega_n|} \;=\; \sum_{\mathfrak{b}\in\mathcal{B}_t}\frac{|\Omega_{n-l(t)}|}{|\Omega_n|} \;=\; |\mathcal{B}_t|\frac{a_{n-l(t)+1}-a_{n-l(t)}}{a_{n+1}-a_n}. \tag{2.8}$$

□

Corollary 2.11. *Consider a strictly increasing sequence of positive integers $\{a_n\}$ with $a_{i+1}-a_i \geq a_{j+1}-a_j$ for all $i \geq j$. Then for $n > \mathcal{L}_\mathcal{S}+\mathcal{L}_\mathcal{T}$, $\mathbb{P}[Z_n=0] \geq 1/|\mathcal{S}|$.*

The proof is a straightforward application of the lemma; see [8, Appendix D].

Finally we consider the variance by first using $\mathbb{E}[K_n]$ to estimate $\mathrm{Var}[K_n]$.

Lemma 2.12. *For large n, $\mathrm{Var}[K_n] > \frac{C^2 l(0)^2}{2|\mathcal{S}|} > 0$.*

Proof. For all $n > \mathcal{L}_\mathcal{S}+\mathcal{L}_\mathcal{T}$, we have

$$\begin{aligned}\mathrm{Var}[K_n] &= \mathbb{E}[K_n^2]-(\mathbb{E}[K_n])^2\\ &= \left(\mathbb{E}[(Z_n-CL_n+f(n-L_n))^2]\right)-(f(n))^2\\ &= \left(\mathbb{E}[(Z_n-CL_n)^2]+\mathbb{E}[2(Z_n-CL_n)\cdot f(n-L_n)]+\mathbb{E}[f(n-L_n)^2]\right)-(f(n))^2.\end{aligned} \tag{2.9}$$

Note $0 \leq L_n \leq \mathcal{L}_\mathcal{S}$ and that $Z_n - aL_n$ is bounded since $-C\mathcal{L}_\mathcal{S} \leq Z_n - CL_n \leq \mathcal{Z}_\mathcal{S}$. Also we know $f(n) = o(1)$. Thus

$$\lim_{n\to\infty}\mathbb{E}[2(Z_n-CL_n)\cdot f(n-L_n)]\;,\;\lim_{n\to\infty}\mathbb{E}[f(n-L_n)^2]\;,\;\lim_{n\to\infty}(f(n))^2 \;=\; 0.$$

Hence

$$\lim_{n\to\infty}\left(\mathrm{Var}[K_n]-\mathbb{E}[(Z_n-CL_n)^2]\right) \;=\; 0. \tag{2.10}$$

On the other hand, for all $n > \mathcal{L}_\mathcal{S}+\mathcal{L}_\mathcal{T}$ we have

$$\begin{aligned}\mathbb{E}[(Z_n-CL_n)^2] &= \sum_{t=0}^{\mathcal{Z}_\mathcal{S}}\mathbb{P}[Z_n=t]\cdot(t-Cl(t))^2\\ &\geq \mathbb{P}[Z_n=0]\cdot(0-Cl(0))^2 \;\geq\; \frac{C^2l(0)^2}{|\mathcal{S}|},\end{aligned} \tag{2.11}$$

where the last inequality comes from Corollary 2.11.

By Equation (2.10), we know there must exist an $N > \mathcal{L}_\mathcal{S}+\mathcal{L}_\mathcal{T}$ such that for all $n > N$, $|\mathrm{Var}[K_n]-\mathbb{E}[(Z_n-CL_n)^2]| < \frac{C^2l(0)^2}{2|\mathcal{S}|}$, so $\mathrm{Var}[K_n]-\mathbb{E}[(Z_n-CL_n)^2] > -\frac{C^2l(0)^2}{2|\mathcal{S}|}$. Then (2.11) implies $\mathrm{Var}[K_n] > \frac{C^2l(0)^2}{2|\mathcal{S}|} > 0$ for all $n > N > \mathcal{L}_\mathcal{S}+\mathcal{L}_\mathcal{T}$. □

Finally, we choose κ. For N as found in Lemma 2.12, define $\hat{N} := \max\{\mathcal{L}_{\mathcal{S}} + \mathcal{L}_{\mathcal{T}} + 2, N\}$. Next let

$$\kappa \ = \ \min\left\{\frac{\mathrm{Var}[Y_{\mathcal{L}_{\mathcal{T}}+2}]}{\mathcal{L}_{\mathcal{T}}+2}, \frac{\mathrm{Var}[Y_{\mathcal{L}_{\mathcal{T}}+3}]}{\mathcal{L}_{\mathcal{T}}+3}, \dots, \frac{\mathrm{Var}[Y_{\hat{N}}]}{\hat{N}}, \frac{C^2 l(0)^2}{2|\mathcal{S}|\mathcal{L}_{\mathcal{S}}}\right\}. \tag{2.12}$$

For all $n > \mathcal{L}_{\mathcal{T}} + 1$, $a_{n+1} - a_n > 1$, so there are at least two integers in $[a_n, a_{n+1})$. Since the $(\mathcal{S}, \mathcal{T})$-legal decomposition of a_n has only one summand while that of $a_n + 1$ has two or more summands, $\mathrm{Var}[Y_n]$ is nonzero when $n > \mathcal{L}_{\mathcal{T}} + 1$. Hence, $\kappa > 0$.

Now we are ready to prove Theorem 2.7.

Proof of Theorem 2.7. We proceed by strong induction.

Basis step: For $n = \mathcal{L}_{\mathcal{T}} + 2, \mathcal{L}_{\mathcal{T}} + 3, \dots, \hat{N}$, $\mathrm{Var}[Y_n] > \kappa n$ by definition of κ.

Induction step: Assume $\mathrm{Var}[Y_r] \geq \kappa r$ for $\mathcal{L}_{\mathcal{T}} + 2 \leq r < n$. We only need to consider the cases when $n > \hat{N} \geq \mathcal{L}_{\mathcal{S}} + \mathcal{L}_{\mathcal{T}} + 2$. So for all $0 \leq t \leq \mathcal{Z}_{\mathcal{S}}$, $n > n - l(t) \geq n - \mathcal{L}_{\mathcal{S}} \geq \mathcal{L}_{\mathcal{T}} + 2$.

By (2.5) we have

$$\begin{aligned}
\mathbb{E}[Y_n^2] \ &= \ \sum_{t=0}^{\mathcal{Z}_{\mathcal{S}}} \mathbb{P}[Z_n = t] \cdot \mathbb{E}[Y_n^2 | Z_n = t] \\
&= \ \sum_{t=0}^{\mathcal{Z}_{\mathcal{S}}} \mathbb{P}[Z_n = t] \cdot \left(\mathbb{E}[Y_{n-l(t)}^2] + 2t[C(n - l(t)) + d + f(n - l(t))] + t^2\right),
\end{aligned} \tag{2.13}$$

and from the inductive hypothesis we have

$$\begin{aligned}
\mathbb{E}[Y_{n-l(t)}^2] \ &= \ \mathrm{Var}[Y_{n-l(t)}] + \left(\mathbb{E}[Y_{n-l(t)}]\right)^2 \\
&\geq \ \kappa(n - l(t)) + \left(C(n - l(t)) + d + f(n - l(t))\right)^2.
\end{aligned} \tag{2.14}$$

Combining (2.13) and (2.14) results in an equation with two parts. One is independent of t, while the other is of the form of $Z_n + f(n - L_n) - CL_n$, which is exactly K_n. We find

$$\begin{aligned}
\mathbb{E}[Y_n^2] \ &\geq \ \sum_{t=0}^{\mathcal{Z}_{\mathcal{S}}} \mathbb{P}[Z_n = t] \Bigg[\kappa(n - l(t)) + \left(C(n - l(t)) + d + f(n - l(t))\right)^2 \\
&\qquad\qquad + 2t[C(n - l(t)) + d + f(n - l(t))] + t^2\Bigg] \\
&= \ (Cn + d)^2 + \kappa n + \sum_{t=0}^{\mathcal{Z}_{\mathcal{S}}} \mathbb{P}[Z_n = t] \cdot (t + f(n - l(t)) - Cl(t))^2 \\
&\qquad + 2(Cn + d) \sum_{t=0}^{\mathcal{Z}_{\mathcal{S}}} \mathbb{P}[Z_n = t] \cdot (t + f(n - l(t)) - Cl(t)) - \kappa \sum_{t=0}^{\mathcal{Z}_{\mathcal{S}}} \mathbb{P}[Z_n = t] \cdot l(t) \\
&= \ (Cn + d)^2 + \kappa n + \mathbb{E}[(Z_n + f(n - L_n) - CL_n)^2] + 2(Cn + d) f(n) - \kappa \mathbb{E}[L_n],
\end{aligned} \tag{2.15}$$

with the last equality coming from (2.6).

Finally, (2.15), the definition of κ, and Lemma 2.12 imply

$$\begin{aligned} \mathrm{Var}[Y_n] - \kappa n &= \mathbb{E}[Y_n^2] - (\mathbb{E}[Y_n])^2 - \kappa n \\ &\geq \mathbb{E}[(Z_n + f(n-L_n) - CL_n)^2] - \kappa\mathbb{E}[L_n] - (f(n))^2 \\ &= \mathbb{E}[K_n^2] - \kappa\mathbb{E}[L_n] - (\mathbb{E}[K_n])^2 \\ &= \mathrm{Var}[K_n] - \kappa\mathbb{E}[L_n] \\ &\geq \mathrm{Var}[K_n] - \kappa\mathcal{L}_{\mathcal{S}} \\ &\geq \frac{C^2 l(0)^2}{2|\mathcal{S}|} - \frac{C^2 l(0)^2}{2|\mathcal{S}|\mathcal{L}_{\mathcal{S}}}\mathcal{L}_{\mathcal{S}} \;=\; 0, \end{aligned} \tag{2.16}$$

and therefore $\mathrm{Var}[Y_n] \geq \kappa n$. □

2.4. **Generating Function for (s,b)-Generacci Legal Decompositions.** Let $p_{n,k}$ (with $n, k \geq 0$) denote the number of $m \in [a_{(n-1)b+1}, a_{nb+1})$ whose (s,b)-Generacci legal decomposition contains exactly k summands, where a_{nb+1} is the first entry in the $(n+1)^{\mathrm{st}}$ bin of size b.

Proposition 2.13. *Let $n, k \geq 0$. Then*

$$p_{n,k} \;=\; \begin{cases} 1 & \text{if } n = k = 0 \\ b & \text{if } 1 \leq n \leq s \text{ and } k = 1 \\ b \cdot q_{n-(s+1),k-1} & \text{if } n \geq s+1 \text{ and } 1 \leq k \leq \frac{n+s}{s+1} \\ 0 & \text{otherwise,} \end{cases} \tag{2.17}$$

where $q_{n,k}$ (with $n, k \geq 0$) is the number of $m \in [0, a_{nb+1})$ whose (s,b)-Generacci legal decomposition contains exactly k summands. Set $F(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{n_*} p_{n,k}x^n y^k$ with $n_* = \lceil \frac{n+s}{s+1} \rceil$. Then

$$F(x,y) \;=\; 1 + \frac{byx}{1 - x - byx^{s+1}}. \tag{2.18}$$

We omit the proof here as the details follow from standard bookkeeping and algebraic manipulation. The proof of this proposition is found in [8, Appendix A].

To complete the proof of Theorem 1.12 we make use the following result from [11].

Theorem 2.14. [11, Theorem 1.8] *Let κ be a fixed positive integer. For each n, let a discrete random variable Y_n in $I_n = \{1, 2, \ldots, n\}$ have*

$$\mathrm{Prob}(Y_n = j) \;=\; \begin{cases} p_{j,n}/\sum_{j=1}^{n} p_{j,n} & \text{if } j \in I_n \\ 0 & \text{otherwise} \end{cases} \tag{2.19}$$

for some positive real numbers $p_{1,n}, p_{2,n}, \ldots, p_{n,n}$. Let $g_n(y) := \sum_j p_{j,n}y^j$.

If g_n has the form $g_n(y) = \sum_{i=1}^{\kappa} q_i(y)\alpha_i^n(y)$ where

(i) *for each $i \in \{1, \ldots, \kappa\}, q_i, \alpha_i : \mathbb{R} \to \mathbb{R}$ are three times differentiable functions which do not depend on n;*
(ii) *there exists some small positive ϵ and some positive constant $\lambda < 1$ such that for all $y \in I_\epsilon = [1-\epsilon, 1+\epsilon], |\alpha_1(y)| > 1$ and $|\frac{\alpha_i(y)}{\alpha_1(y)}| < \lambda < 1$ for all $i = 2, \ldots, \kappa$;*

then

(1) *the mean* μ_n *and variance* σ_n^2 *of* Y_n *both grow linearly with* n*. Specifically,*

$$\mu_n \;=\; Cn + d + o(1), \qquad \sigma_n^2 \;=\; C'n + d' + o(1) \tag{2.20}$$

where

$$\begin{aligned} C &= \frac{\alpha_1'(1)}{\alpha_1(1)},\ d \;=\; \frac{q_1'(1)}{q_1(1)} \\ C' &= \frac{d}{dy}\left(\frac{y\alpha_1'(y)}{\alpha_1(y)}\right)\bigg|_{y=1} \;=\; \frac{\alpha_1(1)[\alpha_1'(1)+\alpha_1''(1)]-\alpha_1'(1)^2}{\alpha_1(1)^2} \\ d' &= \frac{d}{dy}\left(\frac{yq_1'(y)}{q_1(y)}\right)\bigg|_{y=1} \;=\; \frac{q_1(1)[q_1'(1)+q_1''(1)]-q_1'(1)^2}{q_1(1)^2}. \end{aligned} \tag{2.21}$$

Moreover, if

(iii) $\alpha_1'(1) \neq 0$ *and* $\frac{d}{dy}\left[\frac{y\alpha_1'(y)}{\alpha_1(y)}\right]|_{y=1} \neq 0$*, i.e.,* $C, C' > 0$*,*

then

(2) *as* $n \to \infty$*,* Y_n *converges in distribution to a normal distribution.*

To apply Theorem 2.14 we still need some auxiliary results about the function $g_n(y)$ which gives the coefficient of x^n in the expansion of the generating function $F(x,y)$. In fact we need results regarding the partial fraction decomposition of $1/(1-x-byx^{s+1})$.

Lemma 2.15. *Let* $s, b \geq 1$ *and* $y > 0$*. Let* $f(x) = 1 - x - byx^{s+1}$*. Then*

(1) $f(x)$ *has no repeated roots,*
(2) $f(x)$ *has a positive root* $\lambda_1(y)$ *whose modulus is smaller than the modulus of any other root of* $f(x)$*. Moreover,* $\lambda_1(y) < 1$*.*

Proof. (1) Let $h(x) = x^{s+1} + ax - a$, where $a = 1/by$, and suppose that $h(x)$ has a repeated root, say r (note $r \neq 0$). Then $h(r)$ and $h'(r)$ equal 0 yields a contradiction. (2) To find the roots of $f(x) = 1 - x - byx^{s+1}$ we use the change of variable $w = 1/x$ and note that the roots of

$$g(w) \;=\; w^{s+1} - w^s - by \tag{2.22}$$

are the eigenvalues of the companion matrix of the polynomial $g(w)$. This matrix is a non-negative irreducible matrix so by the Perron-Frobenius Theorem, $g(w)$ has a unique positive dominant root $\mu(y)$. Hence $\lambda_1(y) := \frac{1}{\mu(y)}$ is the unique positive root of $f(x)$ with smallest modulus. Now by applying the Intermediate Value Theorem we note that one of the positive roots lies in the interval $[0,1]$. Since $\lambda_1(y)$ is the smallest positive root, then clearly $0 < \lambda_1(y) < 1$. □

Proposition 2.16. *Let* $g_n(y) = \sum_{k=0}^{\infty} p_{n,k}y^k$*, which is the coefficient of* x^n *in the generating function of the* $p_{n,k}$*'s. Then for sufficiently large* n

$$g_n(y) \;=\; \sum_{i=1}^{s+1} q_i(y)\alpha_i^n(y), \tag{2.23}$$

where for $1 \leq i \leq s+1$*,* $\alpha_i(y) = \frac{1}{\lambda_i(y)}$ *with* $\lambda_i(y)$ *the distinct roots of the polynomial* $f(x) = 1 - x - byx^{s+1}$ *and* $q_i(y)$ *are algebraic functions of* y *which depend on these roots.*

Proof. Let $\lambda_1(y), \lambda_2(y), \ldots, \lambda_{s+1}(y)$ be the distinct roots of $f(x) = 1 - x - byx^{s+1}$. Using a partial fraction decomposition of $1/f(x)$,

$$\frac{1}{f(x)} \;=\; \sum_{i=1}^{s+1} \frac{p_i(y)}{x - \lambda_i(y)}, \tag{2.24}$$

where $p_i(y)$ are algebraic functions of y depending on $\lambda_i(y)$. By rewriting the terms and using the geometric sum formula we have that

$$\frac{1}{f(x)} \;=\; \sum_{i=1}^{s+1} \hat{p}_i(y) \frac{1}{1 - \frac{x}{\lambda_i(y)}} \;=\; \sum_{i=1}^{s+1} \sum_{n=0}^{\infty} \hat{p}_i(y) \left(\alpha_i(y) x\right)^n \;=\; \sum_{n=0}^{\infty} \left[\sum_{i=1}^{s+1} \hat{p}_i(y) \alpha_i^n(y)\right] x^n, \tag{2.25}$$

where $\hat{p}_i(y) = -\frac{p_i(y)}{\lambda_i(y)}$ and $\alpha_i(y) = \frac{1}{\lambda_i(y)}$. So

$$F(x, y) \;=\; \frac{1 + x(by - 1) - byx^{s+1}}{f(x)} \;=\; \left(1 + x(by - 1) - byx^{s+1}\right) \sum_{n=0}^{\infty} \left[\sum_{i=1}^{s+1} \hat{p}_i(y) \alpha_i^n(y)\right] x^n. \tag{2.26}$$

Thus for sufficiently large n,

$$g_n(y) \;=\; \sum_{i=1}^{s+1} \alpha_i^n(y) \left[\hat{p}_i + (by - 1)\hat{p}_i \alpha_i^{-1}(y) - by\hat{p}_i \alpha_i^{-s-1}(y)\right] \;=\; \sum_{i=1}^{s+1} q_i(y) \alpha_i^n(y). \tag{2.27}$$

□

Proof of Theorem 1.12. To prove Gaussianity we need only show that $g_n(y)$ satisfies conditions (i)–(iii) in Theorem 2.14.

- Condition (i): For each $i \in \{1, \ldots, s + 1\}$, $q_i(y)$ and $\alpha_i(y)$ are three times differentiable functions as roots of polynomials are differentiable functions of the polynomial coefficients, see [25].
- Condition (ii): Follows from Lemma 2.15.
- Condition (iii): Follows from Theorems 2.2 and 2.7.

Therefore, by satisfying the conditions of Theorem 2.14, we have completed our proof. □

3. Gap Measures for the (s, b)-Generacci Sequences

3.1. **Average Bin Gap Measure.**

Proof of Theorem 1.15. Let $m \in I_n := [a_{(n-1)b+1}, a_{nb+1})$ have legal decomposition

$$m \;=\; a_{\ell_1} + a_{\ell_2} + \cdots + a_{\ell_k} \text{ with } \ell_1 > \ell_2 > \cdots > \ell_k \text{ and } a_{\ell_i} \in \mathcal{B}_{\left\lceil \frac{\ell_i}{b} \right\rceil} \text{ for all } 1 \leq i \leq k. \tag{3.1}$$

Recall that $P_n(g)$ is the fraction of bin gaps that are of length g (i.e., the probability of a bin gap of length g among (s, b)-Generacci legal decompositions of $m \in [a_{(n-1)b+1}, a_{nb+1})$). Clearly $P_n(g) = 0$ whenever $g < s + 1$ since we must skip s bins between summands. For $g \geq s + 1$, define $X_{i,g}$ as the number of $m \in I_n$ whose decompositions contribute a bin gap of length g starting at bin $\mathcal{B}_i$. Then

$$P_n(g) \;=\; \frac{\sum_{i=1}^{n} X_{i,g}}{(\mu_n - 1)([a_{nb+1}] - [a_{(n-1)b+1}])}. \tag{3.2}$$

To compute $X_{i,g}$ note we have a summand from bin $\mathcal{B}_i$ and one from $\mathcal{B}_{i+g}$, and no summands from $\mathcal{B}_{i+1}, \mathcal{B}_{i+2}, \ldots, \mathcal{B}_{i+g-1}$. Moreover since $m \in I_n = [a_{(n-1)b+1}, a_{nb+1})$, m must contain a summand from $\mathcal{B}_n$. Hence there is freedom to choose summands from $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_{i-s-1}$ and then again we are free to choose summands from bins $\mathcal{B}_{i+g+s+1}$, $\mathcal{B}_{i+g+s+2}, \dots, \mathcal{B}_{n-s-1}$.

The number of ways to choose legally from $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_{i-s-1}$ is $a_{(i-s-1)b+1} - 1$. Similarly, the number of ways to choose legally from $\mathcal{B}_{i+g+s+1}, \mathcal{B}_{i+g+s+2}, \dots, \mathcal{B}_{n-s-1}$ is the number of integers in $[0, a_{(n-2s-g-i-1)b+1} - 1)$. As we selected summands from $\mathcal{B}_i, \mathcal{B}_{i+g}$ and $\mathcal{B}_n$,

$$X_{i,g} \;=\; b^3[a_{(i-s-1)b+1} - 1][a_{(n-2s-g-i-1)b+1} - 1]. \tag{3.3}$$

By Equation (1.1) of Theorem 1.3,

$$a_n \;=\; c_1\lambda_1^n(1 + O(\varepsilon^n)), \tag{3.4}$$

where $\varepsilon = |\lambda_2/\lambda_1|$, for some constants c_1, λ_1, and λ_2, where $\lambda_1 > 1$, $c_1 > 0$ and $|\lambda_2| < \lambda_1$. Thus

$$X_{i,g} = b^3c_1^2\lambda_1^{(n-3s-2)b+2}(\lambda_1^b)^{-g}(1 + O(\varepsilon^{(i-s-1)b+1}))(1 + O(\varepsilon^{(n-i-2s-g-1)b+1})). \tag{3.5}$$

We break the sum into three ranges: $i \le 8\log n$, $8\log n < i < n - 8\log n$, and $n - 8\log n \le i \le n$. Note that for $8\log n < i < n - 8\log n$,

$$\varepsilon^{(i-s-1)b+1}, \varepsilon^{(n-i-2s-g-1)b+1} \le \varepsilon^{4\log n}, \tag{3.6}$$

which implies that all lower order terms are negligibly small relative to the main term. On the other hand

$$\begin{aligned}\sum_{1\le i<8\log n} X_{i,g} &= b^3c_1^2\lambda_1^{(n-3s-2)b+2}(\lambda_1^b)^{-g}O(\log n)\\ \sum_{n-8\log n\le i\le n} X_{i,g} &= b^3c_1^2\lambda_1^{(n-3s-2)b+2}(\lambda_1^b)^{-g}O(\log n).\end{aligned} \tag{3.7}$$

Hence

$$\begin{aligned}P_n(g) \;&=\; \frac{\sum\limits_{1\le i<8\log n} X_{i,g} \;+\; \sum\limits_{8\log n\le i<n-8\log n} X_{i,g} \;+\; \sum\limits_{n-8\log n\le i\le n} X_{i,g}}{(\mu_n - 1)([a_{nb+1}] - [a_{(n-1)b+1}])}\\ &=\; \frac{b^3c_1^2\lambda_1^{(n-3s-2)b+2}(\lambda_1^b)^{-g}\left[O(\log n) + (n - 16\log n)\left(1 + O(\varepsilon^{4\log n})\right)\right]}{Cn(c_1\lambda_1^{nb+1} - c_1\lambda_1^{(n-1)b+1})}\\ &=\; \frac{b^3c_1^2\lambda_1^{(n-3s-2)b+2}}{Cnc_1\lambda_1^{(n-1)b+1}(\lambda_1^b - 1)}(\lambda_1^b)^{-g}\left[n + O(\log n)\right].\end{aligned} \tag{3.8}$$

Taking the limit as $n \to \infty$ yields

$$P(g) \;=\; \frac{b^3c_1}{C(\lambda_1^b - 1)\lambda_1^{(3s+1)b-1}}(\lambda_1^b)^{-g}. \tag{3.9}$$

As $P(g)$ defines a probability distribution and $P(g) = 0$ for $g < s+1$, $\sum_{g=s+1}^{\infty} P(g) = 1$. Evaluating the geometric series and using λ_1 is a root of $x^{(s+1)b} - x^{sb} - b = 0$ yields

$$\frac{b^3c_1}{C(\lambda_1^b - 1)\lambda_1^{(3s+1)b-1}} \;=\; b. \tag{3.10}$$

Thus $P(g) = b(\lambda_1^b)^{-g}$. □

3.2. **Spacing Bin Gap Measure.** We prove Theorem 1.16 by checking that the conditions of [12, Theorem 1.1] are satisfied by the spacing bin gap measure of the (s, b)-Generacci sequence; note we are working with gaps between bins and not summands, but by collapsing a bin we find the arguments are identical. We restate [12, Theorem 1.1] below for ease of reference.

Theorem 3.1. [12, Theorem 1.1] *For $z \in I_n := [a_{c_1 n + d_1}, a_{c_2 n + d_2})$, the individual gap measures $\nu_{z,n}(x)$ converge almost surely in distribution to the average gap measure $\nu(x)$ if the following hold.*

(1) *The number of summands for decompositions of $z \in I_n$ converges to a Gaussian with mean $\mu_n = c_{\rm mean} n + O(1)$ and variance $\sigma_n^2 = c_{\rm variance} n + O(1)$, for constants $c_{\rm mean}, c_{\rm variance} > 0$, and $k(z) \ll n$ for all $z \in I_n$.*

(2) *We have the following, with $\lim_{n\to\infty} \sum_{g_1,g_2} {\rm error}(n, g_1, g_2) = 0$:*

$$\frac{2}{|I_n|\mu_n^2} \sum_{j_1<j_2} X_{j_1,j_1+g_1,j_2,j_2+g_2}(n) \ = \ P(g_1)P(g_2) + {\rm error}(n, g_1, g_2). \tag{3.11}$$

(3) *The limits in Equation* (1.19) *exist.*

In [12], the authors used the following definition: for $g_1, g_2 \geq 0$

$$X_{j_1,j_1+g_1,j_2,j_2+g_2}(n) \ := \ \# \left\{ z \in I_n : {\scriptstyle b_{j_1},\ b_{j_1+g_1},\ b_{j_2},\ b_{j_2+g_2} \text{ in } z\text{'s decomposition,} \atop \text{but not } b_{j_1+q},\ b_{j_2+p} \text{ for } 0<q<g_1,\ 0<p<g_2} \right\}. \tag{3.12}$$

Since we are concerned with the gaps between bins we will compute $X_{j_1,j_1+g_1,j_2,j_2+g_2}(n)$ by counting $z \in I_n$ whose decomposition has a summand from bins $\mathcal{B}_{j_1}$ and $\mathcal{B}_{j_1+g_1}$ (with no bins used in between) and again from bins $\mathcal{B}_{j_2}$ and $\mathcal{B}_{j_2+g_2}$ (with no bins used in between).

Proposition 3.2. *We have*

$$\frac{2}{|I_n|\mu_n^2} \sum_{j_1<j_2} X_{j_1,j_1+g_1,j_2,j_2+g_2}(n) \ = \ P(g_1)P(g_2) + error(g_1, g_2, n) \tag{3.13}$$

where the error as $n \to \infty$ summed over all pairs (g_1, g_2) goes to zero.

Proof. Assume $j_1 < j_2$. We compute $X_{j_1,j_1+g_1,j_2,j_2+g_2}(n)$: We take a summand each from bins $\mathcal{B}_{j_1}$ and $\mathcal{B}_{j_1+g_1}$ and again from bins $\mathcal{B}_{j_2}$ and $\mathcal{B}_{j_2+g_2}$, and finally since $z \in I_n = [a_{(n-1)b+1}, a_{nb+1})$, z must contain a summand from bin $\mathcal{B}_n$. Additionally, we have freedom in selecting summands from bins $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_{j_1-(s+1)}$, then from bins $\mathcal{B}_{j_1+g_1+(s+1)}, \mathcal{B}_{j_1+g_1+(s+2)}, \dots, \mathcal{B}_{j_2-(s+1)}$, and lastly from bins $\mathcal{B}_{j_2+g_2+(s+1)}, \mathcal{B}_{j_2+g_2+(s+2)}, \dots, \mathcal{B}_{n-(s+1)}$.

The number of ways to choose summands legally from $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_{j_1-(s+1)}$ is $a_{(j_1-s-1)b+1} - 1$; the number of ways to choose summands legally from $\mathcal{B}_{j_1+g_1+(s+1)}, \mathcal{B}_{j_1+g_1+(s+2)}, \dots, \mathcal{B}_{j_2-(s+1)}$ is $a_{(j_2-j_1-g_1-2s-1)b+1} - 1$; the number of ways to choose summands legally from $\mathcal{B}_{j_2+g_2+(s+1)}, \mathcal{B}_{j_2+g_2+(s+2)}, \dots, \mathcal{B}_{n-(s+1)}$ is given by $a_{(n-j_2-g_2-2s-1)b+1} - 1$. Hence

$$X_{j_1,j_1+g_1,j_2,j_2+g_2}(n) \ = \ b^5[a_{(j_1-s-1)b+1} - 1][a_{(j_2-j_1-g_1-2s-1)b+1} - 1][a_{(n-j_2-g_2-2s-1)b+1} - 1]. \tag{3.14}$$

Using the explicit form for the terms of the (s, b)-Generacci sequence given in Equation (3.4), Equation (3.14) yields

$$\begin{aligned} & X_{j_1,j_1+g_1,j_2,j_2+g_2}(n) \\ & \quad = b^5 c_1^3 \lambda_1^{(n-5s-3)b+3} \left(\lambda_1^b\right)^{-(g_1+g_2)} (1 + O(\varepsilon^{j_1 b}))(1 + O(\varepsilon^{(j_2-j_1)b}))(1 + O(\varepsilon^{(n-j_2)b})), \end{aligned} \tag{3.15}$$

where it is important to recall that $\varepsilon < 1$.

$$\begin{aligned} \text{Let}\quad S_n = &\ \{(j_1, j_2):\ 1 \le j_1 < j_2 \le n\}, \text{and} \\ T_n = &\ \{(j_1, j_2) \in S_n:\ 8\log n \le j_1 < j_2 < n - 8\log n,\ j_2 - j_1 > 8\log n\}. \end{aligned}$$

Then for $(j_1, j_2) \in T_n$, $\varepsilon^{j_1 b} \le \varepsilon^{8\log n}$, $\varepsilon^{(j_2-j_1)b} \le \varepsilon^{8\log n}$, $\varepsilon^{(n-j_2)b} \le \varepsilon^{8\log n}$, which implies that all lower order terms are negligibly small relative to the main term. Also, note that the sum of 1 over all $(j_1, j_2) \in S_n \setminus T_n$ is of order $n \log n$. Thus

$$\begin{aligned} &\frac{2 \sum\limits_{(j_1,j_2)\in S_n} X_{j_1,j_1+g_1,j_2,j_2+g_2}(n)}{|I_n|\mu_n^2} \\ &\qquad = \frac{2\left(\sum\limits_{(j_1,j_2)\in T_n} X_{j_1,j_1+g_1,j_2,j_2+g_2}(n) \ + \sum\limits_{(j_1,j_2)\in S_n\setminus T_n} X_{j_1,j_1+g_1,j_2,j_2+g_2}(n)\right)}{|I_n|\mu_n^2} \\ &\qquad = \frac{b^6 c_1^2}{C^2 \lambda_1^{(6s+2)b-2}(\lambda_1^b - 1)^2} (\lambda_1^b)^{-g_1} (\lambda_1^b)^{-g_2} \frac{\lambda_1^{sb}(\lambda_1^b - 1)}{b} \left[1 + O(\log n/n)\right] \\ &\qquad = P(g_1)P(g_2)\left[1 + O(\log n/n)\right], \end{aligned} \tag{3.16}$$

the last equality follows immediately from (3.9) and the fact that λ_1 is the largest root of the characteristic equation $x^{(s+1)b} - x^{sb} - b = 0$, the defining relation of our sequence $\{a_n\}$. As $P(g_1)P(g_2)$ sums to 1, the sum of the error term over all pairs (g_1, g_2) goes to zero as required. □

Proof of Theorem 1.16. We simply need to check that Conditions (1)–(3) of Theorem 3.1 hold. First we note that letting $c_1 = b$, $d_1 = 1 - b$ and $c_2 = b$ and $d_2 = 1$, implies that the interval of interest is $I_n = [a_{b(n-1)+1}, a_{bn+1})$. Then Theorem 1.12 shows the first part Condition (1) is satisfied. Now note that there are $n - 1$ allowable bins from which to select summands and any $z \in I_n$ will have at most $\left\lceil \frac{n-1}{s+1} \right\rceil$ summands as there must be s bins between each summand selected. Hence for any $z \in I_n$, $k(z) \le \left\lceil \frac{n-1}{s+1} \right\rceil < n$ which completes the proof that Condition (1) is satisfied. Condition (3) follows from Theorem 1.15. Finally, Condition (2) follows from Proposition 3.2. □

4. Gaussianity and Gap Measures for Fibonacci Quilt

The $(4, 1)$-Generacci sequence yields Gaussian and Gap Measure results for Greedy-6 decompositions. The Greedy-6 decomposition is almost the same as the legal decomposition from the $(4, 1)$-Generacci sequence as the gap between almost all summands in a Greedy-6 decomposition is at least 5. The only difference is that for the Greedy-6 decomposition the last two summands can have indices differing by 2 (if that happens the subsequent index is at least 6 larger). This possible gap of length 2 does not matter in the limit.

Proof of Theorem 1.14. As the two decompositions are so similar, the Gaussianity result for the Greedy-6 decomposition follows from that for the $(4, 1)$-Generacci sequence. We partition our integers m into two distinct sets where the Greedy-6 decomposition $\mathcal{G}(m)$ starts with q_n and either:

- ends with $q_4 + q_2$ and the third smallest summand is at least q_{10}; or

- all indices differ by at least 5.

Both of these cases have Gaussian behavior by Theorem 1.12 specified to the $(4,1)$-Generacci Sequence.

In the first case, Greedy-6 decompositions must have the summand q_n as well as q_4 and q_2. Thus we do not have $q_1, q_3, q_5, q_6, q_7, q_8$ or q_9, but q_{10} is possible. Define

$$Q_{n,\alpha} := \{m \in [q_n, q_{n+1}) \mid q_2 \text{ and } q_4 \text{ are summands in the Greedy-6 decomposition of } m\}.$$

Consider the $(4,1)$-Generacci sequence $\{a_n\}$. Define the set of integers

$$J_{n,\alpha} := \{\omega \in [a_n, a_{n+1}) \mid a_1, a_2, \dots, a_9 \text{ are not in the decomposition of } \omega\}.$$

As the integers in $J_{n,\alpha}$ decompose with a_n as the largest summand and any legal set of summands from $\{a_{n-5}, a_{n-6}, \dots, a_{10}\}$, we have $|J_{n,\alpha}| = a_{n-14} - 1$. Moreover, the bijection between the sets $J_{n,\alpha}$ and $[1, a_{n-14})$ preserves the number of summands in a decomposition. As the number of summands in the $(4,1)$-Generacci legal decomposition of an integer from $[1, a_{n-14})$ is Gaussian, the number of summands in the $(4,1)$-Generacci legal decomposition of an integer from $J_{n,\alpha}$ is Gaussian. There is a bijection between the sets $J_{n,\alpha}$ and $Q_{n,\alpha}$ that exactly increases the number of summands in a decomposition by 2, hence the number of summands in the Greedy-6 legal decomposition of an integer chosen uniformly at random from $Q_{n,\alpha}$ is Gaussian. The mean and variance of each of this Gaussian will differ from the mean and variance of the $(4,1)$-Generacci sequence in the constant term, but as the mean and the variance are of the form $An + B + o(1)$ and $Cn + D + o(1)$, this shift does not matter in the limit.

All m in the second case are in a bijection with all $\omega \in [a_n, a_{n+1})$ that precisely preserves the indices in the decompositions of m and ω. Hence the number of summands of such an m is Gaussian.

Combining these two Gaussians distributions results in an overall Gaussian. □

Proof of Theorem 1.17. We first note that the proportion of gaps of length 2 is negligibly small as $n \to \infty$. The number of gaps of a typical element is strongly concentrated on the order of n, so one extra gap of length 2 is proportionally only on the order of $1/n$, and thus in the limit will have zero probability.

For the remaining gap sizes, we break this problem into two cases as we did in the proof of Theorem 1.14. We then argue identically as in the $(4,1)$-Generacci case, and note that our proofs were entirely combinatorial; all that mattered was the number of ways to choose summands satisfying the legal rule. □

Remark 4.1. *Note the utility of this perspective suggests some natural future questions: as the Fibonacci Quilt's Greedy-6 decomposition is just the $(4,1)$-Generacci with a tweak in the beginning, do other tweaks lead to geometrically interesting sequences?*

5. Range of Number of Summands for Fibonacci Quilt Decompositions

We introduce the notion of gap strings to clean up manipulations by eliminating the need for cluttering the paper with sums and subscripts.

Definition 5.1. *Let $q_{\ell_1} + q_{\ell_2} + \cdots + q_{\ell_t}$ be any decomposition of m with $q_{\ell_i} \geq q_{\ell_{i+1}}$ for $i = 1, 2, \dots, t-1$. The* gap string *of the decomposition is the $(t-1)$-tuple*

$$(\ell_1 - \ell_2, \ell_2 - \ell_3, \dots, \ell_{t-1} - \ell_t). \tag{5.1}$$

From Theorem 1.11 we know the number of summands in the Greedy-6 decomposition of any m is minimal and the corresponding gap string $(x_1, x_2, \dots, x_{k_{\min}(m)-1})$ has $x_i \geq 5$ for all i except possibly $x_{k_{\min}(m)-1} = 2$ (i.e., the Greedy-6 decomposition used $q_4 + q_2$).

Proof of Theorem 1.19. From Theorem 1.11 the Greedy-6 decomposition is a minimal decomposition (i.e., no other legal Fibonacci Quilt decomposition uses fewer summands). We investigate how many $m \in [q_n, q_{n+1})$ have $k_{\max}(m) - k_{\min}(m) \geq g(n)$ for a fixed function $g(n)$, and then see how large we may take it while ensuring the inequality holds for almost all m in the interval. The argument below was chosen as it gives the optimal growth rate of $g(n)$ but not the optimal constant; with a little more work the value of $C_{\rm FQ}$ could be slightly increased, but a growth rate of essentially $\log(n)$ is the natural boundary of this approach.

Let $\mathcal{G} = (5, 5, 10, 5, 5, 10, \dots, 5, 5, 10)$ be a fixed gap pattern among $3g(n)+1$ addends. Note that the number of summands in a decomposition of $m \in I_n$ can be increased by $g(n)$ if the decomposition has a gap string that contains the substring $\mathcal{G}$ beginning at $q_{A+20g(n)}$ with $10 + 20g(n) \leq A + 20g(n) \leq n$. Using recurrence relations ($q_n + q_{n-2} = q_{n+1} + q_{n-5}$ and $q_n + q_{n-4} = q_{n+1}$ proved in [7]) we get a new FQ-legal decomposition of m where the only difference is that substring $\mathcal{G}$ is replaced with the substring $\mathcal{G}' = (6, 2, 7, 5, 6, 2, 7, 5, \dots, 6, 2, 7, 5)$.[2] The starting and ending summands remain the same but there are now $4g(n) + 1$ summands indicated by the gap substring. Hence for such m, $k_{\max}(m) - k_{\min}(m) \geq g(n)$.

We break the set of Fibonacci Quilt summands $\{q_n, \dots, q_1\}$ into adjacent and non-overlapping blocks of length $20g(n) + 1$; the number of such complete blocks is $\lfloor \frac{n}{20g(n)+1} \rfloor$. There are $2^{20g(n)+1}$ ways to choose which summands in a given block we take, and at least one of them is the desired gap pattern $\mathcal{G}$. Thus the probability that a given decomposition has pattern $\mathcal{G}$ is at least $1/2^{20g(n)+1}$, so the probability that we do not have $\mathcal{G}$ is at most $1 - 1/2^{20g(n)+1}$. Therefore the probability that the pattern occurs at least once is

$$\Pr(\text{gap substring } \mathcal{G} \text{ occurs in the gap string of } m) \ \geq\ 1 - \left(1 - 1/2^{20g(n)+1}\right)^{\lfloor \frac{n}{20g(n)+1} \rfloor}. \quad (5.2)$$

To show this tends to 1 we just need to show the subtracted quantity tends to zero, or equivalently that its logarithm tends to $-\infty$; for large n this is

$$\left\lfloor \frac{n}{20g+1} \right\rfloor \log\left(1 - 1/2^{20g(n)+1}\right) \ \leq\ -\frac{n}{21g(n)} \frac{1/2}{2^{20g(n)}} \ =\ -\frac{2}{21} \frac{n}{g(n) e^{20g(n)\log(2)}}. \quad (5.3)$$

If we take $g(n) = C_{\rm FQ} \log(n)$ then

$$\left\lfloor \frac{n}{20g+1} \right\rfloor \log\left(1 - 1/2^{20g(n)+1}\right) \ \leq\ -\frac{2}{21 C_{\rm FQ}} \frac{n}{n^{20 C_{\rm FQ} \log 2} \log(n)}, \quad (5.4)$$

which tends to $-\infty$ so long as $C_{\rm FQ} < 1/20 \log 2$, completing the proof. □

Remark 5.2. *We could increase the constant $C_{\rm FQ}$ slightly if we replace $2^{20g(n)+1}$ by the number of legal decompositions there are involving the $20g(n) + 1$ summands. This is on the order of $q_{20g(n)+1}$; while this is an exponentially growing sequence, it has a smaller base. If we wish to increase the constant by replacing the inequality with an equality we would then have to worry about the logarithm in the denominator. While this could be done at the cost of a more complicated expression, as it is essentially the same size we do not pursue that here.*

[2]For example, replacing string $(5, 5, 10)$ with $(6, 2, 7, 5)$ can be seen as $q_{30+\ell} + q_{25+\ell} + q_{20+\ell} + q_{10+\ell} = q_{30+\ell} + q_{24+\ell} + q_{22+\ell} + q_{15+\ell} + q_{10+\ell}$.

6. Future Research

We end with a list of additional problems to study for the Fibonacci Quilt; this is a particularly appealing sequence to investigate as it is similar to a PLRS, but is not and has already been shown to have the same behavior for some problems but very different in others. Recall $d(m)$ denotes the number of legal decompositions of m by the Fibonacci quilt.

- Can we solve $d(m) = \ell$ for fixed ℓ? What about $d(m) \leq w(m)$ for some fixed increasing function w?
- How rapidly does $\max_{m \leq N} d(m)$ go to infinity?
- For $m \leq N$, what does the distribution of $d(m)$ look like?
- Let $K_{\min}(m)$ be the fewest number of summands needed in a Fibonacci quilt legal decomposition of m (and similarly define $K_{\max}$, K_{ave}). What can we say about $K_{\min}$ and $K_{\max}$?
- Find all m such that $K_{\min}(m) = K_{\max}(m)$.
- How does $K_{\text{ave}}(m)$ compare to $K_{\min}$ and $K_{\max}$? Is is closer to one or the other for all m?

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