CODEE Expects that every Mathematician will do their ODEs: From the Battle of Trafalgar to Calculus (or Nelson to Newton)

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Abstract: Using the historical Battle of Trafalgar as a case study, we introduce students to mathematical modeling to describe and explore real-world systems. Starting with a complex optimization problem—how do we strategize to win the battle?—we show how to use a blend of mathematics and computational tools to break this down and come up with a reasonable answer. Beginning by defining a simple differential equation model for combat dynamics, we analyze its behavior using simple and accessible techniques from pre-calculus. When direct analysis is too complex to carry out, or conjectures of behavior need to be established, we show how to use computational methods and simulations to do so in a systematic and visual way. This material could serve as the foundation of a lesson plan which integrates historical background, computational tools, and mathematical theory to motivate students and foster an interest in using mathematical modeling to solve complex, real-world problems.

1 Introduction

There are numerous challenges facing anyone teaching college mathematics, many of which have only grown worse following the pandemic response. These range from the difficulties of classes of students with markedly different skill sets and backgrounds, to their varied interests and goals for their courses and beyond. In this note we[1] share some

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[1] The authors met at the 2023 Williams College SMALL REU, where the first named author was a student researcher in the Chip-Firing Group and the second named author was the program director and lead

lessons we've learned and insights we've gained over the years in creating engaging units that can be used in a variety of classes to effectively convey some of the most important skills we wish 21st century students to learn: how to create and investigate mathematical models that describe problems of interest in the real world.

The mathematical landscape is undergoing a rapid transformation, and while it's unclear where it'll settle (older readers may remember some of the fears that Massive Open Online Courses would fundamentally change the college experience), the tool sets available to students and practitioners creates opportunities unavailable before. This opens up both the opportunity and the need to rethink what we're teaching and why. We have a limited amount of contact hours; which material should be emphasized and why? What skills and perspectives do we wish students to master, especially considering that few if any in most of our classes will become professional mathematicians?

When the second named author was an undergraduate he was advised to have two tables of integrals: a small portable book that he would keep with him all the time, and a big monster at home! Not only are all of these available for free online (see for example [1, 5]), but gone is the need in knowing how to convert integrals to a form that can be evaluated, as this can be easily be done with computer programs.

Our focus here is on introducing students to mathematical modeling, creating a system of equations that describe a system which can the be fed into a program to solve. We focus on a non-standard example, Nelson's pivotal victory in the Battle of Trafalgar during the Napoleonic Wars. We chose this example for several reasons.

- This is not an artificial problem created to make sure students have learned the mechanics of a subject, but one of great historical significance.
- Frequently problems are chosen from pure mathematics, physics, mathematical biology (especially after the covid pandemic response) or sports; it's nice to increase the range of problems seen as students are interested in more than just these topics.
- The mathematical pre-requisites are modest; a simpler version can be taught as early as pre-calculus or Algebra II, while the full model can be easily worked into Calculus I, II, III, Linear Algebra or Probability (among other courses).
- There are numerous opportunities for coding and simulation, allowing students to see how the solution depends on the parameter values and initial conditions, and there are many subtleties to the problem which highlight pitfalls to be avoided.
- For those wishing to learn more, there are numerous articles, lectures and code implementations freely available.

We first describe the general problem and its historical significance, then turn to discussing our experiences in introducing students to model creation and exploration, and end with some comments on the interplay between theory and simulation. As our goal is pedagogical, while we prove many we do not prove all the results; we encourage readers

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professor in the Probability and Analytic Number Theory group.

to solve the model. This paper is the outgrowth of talks[2] by the second named author at several classes at Williams College (especially probability and differential equations) and at an AMS Special Session on Modeling to Motivate the Teaching of Mathematics of Differential Equations at the 2024 Joint Mathematical Meetings and at SIMIODE EXPO 2024. The presentation was inspired by the work of Lanchester [7, 8]; these works are available online, which we strongly encourage you to read. See [2, 6] for more on these and related problems.[3]

2 History

The Battle of Trafalgar was one of the most decisive military conflicts of all time. It took place in 1805, near the start of more than a decade of warfare in Europe. Napoleon needed control of the English Channel to successfully invade Britain, but doing so required defeating the powerful British fleet. We quote from the Wikipedia page on the battle.

> *As part of Napoleon's plans to invade the United Kingdom, the French and Spanish fleets combined to take control of the English Channel and provide the Grande Armée safe passage. The allied fleet, under the command of the French admiral, Pierre-Charles Villeneuve, sailed from the port of Cádiz in the south of Spain on 18 October 1805. They encountered the British fleet under Lord Nelson, recently assembled to meet this threat, in the Atlantic Ocean along the southwest coast of Spain, off Cape Trafalgar. Nelson was outnumbered, with 27 British ships of the line to 33 allied ships....*

While much more can be said to set the stage, the above suffices for our purposes and can be done quickly and easily in a lecture. There are many questions one can use to guide a discussion; the following provide a great start.

- What strategy should Nelson pursue, given that his forces are outnumbered?
- What are the right metrics to measure the strength of the two sides? A hundred soldiers with arrows are numerically superior to ten with machine guns, but that advantage will not translate to victory.
- How can one model a battle between two combatants? How much detail is needed?[4]

For the first, after some discussion students frequently discover a fundamental tenant of war: concentrate superior numbers on the enemy; divide and conquer. In other words,

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[2] Talks and slides available here: https://youtu.be/rRBWahHdMS4 and https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/AMSexpectsallmathtodobest_ModelingDiffEq_SIMIODE2024.pdf.

[3] See https://colab.research.google.com/drive/1nGWDTs4CNqelxegTSMC0f06KT3bYjNV4#scrollTo=kRPocS_DOIeK for code.

[4] For example, does it matter where the fleets are relative to land and islands, as that can affect maneuverability? The answer is almost surely yes. Hopefully we can ignore much of this for a good first approximation, but note that this is not always a reasonable assumption; the battle of Gettysburg was decided by the North repulsing the South's efforts to gain the high ground for their cannon.

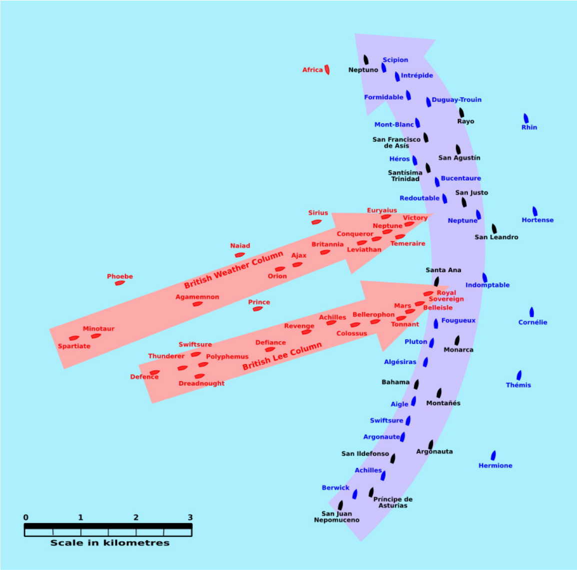

Figure 1: From Wikipedia: *This map of the Battle of Trafalgar shows the approximate position of the two fleets at 1200 hours during the battle as the Royal Sovereign was breaking into the Franco-Spanish line. North is to the top, and Cape Trafalgar is 10 miles to the northeast.* Image by Pinpin from Wikimedia Commons.

if possible break the larger force into smaller ones, and fight a sequence of engagements where you have the advantage. Of course, the enemy won't passively stand by and let you do this, so instead of the ideal string of combats you must instead work towards one that is not your ideal, but better than all at once.

The simplest solution is to use your force to break the enemy into two groups, which is what Nelson did (though an earlier battle plan called for splitting into three groups); see Figure 1.

A small part of his force was to fight a delaying action, keeping many of the French and Spanish ships busy, while the bulk of his fleet engaged the rest; the plan was to win that battle and then have the ships pivot and join the smaller force. Ignoring for the moment *how* to maneuver so that there'll be two smaller engagements, as mathematicians we can ask a simple optimization problem: how should we split our forces, and how do we hope to split the opposing ones, to maximize our chance for victory? This leads to great discussions on optimization, and while we can solve this mathematically we'll see later that it's possible to model it well enough to guess an answer.[5]

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[5] It's worth noting that successful practitioners in the real-world do not write down equations and solve them to determine their behavior; lifeguards and centerfielders do not do calculus to find the optimal paths to drowning swimmers and fly balls. Years of experience have taught them how to quickly size up the situation and move to a nearly optimal path.

3 Model

It's now time to create a model. While we could model the battle as a discrete system, given that students often have taken years of calculus it's nice to show its utility! It's often a lot easier to get explicit, closed form solutions from continuous systems over discrete ones, so we'll proceed accordingly.

We'll consider the general problem of two forces fighting, with $r(t)$ denoting how many red units there are at time t and $b(t)$ the number of blue units. For simplicity we assume each red unit has the same combat effectiveness as the other reds, but not necessarily the same as the blue units. This assumption can be removed by instead looking at a vector

$$r(t) \;=\; (r_1(t),\, r_2(t),\, \dots,\, r_k(t)).$$

How should $r(t)$ change with time? It's reasonable that the more blue units they face, the more their numbers should decrease. Thus the simplest model is $r'(t) = -S_b b(t)$, where S_b is a constant that represents how potent blue's forces are (the negative sign is because the red forces decrease as they engage the blue). Similarly $b'(t) = -S_r r(t)$, where S_r and S_b need not be the same; for example, if red has superior ships and crews, we would expect $S_r > S_b$. We thus have to solve

$$\begin{aligned} r'(t) &= -S_b b(t) \\ b'(t) &= -S_r r(t), \end{aligned} \tag{3.1}$$

subject to initial conditions that at time zero there are $r(0)$ red units and $b(0)$ blue units.

3.1 Testing the Model

Before solving, we can ask students if there are any cases where they have intuition for the results. Usually someone says if $S_b = S_r$ then whichever force is larger wins. This is a great observation, and accurately states what happens, but it leaves an important quantity unspecified: how long will it take?

Notice something interesting happens if $S_r = S_b$; for convenience we'll assume the common value is 1. Our system reduces to

$$\begin{aligned} r'(t) &= -b(t) \\ b'(t) &= -r(t). \end{aligned} \tag{3.2}$$

If we let $u(t) := r(t) + b(t)$ represent the total number of units in the battle, then we obtain the simpler

$$u'(t) \;=\; -u(t), \tag{3.3}$$

with initial condition $u(0) = r(0) + b(0)$. While the general system of equations may be hard for students to solve, this is the famous differential equation for the exponential function, with the well known solution

$$u(t) \;=\; u(0)e^{-t}. \tag{3.4}$$

This toy problem provides an excellent opportunity to discuss a common mistake made in modeling: be careful to use equations only where they're valid; be careful about extrapolating. A terrific example of this comes from physics. In everyday interactions, speed looks to be additive: if we're on a train traveling at 100mph past a station and we're running at 10mph in the train in the direction it's moving, then to an observer at a station we appear to be moving at 110mph. Classical physics says that we *are* moving at 110mph, but special relativity says there is a slight correction and the true speed is $\frac{v_1+v_2}{1+v_1v_2/c^2}$ where the v_i's are the two speeds and c is the speed of light. The actual answer? We appear to be moving at approximately 109.999999999999758mph. As this difference is indistinguishable with the tools available before the 20th century, it's not surprising people thought speed was additive, and extrapolated (incorrectly) from speeds being additive at small values to large ones. If instead our two speeds are .25 and .10 times the speed of light then the sum is about .34 times the speed of light, not .35. The difference becomes more pronounced the greater the speeds; if $v_1 = v_2 = 1/2$ the speed of light then the sum is not 1 but rather 4/5ths the speed of light.[6]

How is this relevant to our problem at hand? Remember $u(t)$ denotes the total number of units in the field. If $r(0) > b(0)$ then the red forces triumph, and at some point there are no longer any blue forces. From this point onward there cannot be any additional changes, and however many red units are left, that's the value of $u(t)$ till the end of time; however, our solution of $u(t) = u(0)e^{-t}$ has $u(t)$ tending exponentially quickly to zero as t goes to infinity.

Thus our simple example illustrates a key point: we weren't careful enough in writing down our model. Our system of differential equations only applies so long as $r(t)$ and $b(t)$ are positive! If we continue to use these equations once one side is wiped out, we could even be in the absurd situation of one side becoming negative while the other side grows *stronger* in the midst of a battle![7]

3.2 Solving the Model, I

In the previous subsection we analyzed a special case of the model. We showed if $S_r = S_b = 1$ (i.e., if the two forces have the same fighting effectiveness) then the total number of units is exponentially decreasing in time. Further, we saw that we needed to be careful and *only* use the differential equations when $r(t)$ and $b(t)$ are positive, as otherwise we have the absurdity where after the more numerous forces wipes out the smaller one, the total number of forces decays to zero. For definiteness, say $r(0) > b(0)$ so the red fleet triumphs. In the limit we know the combined number of units tends to zero, but it's possible that $r(t)$ converges to say $c > 0$ and $b(t)$ to $-c$!

Our analysis there crucially used $S_r = S_b = 1$ to combine the two equations to give $u'(t) = -u(t)$; if $S_r \neq S_b$ we would have

$$\frac{d}{dt}\left[r(t) + b(t)\right] \;=\; -S_r r(t) - S_b b(t). \tag{3.5}$$

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[6] One of the postulates of special relativity is nothing travels faster than the speed of light; the speed addition formula shows that if each $v_i < c$ then so too is their 'sum'.

[7] Interestingly, that did happen at Trafalgar. In one of the most lopsided victories of all time, the British did not lose any ships but captured 17 from the Franco-Spanish armada!

This difficulty illustrates a common feature: it can be dangerous to look at special cases, as you can see non-generic behavior, or the techniques that work for that problem may not work in general.

Fortunately these equations can easily be uncoupled to give two equations, one involving just $r(t)$ and its derivatives and one similarly for $b(t)$ and its derivatives. We know $b'(t) = -S_r r(t)$, and $r'(t) = -S_b b(t)$. Thus if we differentiate $b'(t) = -S_r r(t)$ we have $b''(t) = -S_r r'(t)$, and then substituting for $r'(t)$ yields

$$\begin{aligned} r''(t) &= S_r S_b r(t) \\ b''(t) &= S_r S_b b(t) \end{aligned} \tag{3.6}$$

(the second equation is derived analogously as the first), with initial conditions $r(0)$ and $b(0)$.

This should be a little troubling as we have two second order differential equations but each has just one initial condition; do we have enough information to determine the solution? There are two solutions to the differential equation $v''(t) = -av(t)$ with $a > 0$ and initial condition $v(0)$, namely $v(0)e^{\sqrt{a}t}$ and $v(0)e^{-\sqrt{a}t}$, and any solution is a linear combination of the two such that when $t = 0$ we get $v(0)$. Let's write it as

$$v(t) \ = \ \alpha_1 e^{\sqrt{a}t} + \alpha_2 e^{-\sqrt{a}t}, \tag{3.7}$$

with $\alpha_1 + \alpha_2 = v(0)$. In many problems we can exclude a solution from physical grounds; for example, oftentimes we get a quadratic equation for the side of a rectangle in an optimization problem, and if one of the roots is negative we know that cannot be the answer as lengths are positive.

For a simple example of the phenomenon of algebra introducing extra candidate solutions which do not work, consider the following problem: solve

$$\sqrt{x+2} \ = \ x - 4. \tag{3.8}$$

If we square both sides we get

$$x + 2 \ = \ x^2 - 8x + 16 \quad \text{or} \quad x^2 - 9x + 14 \ = \ 0. \tag{3.9}$$

It factors as $(x-7)(x-2) = 0$, and thus the solutions seem to be $x = 2$ and $x = 7$; while $x = 7$ works if we substitute $x = 2$ we get $\sqrt{4} = -2$. The extraneous (or phantom) solution comes from squaring both sides; we introduced an extra root in doing so.[8]

Can we do a similar argument here, saying that the exponentially growing solution is impossible as in a battle your fleet cannot grow over time[9], and we must therefore have $v(t) = v(0)e^{-\sqrt{a}t}$? If yes then we would have the solution is

$$r(t) \ = \ r(0)e^{-\sqrt{S_r S_b}t}, \quad b(t) \ = \ b(0)e^{-\sqrt{S_r S_b}t}. \tag{3.10}$$

As always, this is a great opportunity to ask the class if the answer is reasonable. Several issues should be apparent.

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[8] For a truly trivial example of this, solve $x = 1$ by squaring both sides to get $x^2 = 1$, or $x^2 - 1 = (x-1)(x+1) = 0$ and thus $x = 1$ or -1!

[9] In naval battles centuries ago it did happen that fleets captured opposing ships and thus their numbers could increase, but even in this situation they could only increase to a certain point!

- The battle never ends; at any moment in time both forces are positive.
- It doesn't matter how superior one force is to another, in the limit both fleets converge to being completely destroyed!

3.3 Solving the Model, II

Obviously something went wrong. The issue is we (again) forgot our differential equations only apply until one fleet is destroyed! We can't eliminate the exponentially growing term as we don't have time going off to infinity, and our solution is

$$\begin{aligned} r(t) &= \rho_1 e^{\sqrt{S_r S_b}t} + \rho_2 e^{-\sqrt{S_r S_b}t}, \quad \rho_1 + \rho_2 &= r(0) \\ b(t) &= \beta_1 e^{\sqrt{S_r S_b}t} + \beta_2 e^{-\sqrt{S_r S_b}t}, \quad \beta_1 + \beta_2 &= b(0). \end{aligned} \tag{3.11}$$

Unfortunately we have four parameters but only two equations, and there are thus infinitely many solutions, which is absurd. The problem is well determined: we know the initial concentration of forces, we know how they change over time, there should be a unique solution.

Similar to how squaring introduces an extraneous solution to the equation $\sqrt{x+2} = x-4$, taking additional derivatives of (3.1) caused extra solutions to emerge. Thus let's revisit (3.1) and incorporate the first order derivatives there by using (3.11). We find

$$\begin{aligned} \sqrt{S_r S_b}\beta_1 e^{\sqrt{S_r S_b}t} - \sqrt{S_r S_b}\beta_2 e^{-\sqrt{S_r S_b}t} &= -S_r\rho_1 e^{\sqrt{S_r S_b}t} - S_r\rho_2 e^{-\sqrt{S_r S_b}t} \\ \sqrt{S_r S_b}\rho_1 e^{\sqrt{S_r S_b}t} - \sqrt{S_r S_b}\rho_2 e^{-\sqrt{S_r S_b}t} &= -S_b\beta_1 e^{\sqrt{S_r S_b}t} - S_b\beta_2 e^{-\sqrt{S_r S_b}t}. \end{aligned} \tag{3.12}$$

The only way these equations are satisfied is if the coefficients of the exponentials on both sides are the same; otherwise by regrouping we have a non-zero multiple of an exponentially increasing function equals one of an exponentially decreasing one. Thus these equations and our initial condition give

$$\begin{aligned} \rho_1 &= -\sqrt{\frac{S_b}{S_r}}\beta_1, \quad \rho_2 = \sqrt{\frac{S_b}{S_r}}\beta_2 \\ r(0) &= \rho_1 + \rho_2 \\ b(0) &= \beta_1 + \beta_2;. \end{aligned} \tag{3.13}$$

note the two equations in (3.12) produce the same relations between the ρ's and the β's (which is to be expected given their symmetry; note this leaves us with four equations and four unknowns). It's a simple matter to solve[10] and we find

$$\begin{aligned} \rho_1 &= \frac{r(0)}{2} - \sqrt{\frac{S_b}{S_r}}\frac{b(0)}{2}, \quad \rho_2 = \frac{r(0)}{2} + \sqrt{\frac{S_b}{S_r}}\frac{b(0)}{2} \\ \beta_1 &= \frac{b(0)}{2} - \sqrt{\frac{S_r}{S_b}}\frac{r(0)}{2}, \quad \beta_2 = \frac{b(0)}{2} + \sqrt{\frac{S_r}{S_b}}\frac{r(0)}{2}. \end{aligned} \tag{3.14}$$

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[10] If we add ρ_1 and ρ_2 we get a multiple of $-\beta_1 + \beta_2$, and we then multiply $b(0) = \beta_1 + \beta_2$ by that constant and add, which gives us $2\beta_2$, from which everything follows easily.

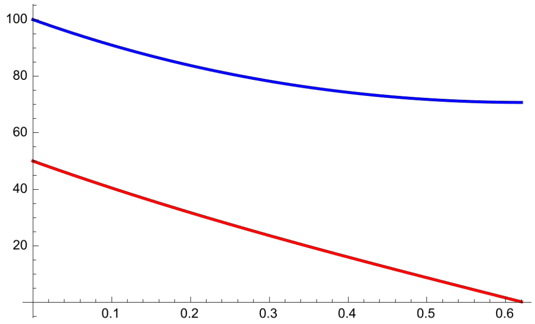

Figure 2: Battle between a Red force of 50 units and a Blue force of 100, where Red's effectiveness is $S_r = 2$, twice Blue's effectiveness of $S_b = 1$.

Remark 3.1. Solving the above system of equations for the constants $\rho_1, \rho_2, \beta_1, \beta_2$ provides an excellent opportunity to reiterate a valuable lesson: always ask if the answer is reasonable. Are there simple tests we can do? While the magnitudes of these parameters are hard to gauge in advance, their *signs* are much easier. We first note that we cannot have both ρ's or both β's negative, as then we would have a negative force size; thus at most one of each is negative. We have ρ_1, β_1 associated to the exponentially growing term while ρ_2, β_2 go with the exponentially decaying one. Note both ρ_2 and β_2 are positive, and since $\rho_1 = -\sqrt{\frac{S_b}{S_r}}\beta_1$ we see ρ_1 and β_1 are of opposite signs. This is reasonable; if both were positive then neither fleet would be destroyed, thus we need at least one of them to be negative. While these arguments do not prove that we've done the algebra correctly, this is a great consistency check.

3.4 Example

In classes it's helpful to do examples with good choices parameters to give students a feel for the answer. We can of course collect all our values above and write down a closed form solution. What would be a good case to try? We could start with equal forces but red's ships more effective; the answer should clearly have red winning.

For a more interesting case, imagine we have 50 red units and 100 blue units, but $S_r = 2$ while $S_b = 1$; thus the red forces are twice as effective as the blue but half as numerous. The students can be surveyed before we write down the solution: do you think red wins, blue wins, or it's a draw?

As this is the 21st century, we can have Mathematica solve the system

```
DSolve[{r'[t] == - b[t], b'[t] == -2 r[t], r[o] == 50, b[o] == 100}, {r[t], b[t]}, t];
```

we plot the results in Figure 2

The answer is not what many students expect - the numerical advantage of blue more than makes up for the better effectiveness of red.[11] We end the class with the following

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[11] A quick calculation provides a nice justification. Initially the two forces are losing ships at the same rate as $b(0) = 2r(0)$. Imagine that they always lose ships at the same rate; then after a bit there are 90 blue and 40 red ships, but now $-b(t) = -90$ while $-2r(t) = -80$, and thus blue's forces decrease at a slower

questions, asking students to solve in general if possible, if not for specific values of the parameters such as the above.

- How long does it take for the battle to end?
- If one side has twice as many forces as another, how much more effective must the other ships be so that the battle ends in a draw?
- More generally, given $r(0)$ and $b(0)$ what should S_r and S_b equal to ensure a draw? Without loss of generality argue that we may normalize so that $r(0) = 1$.

We can write some simple Mathematica code to help with the exploration.

```
srb[sr_, sb_] := Sqrt[sr * sb];
srob[sr_, sb_] := Sqrt[sr/sb];
sbor[sr_, sb_] := Sqrt[sb/sr];
rho1[t_, sr_, sb_, r0_, b0_] :=  r0/2 - sbor[sr, sb]  b0/2;
rho2[t_, sr_, sb_, r0_, b0_] :=  r0/2 + sbor[sr, sb]  b0/2;
beta1[t_, sr_, sb_, r0_, b0_] :=  b0/2 - srob[sr, sb]  r0/2;
beta2[t_, sr_, sb_, r0_, b0_] :=  b0/2 + srob[sr, sb]  r0/2;
r[t_, sr_, sb_, r0_, b0_] :=
  rho1[t, sr, sb, r0, b0]  Exp[srb[sr, sb]  t] +
   rho2[t, sr, sb, r0, b0]  Exp[-srb[sr, sb]  t];
b[t_, sr_, sb_, r0_, b0_] :=
  beta1[t, sr, sb, r0, b0]  Exp[srb[sr, sb]  t] +
   beta2[t, sr, sb, r0, b0]  Exp[-srb[sr, sb]  t];
Manipulate[
 Plot[{r[t, sr, sb, r0, b0], b[t, sr, sb, r0, b0]}, {t, 0, tmax},
  PlotStyle -> {Red, Blue}], {tmax, .1, 10}, {sr, 1, 10}, {sb, 1,
  10}, {r0, 100, 1000}, {b0, 100, 1000}]
```

This code *almost* does what is needed. The problem is that it has the wrong definition for $r(t)$ and $b(t)$ after the weaker fleet is destroyed. We leave it to the reader to adjust the code to fix the definition of the functions (a good way to do that is to determine the value of t when the battle ends, say t_{end}, and then for $t \geq t_{\text{end}}$ set $r(t) = r(t_{\text{end}})$, and similarly for $b(t)$). Mathematica's `Manipulate` function does an excellent job allowing one to explore the consequences of changing parameter values in real time; see Figure 3.

4 Exploring Model Implications

We end by discussing how to analyze models. In particular, how do we deal with a large number of parameters in a systematical way to illuminate the structure and dependencies? How can we use data from special cases to make natural conjectures?

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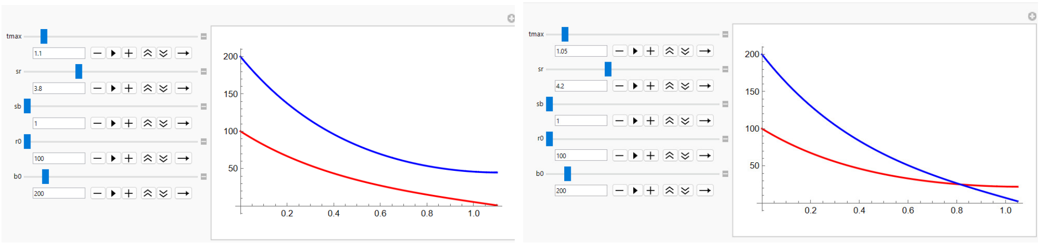
rate; as the battle continues this difference becomes more and more pronounced, and we expect blue to comfortably triumph.


tmax 1.1
sr 3.8
sb 1
r0 100
b0 200

tmax 1.05
sr 4.2
sb 1
r0 100
b0 200

Figure 3: Two battles between a Red force of 100 units and a Blue force of 200, where Blue's effectiveness is fixed as $S_b = 1$ in both battles but Red's effectiveness is $S_r = 3.8$ and $S_r = 4.2$ on the left and right respectively. Red loses in the battle corresponding to the left figure but wins the battle on the right, suggesting some "tipping point" of roughly 4 for the multiplicative skill advantage necessary for Red to win rather than 2.

4.1 Square Law Discovery: Empirically and Analytically

As students are changing the model's parameters and exploring the consequences, as discussed a natural question is predicting which side wins. In particular, is there a clear threshold which separates winning configurations from losing ones that we can either describe with an exact formula, or some approximate heuristic? While simple numerical tests may produce a lot of data points, students can take a more systematic approach to trying to discover this relationship for themselves numerically, which we develop below.

A key aspect of modeling is to *keep it simple*. While on the surface there are 4 variables affecting the battle's outcome (namely $r(0), b(0), S_r, S_b$), experimentation suggests that for fixed S_r and S_b the winner of the battle does not change so long as the ratio $b(0)/r(0)$ is constant. As our model is linear, with

$$\begin{pmatrix} r'(t) \\ b'(t) \end{pmatrix} = \begin{pmatrix} 0 & -S_b \\ -S_r & 0 \end{pmatrix} \begin{pmatrix} r(t) \\ b(t) \end{pmatrix}, \tag{4.1}$$

this makes sense: if we double both initial concentrations, we can view it as two separate battles. Thus there are at most 3 variables determining who wins: $b(0)/r(0)$, S_r and S_b.

Do we need both S_b and S_r, or does the answer only depend on their ratio (or some other relationship)? If yes, it would be sufficient to specify just two numbers to determine the result! We use our code to consider a battle with fixed $k := b(0)/r(0)$ and S_b values, and easily determine an approximate minimum value of S_r that allows the red forces to win; see Figure 4.

Figure 4 suggests that the set of points appears to correspond to lines with y-intercept of 0. Empirically the winner of the battle seems to be invariant among different choices of S_r and S_b which preserve the ratio S_r/S_b, just as was the case for the ratio of initial force sizes. The logic for this is that if the points in the plot corresponding to $b(0)/r(0) = k$ perfectly fit a line with y-intercept 0 and slope m_k , then the minimum value of S_r allowing the red force to win when $b(0)/r(0) = k$, denoted as $(S_r)_{\min}$ below, is given by

$$(S_r)_{\min} = m_k S_b \quad \text{for any } S_b > 0 \text{ considered,} \tag{4.2}$$

and hence the red force wins when $S_r > m_k S_b$, or equivalently when $S_r/S_b > m_k$. Armed with this new revelation, we now only have two variables to consider when wishing to

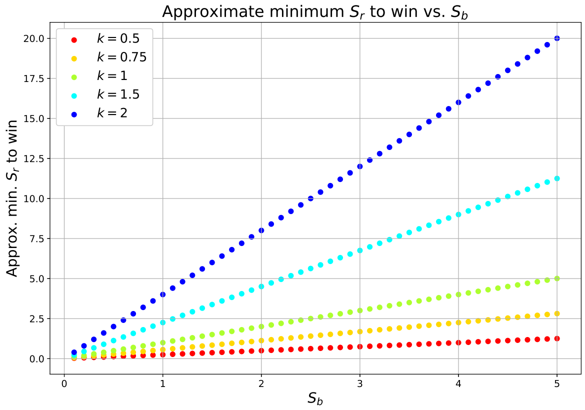


Figure 4: Scatter plot of the approximate minimum value of S_r required for the red forces to win as a function of S_b, plotted for various settings of $k = b(0)/r(0)$. Note that for each k, the associated data seems to fall on a line passing through the origin.

determine whether or not the red force wins: $b(0)/r(0)$ and S_r/S_b. We've reduced our parameter space from 4 to 2, illustrating an important point about modeling and statistics: while it is often easy to write our models as functions of various quantities, in the end not all matter equally (or perhaps it's better to say that many of the initial parameters can be combined, reducing the complexity of the space to study).[12]

In an entirely analogous fashion, students can approximately determine the minimum value of S_r/S_b needed to win as a function of $b(0)/r(0)$ for various values of $b(0)/r(0)$ and plot the corresponding data, generating a plot similar to one on the left-hand size of Figure 5.

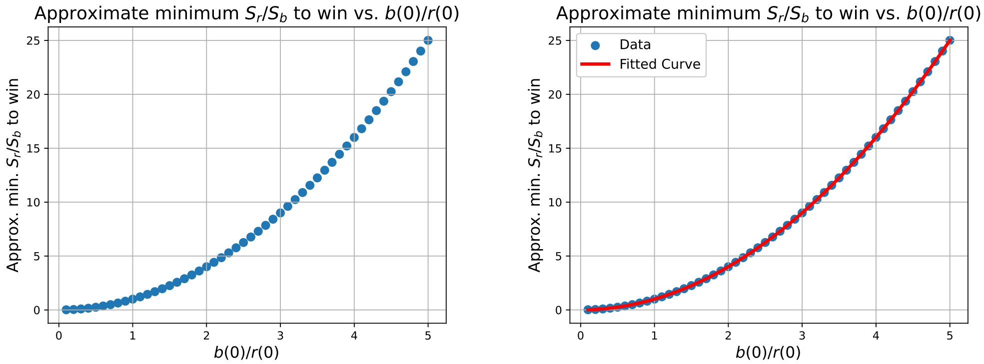


Figure 5: Left: Plot of the minimum value of S_r/S_b as a function of $b(0)/r(0)$. Right: Curve fit to the points, showing the function $y = x^2$.

By using curve fitting tools[13] such as FindFit in Mathematica or scipy.optimize.curve_fit

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[12] Note that $b(0)/r(0)$ has the term corresponding to blue in the numerator while for S_r/S_b it is in the denominator; this aligns better with some of the later algebra.

[13] As a goal of this article is to emphasize opportunities for the classroom, we note that for those who

in Python to fit a curve to this function, as seen on the right-hand side of Figure 5, we find that the points approximately fit the curve $y = x^2$, and hence we empirically observe that the minimum value of S_r/S_b allowing the red forces to win is $(b(0)/r(0))^2$, so that the red forces win when

$$S_r/S_b \ > \ (b(0)/r(0))^2, \quad \text{i. e.,} \quad S_r r(0)^2 \ > \ S_b b(0)^2. \tag{4.3}$$

There is another approach, frequently suggested by students: As we have a closed-form expression for both $r(t)$ and $b(t)$, why not utilize this directly? In particular, the red forces lose if and only if at some time $t_{\text{loss}} > 0$ we have $r(t_{\text{loss}}) = 0$ and $b(t_{\text{loss}}) > 0$. Since

$$r(t) \ = \ \rho_1 e^{\sqrt{S_r S_b}t} \ + \ \rho_2 e^{-\sqrt{S_r S_b}t}, \tag{4.4}$$

we have that the red forces losing is equivalent to the condition that there exists some unique time t_{loss} such that

$$0 \ = \ \rho_1 e^{\sqrt{S_r S_b}t_{\text{loss}}} \ + \ \rho_2 e^{-\sqrt{S_r S_b}t_{\text{loss}}} \tag{4.5}$$

(and of course then checking to make sure the blue force size is positive here). Rearranging the above yields

$$e^{2\sqrt{S_r S_b}t_{\text{loss}}} \ = \ -\frac{\rho_2}{\rho_1}. \tag{4.6}$$

As the exponential function is a 1-to-1 map from the positive reals to the positive reals, there is a solution if and only if $-\rho_2/\rho_1$ is positive; that solution is

$$-\frac{\rho_2}{\rho_1} \ = \ \frac{\frac{r(0)}{2} \ + \ \sqrt{\frac{S_b}{S_r}}\frac{b(0)}{2}}{\sqrt{\frac{S_b}{S_r}}\frac{b(0)}{2} \ - \ \frac{r(0)}{2}}. \tag{4.7}$$

As the numerator is strictly positive, the above expression is positive exactly when the denominator is positive, which clearly occurs if and only if $\sqrt{S_b/S_r}(b(0)/2) \ > \ r(0)/2$. Rearranging, this is equivalent to the expression $(b(0)/r(0))^2 < S_b/S_r$, and red forces lose if and only if

$$\left(\frac{r(0)}{b(0)}\right)^2 \ < \ \frac{S_b}{S_r}.$$

This is equivalent to the expression we derived empirically above: the red forces win if and only if

$$\frac{S_r}{S_b} \ > \ \left(\frac{b(0)}{r(0)}\right)^2 \quad \text{or} \quad S_r r(0)^2 \ > \ S_b b(0)^2. \tag{4.8}$$

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know linear regression, we can do a log-log transformation of the data and then use the Method of Least Squares to find the best fit line; see [9, 10]. This is a wonderful opportunity to highlight the value of logarithms, a point sadly often skipped when students first meet this function. Logarithms transform power relationships to linear ones, making powerful statistical techniques applicable.

Of course, a similar argument yields an equivalent condition for blue to triumph, and we deduce that the two forces mutually annihilate each other if

$$S_r r(0)^2 = S_b b(0)^2. \tag{4.9}$$

The above is known as the *Square Law*. It's always good to check degenerate cases and see if they agree with intuition; here if $S_r = S_b$ then the battle is a draw if and only if the initial numbers are equal, which is quite reasonable. Similar power relationships hold in other systems.

4.2 Deriving Expressions for Surviving Forces and Time of Battle

In the previous section, once we formulated the closed-form expression for force size as a function of time, students were able to see how to analytically derive an expression to determine which side would win by simply recognizing that a side loses if and only if there exists some point at which their force size is zero. In particular, setting the closed-form expression derived previously equal to zero, the Square Law emerges as the necessary and sufficient condition for a solution to exist. A reasonable question to pose to students now is what else can gleamed from analyzing zeros of this closed-form expression? In the event that blue wins (which can be determined via the Square Law), students should be able recognize that the following relationships hold:

- The unique time t' such that $r(t') = 0$ corresponds to the end of the battle.
- Denoting such a zero as t_{end}, it follows that $b(t_{\text{end}})$ is the number of surviving forces.

Using these facts, it is now a matter of simple algebraic manipulation to compute these corresponding quantities. In particular, simplifying $r(t_{end}) = 0$ as in the previous section yields:

$$e^{2\sqrt{S_r S_b}t_{\text{end}}} = -\frac{\rho_2}{\rho_1}. \tag{4.10}$$

However, note

$$e^{2\sqrt{S_r S_b}t_{\text{end}}} = -\frac{\rho_2}{\rho_1} \iff t_{\text{end}} = \frac{1}{2\sqrt{S_r S_b}} \ln\left(-\frac{\rho_2}{\rho_1}\right) \tag{4.11}$$

where

$$\rho_1 = \frac{r(0)}{2} - \sqrt{\frac{S_b}{S_r}}\frac{b(0)}{2} \quad \text{and} \quad \rho_2 = \frac{r(0)}{2} + \sqrt{\frac{S_b}{S_r}}\frac{b(0)}{2}. \tag{4.12}$$

From the simple realization about the significance and uniqueness of zeros of our closed-form expression for force size, all it took was some basic algebra to get a closed-form expression for the time of battle! Likewise, by the second realization above, this new closed-form expression should allow students to easily derive the closed form expression for the number of blue forces remaining by plugging this value into the closed form expression of $b(t)$:

$$\begin{aligned} b(t_{\text{end}}) &= \beta_1 e^{\sqrt{S_r S_b} t_{\text{end}}} + \beta_2 e^{-\sqrt{S_r S_b} t_{\text{end}}} \\ &= \beta_1 e^{\frac{1}{2}\ln\left(-\frac{\rho_2}{\rho_1}\right)} + \beta_2 e^{-\frac{1}{2}\ln\left(-\frac{\rho_2}{\rho_1}\right)} \\ &= \beta_1 \sqrt{-\frac{\rho_2}{\rho_1}} + \beta_2 \sqrt{-\frac{\rho_1}{\rho_2}}. \end{aligned} \tag{4.13}$$

Substituting in for ρ_1, ρ_2 and manipulating, students can find that the surviving number of blue forces is

$$\sqrt{b(0)^2 - \frac{S_r}{S_b} r(0)^2} \tag{4.14}$$

when blue wins (and similarly when red wins the surviving number of red forces is $\sqrt{r(0)^2 - \frac{S_b}{S_r} b(0)^2}$). We now have all the needed ingredients to describe and solve the optimization problem: how to split your forces and your enemy's forces.

4.3 Putting Everything Together: Optimizing a Divide-and-Conquer Strategy

Returning to the Battle of Trafalgar, as General Nelson had 27 ships while the Franco-Spanish fleet had 33, the Square Law implies Nelson's forces would need to have a relative strength per unit of $(33/27)^2 \approx 1.49$ over that of their opponents to win in a direct fight - possibly an unreasonably large skill advantage.[14] Amazingly, Nelson won without losing a single ship[15], due to his masterful use of divide and conquer. Rather than one massive engagement, Nelson was able to split the battle into smaller parts to his advantage. But this begs the question - what is the best way to do so?

To model the combat, we assume that Nelson sends β of his ships to fight α of the enemy ships in one skirmish, with the remaining forces on each side battling in another separate skirmish. Once one skirmish ends, the surviving ships join their compatriots in the ongoing battle. The question of optimal battle strategy now boils down to finding the feasible values of α, β maximizing the number of surviving British ships. [16]

This optimization problem can be approached in many ways. Given our closed-form expressions for determining the winner, number of survivors and total battle time, it's possible to set up an explicit optimization problem and attempt to solve this either numerically or analytically. For this particular problem, given there are not many values

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[14] For many years the British policy was to be able to fight the fleets of the next two most powerful nations; this was reduced to a 60% numerical superiority over the next largest nation in the naval races before WWI; see [3, 4] for a discussion of this as well as the conversations on the development of different types of ships and armaments.

[15] Factors our simple model doesn't account for include Nelson's use of the element of surprise, and the chaos of war.

[16] We assume that Nelson can ensure any division he wishes, which may not be a realistic assumption in battle.

of β and α a reasonable approach is to write code and simulate each pair. Thus for each $(\alpha, \beta) \in \{1, \ldots, 32\} \times \{1, \ldots, 26\}$, we use the above formulas to determine the time at which some skirmish first concludes, and how many ships remain. Then, we add those ships to the other skirmish and use again the equation for determining the winner and number of survivors to deduce the final state of the battle. In addition to not requiring tedious analytical computation or reasoning (which is not possible for all models), a major benefit of this method is that it allows us to visualize the data. Particularly, rather than just computing the optimal way to divide forces, without knowing anything about the alternatives, we deduce for each possible division exactly how many British forces survive, and can plot these values in the form of a heat map.

Interestingly, we run into an issue when we execute our code. Something similar to Figure 6 should appear, which appears to show that the final battle result is independent of how the forces are initially divided! This is a wonderful opportunity to stop and ask the class if this is reasonable: Is divide-and-conquer no better than a head on battle under our assumptions, and if so, why?

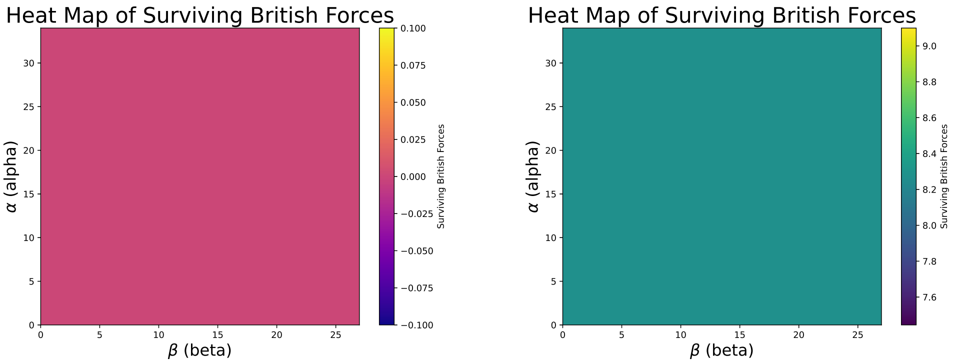

Figure 6: Plots showing a constant number of surviving British forces for all values of α, β assuming instantaneous reunion of forces, for $S_b/S_r = 1.25$ (Left) and $S_b/S_r = 1.75$ (Right).

Given that the outcome always matches the case in which both $\beta, \alpha = 0$ (when no divide-and-conquer occurred), our divide-and-conquer model seems to coincide exactly with that of our original model. If we denote $r_1(t), r_2(t)$ to be the sizes of red forces of skirmishes 1 and 2 respectively prior to the end of one skirmish, and likewise for $b_1(t), b_2(t)$, it follows that

$$r_1'(t) = S_b b_1(t) \quad \text{and} \quad r_2'(t) = S_b b_2(t)$$

$$\Longrightarrow \; r'(t) = r_1'(t) + r_2'(t) = S_b(b_1(t) + b_2(t)) = S_b b(t).$$

Once one skirmish concludes, and assuming an instantaneous reunion of forces, it then follows that the battle follows the same dynamics as discussed previously. As a result, at any point in time during the battle necessarily $r'(t) = S_b b(t)$, and by analogous reasoning $b'(t) = S_r r(t)$, so that the battle outcome observed as a result of a divide-and-conquer strategy with no time-delay of ship reunion should be identical to that of our original model, which explains our observation!

What happens if we now consider the entirely opposite extreme: forces do not reunite until both skirmishes have individually concluded (which can be modeled as an “infinite reunion time”)? Modifying our simulation for this case, we observe outcomes as in Figure 7.

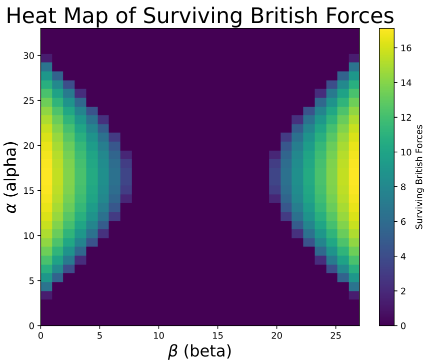

Figure 7: Plot showing the number of surviving British forces for all values of α, β assuming reunion does not occur until after both skirmishes individually conclude, with $S_b/S_r = 1.25$.

Empirically, it appears that the best option for Nelson is to divide the enemy forces in half while keeping his own forces together. [17] Students should once again question *“Does this make sense?”*. One reasonable explanation is the fact that, as forces cannot engage until both skirmishes have individually concluded, this strategy effectively forces half of the forces to “sit out” on the majority of the battle. But what about for other, less extreme time values? If we assume that $S_b = .125$ and $S_r = .1$, we get the following graphs for various values of reunion times as seen in Figure 8.

This simulation, providing very visual output, allows students to think intuitively about the relationship between different variables affecting the battle. In Figure 8 for example, we not only have numerically verified the efficacy of a divide-and-conquer strategy in allowing the British to gain the upper hand, but see that this effect becomes more pronounced as the time required for forces to reunite increases. For all reunion times, we see that setting $\beta = 0$ is optimal for Nelson, while the optimal α corresponding to $\beta = 0$ decreases and converges to approximately half of the total opponent force size as t_{reunion} increases. Furthermore, conditioning on the first skirmish having β be less than half the British force size, smaller values of β are more optimal than larger ones, regardless of the chosen α. Thus, our model suggests that Nelson’s best strategy is to use as little British force as feasibly possible to divert at least half of the opposing forces into a separate skirmish, with the optimal size of this opposing force diverted decreasing towards $1/2$ the total opponent force size as t_{reunion} grows large. Additionally, if we were Nelson and had to choose a battle plan with the estimate that $S_b = .125, S_r = .100$, from the above we can immediately see the following.

- If given conditions allow us to assume t_{reunion} is large, say at least 6, then a great strategy is to set α to be 16 and β to be as small as feasibly possible. Then, our model

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[17] Although forcing half of the enemy to not engage in the battle seems impossible from a practical manner of course.

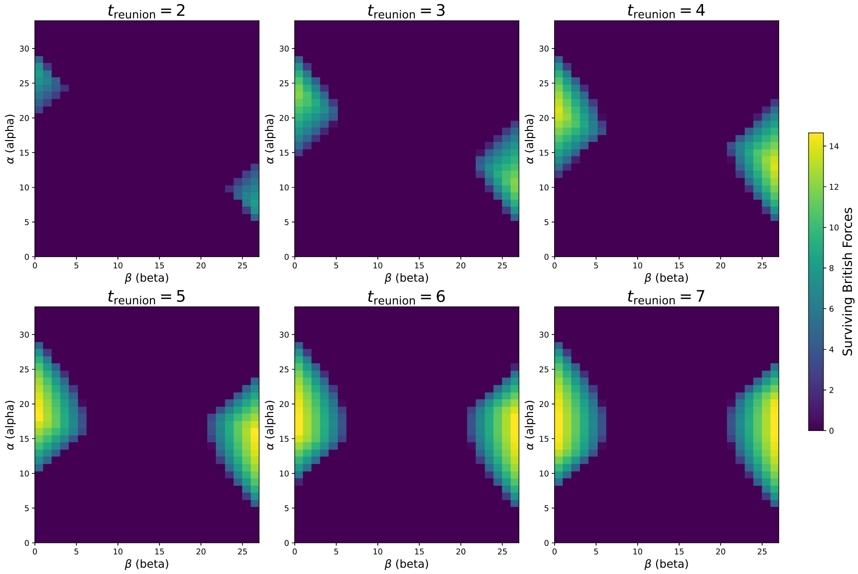

Figure 8: Plots showing the number of surviving British forces for all values of α, β for various reunion times. For these plots, $S_b = .125$ and $S_r = .100$.

predicts that choosing $\beta \leq 6$ would lead to British victory (with at least 10 British ships surviving when $\beta = 2$ for example).

- If we are highly uncertain about t_{reunion}, or our ability to set α close to 0, a safer division is $\beta = 24$ and $\alpha = 4$, which our model predicts will allow the British to win so long as $t_{\text{reunion}} \geq 2$. While this is far from optimal in the case $t_{\text{reunion}} = 7$, our model predicts it is more likely to result in a British victory (albeit a significantly more narrow one).

As this model allows for such a visual representation of the battle outcomes, it has allowed us to see the trends above very blatantly, and hence accommodates a more dynamic notion of "optimal" dependent on Nelson's knowledge of uncertain conditions.[18]

Optimization is a broad and important field, with a rich theory and seemingly limitless practical implications. It basically addresses the question: What is the best thing we can do subject to some constraints? It's hard to think of an applied field for which questions of this type aren't relevant! Through modeling the Battle of Trafalgar, students see the power of both pure analysis and simulation, and the challenges in creating a tractable model and the need to often make simplifying assumptions to turn an optimization problem into one which is simpler to solve. Beyond this, students have seen the importance of trouble-shooting and critically thinking about unexpected results via extreme cases, and the importance of deciding how best to visualize data to understand relationships between variables and systematically spot trends, particularly when the typical analytic tools learned in math class (e.g. calculus) are either too involved to utilize directly or not clearly applicable.

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[18] One such example is wind speeds, which would affect how fast ships could move and hence how fast they could reunite

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