Benford's Law under Zeckendorf expansion

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abstract In the literature, Benford's Law is considered for base-b expansions where $b > 1$ is an integer. In this paper, we investigate the distribution of leading "digits" of a sequence of positive integers under other expansions such as Zeckendorf expansion, and declare what Benford's Law should be under generalized Zeckendorf expansion.

1 Introduction

Introduced in [2, 18] is a probability distribution of the leading decimal digits of a sequence of positive integers, known as *Benford's Law*, and the exponential sequences such as $\{3^n\}$ are standard examples of sequences that satisfy Benford's Law. Given $d \in \{1,2,3,\dots,9\}$, the probability of having the leading digit d in the decimal expansion of 3^n is $\log_{10}\frac{d+1}{d}$, and this distribution is Benford's Law. In fact, given a block B of digits of any length, the probability of having the leading block B in the decimal expansion of 3^n is given by a similar logarithmic formula as well, and this is known as *strong Benford's Law;* see Example 1.9. It is indeed a special property that a sequence has convergent proportions for each leading digit. For example, the proportion of odd integers $2n-1 \le M$ with leading digit d oscillates, and does not converge as $M \to \infty$; see Section 4.10.

In the literature, Benford's Law is considered for base-b expansions where $b > 1$ is an integer. For example, the probabilities of the binary expansions of integer powers of 3 having the leading binary digits 100_2 and 101_2 are $\log_2\frac{2^2+1}{2^2}$ and $\log_2\frac{2^2+2}{2^2+1}$, respectively; for later reference, we may rewrite the values as follows:

$$\log_2\frac{1+2^{-2}}{1} \approx 0.322, \quad \log_2\frac{1+2^{-1}}{1+2^{-2}} \approx 0.264. \tag{1}$$

In this paper, we shall consider the distribution of leading "digits" of a sequence of positive integers under other expansions such as Zeckendorf expansion [19]. For example, let $\{F_n\}_{n=1}^{\infty}$ for $n \ge 1$ be the shifted Fibonacci sequence, i.e., $F_{n+2} = F_{n+1} + F_n$ for all $n \in \mathbb{N}$ and $F_1 = 1$ and $F_2 = 2$, and consider two Zeckendorf expansions: $3^5 = F_{12} + F_5 + F_2$ and $3^8 = F_{18} + F_{16} + F_{14} + F_{11} + F_7 + F_5$. Similar to the way the binary expansions are denoted, we may write

$$3^5 = 100000010010_F, \quad 3^8 = 101010010001010000_F$$

where 1's are inserted at the kth place from the right if F_k is used in the expansions.

Definition 1.1. Let $A = \{0,1\}$. Given $\{s,n\} \subset \mathbb{N}$, let $n = \sum_{k=1}^{M} a_k F_{M-k+1}$ be the Zeckendorf expansion of n (where $a_1 = 1$). We define $\mathrm{LB}_s(n) := (a_1,\dots,a_s) \in A^s$ if $M \ge s$; otherwise, $\mathrm{LB}_s(n)$ is undefined. The tuple $\mathrm{LB}_s(n)$ is called *the leading block of n with length s under Zeckendorf expansion*.

For example, $\mathrm{LB}_3(3^5) = (1,0,0)$, $\mathrm{LB}_3(3^8) = (1,0,1)$, and $\mathrm{LB}_6(3^8) = (1,0,1,0,1,0)$. Notice that by Zeckendorf's Theorem, $\mathrm{LB}_2(n) = (1,0)$ for all integers $n \ge 2$, and hence, it is only meaningful to consider the first three or more Zeckendorf digits. We prove Theorem 1.3 in this note.

Definition 1.2. Given a conditional statement $P(n)$ where $n \in \mathbb{N}$, and a subset A of $\mathbb{N}$, let us define

$$\mathrm{Prob}\left\{ n \in A : P(n) \text{ is true} \right\} := \lim_{n\to\infty} \frac{\#\{k \in A : P(k) \text{ is true, } k \le n\}}{\#\{k \in A : k \le n\}}.$$

For example, if $A = \{n \in \mathbb{N} : n \equiv 2 \mod 3\}$, then $\mathrm{Prob}\left\{ n \in A : n \equiv 1 \mod 5 \right\} = \frac{1}{5}$. If A is finite, the limit always exists.

Let ϕ be the Golden ratio. The following is an analogue of Benford's Law under binary expansion demonstrated in (1).

Theorem 1.3. *Let $a > 1$ be an integer.*

$$\begin{aligned}
\mathrm{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_3(a^n) = (1,0,0) \right\} &= \log_\phi(1+\phi^{-2}) \approx .672,\\
\mathrm{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_3(a^n) = (1,0,1) \right\} &= \log_\phi \frac{\phi}{1+\phi^{-2}} \approx .328.
\end{aligned}$$

In particular, they exist! Although the probabilities are different from the binary cases, the structure of the log expressions in Theorem 1.3 is quite similar to that of the binary expansions in (1), i.e., the denominators of the quotients express the leading digits in power expansions with respect to their bases. The exponential sequences $\{a^n\}_{n=1}^\infty$ where $a > 1$ is an integer are standard sequences that satisfy Benford's Law under base-b expansion. Motivated from these standard examples, we define Benford's Law under Zeckendorf expansion to be the above distribution of the leading blocks $(1,0,0)$ and $(1,0,1)$ under Zeckendorf expansion; see Definition 3.6.

The exponential sequences $\{a^n\}_{n=1}^\infty$ are standard sequences for so-called *strong Benford's Law under base-b expansion* as well; see Example 1.9. We introduce below the probability of the leading Zeckendorf digits of a^n with arbitrary length, which is a generalization of Theorem 1.3; this result is rewritten in Theorem 3.8 with more compact notation.

Definition 1.4. Let $A = \{0,1\}$, and let $s \ge 2$ be an integer. Let $\mathbf{b} = (b_1, b_2, \ldots, b_s) \in A^s$ such that $b_1 = 1$ and $b_k b_{k+1} = 0$ for all $1 \le k \le s-1$. We define $\widetilde{\mathbf{b}}$ to be a tuple $(\widetilde{b}_1, \ldots, \widetilde{b}_s) \in A^s$ as follows. If $1 + \sum_{k=1}^s b_k F_{s-k+1} < F_{s+1}$, then $\widetilde{b}_k$ for $1 \le k \le s$ are defined to be integers in A such that $1 + \sum_{k=1}^s b_k F_{s-k+1} = \sum_{k=1}^s \widetilde{b}_k F_{s-k+1}$ and $\widetilde{b}_k \widetilde{b}_{k+1} = 0$ for all $1 \le k \le s-1$. If $1 + \sum_{k=1}^s b_k F_{s-k+1} = F_{s+1}$, then $\widetilde{b}_1 := \widetilde{b}_2 := 1$, and $\widetilde{b}_k := 0$ for all $3 \le k \le s$.

For the case of $1 + \sum_{k=1}^s b_k F_{s-k+1} < F_{s+1}$, the existence of the tuple $\widetilde{\mathbf{b}}$ is guaranteed by Zeckendorf's Theorem.

Theorem 1.5. *Let $a > 1$ and $s \ge 2$ be integers. Let $\mathbf{b}$ and $\widetilde{\mathbf{b}}$ be tuples defined in Definition 1.4. Then,*

$$\mathrm{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_s(a^n) = \mathbf{b} \right\} = \log_\phi \frac{\sum_{k=1}^s \widetilde{b}_k \phi^{-(k-1)}}{\sum_{k=1}^s b_k \phi^{-(k-1)}}.$$

For example,

$$\begin{aligned}
\text{Prob}\left\{\, n \in \mathbb{N} : \text{LB}_6(a^n) = (1,0,0,0,1,0) \,\right\} &= \log_\phi \frac{1+\phi^{-3}}{1+\phi^{-4}} \approx 0.157 \\
\text{Prob}\left\{\, n \in \mathbb{N} : \text{LB}_6(a^n) = (1,0,1,0,1,0) \,\right\} &= \log_\phi \frac{1+\phi^{-1}}{1+\phi^{-2}+\phi^{-4}} \\
&= \log_\phi \frac{\phi}{1+\phi^{-2}+\phi^{-4}} \approx 0.119.
\end{aligned}$$

As in Benford's Law under Zeckendorf expansion, we define the probability distributions described in Theorem 3.8 to be *strong Benford's Law under Zeckendorf expansion*; see Definition 3.9.

Exponential sequences are standard examples for Benford's Laws, but some exponential sequences do not satisfy Benford's Law under some base-b expansion. Let us demonstrate similar examples under Zeckendorf expansion. Let $\{G_n\}_{n=1}^{\infty}$ be the sequence given by $G_k = F_{2k} + F_k$ for $k \in \mathbb{N}$. Then, given an integer $s > 1$, the s leading Zeckendorf digits of G_k is $100\cdots 00_F$ as $k \to \infty$ since the gap $2k - k = k$ between the indices of F_{2k} and F_n approaches ∞. Thus, $\text{Prob}\left\{\, n \in \mathbb{N} : \text{LB}_s(G_n) = (1,0,0,\ldots,0) \,\right\} = 1$ for all $s \in \mathbb{N}$, and the probabilities of other digits of length s are all (asymptotically) 0. Similar probability distributions occur for the Lucas sequence $\{K_n\}_{n=1}^{\infty}$ given by $K_{k+2} = K_{k+1} + K_k$ for $k \in \mathbb{N}$ and $(K_1, K_2) = (2,1)$. Given $s \in \mathbb{N}$, the probabilities of having leading Zeckendorf digits of length s are entirely concentrated on one particular string of digits. For example, $\text{Prob}\left\{\, n \in \mathbb{N} : \text{LB}_{10}(K_n) = (1,0,0,0,1,0,0,0,1,0) \,\right\} = 1$, and the probabilities of having other digits of length 10 are all (asymptotically) 0; see Example 5.10 for full answers.

Generalized Zeckendorf expansions are introduced in [10, 17]. In Section 6, we prove Theorem 6.9 on the probability of the leading digits of a^n with arbitrary length under generalized Zeckendorf expansion, and define these probability distributions to be strong Benford's Law under generalized Zeckendorf expansion; see Definition 6.10. As in the concept of *absolute normal numbers* [12], we introduce in Definition 6.15 the notion of *absolute Benford's Law*, which is the property of satisfying strong Benford's Law under all generalized Zeckendorf expansions. For example, the sequence given by $K_n = \left\lfloor \frac{\phi}{\sqrt{5}} (\frac{89}{55})^n \right\rfloor$ for $n \in \mathbb{N}$ satisfies strong Benford's Law under all generalized Zeckendorf expansions; see Example 6.18. Its first fifteen values are listed below:

$$(1,1,3,4,8,12,21,34,55,89,144,233,377,610,988).$$

They are nearly equal to the Fibonacci terms as $\frac{89}{55}$ is the 10th convergent of the continued fraction of ϕ. The differences amplify as we look at higher terms, and even under Zeckendorf expansion, this sequence satisfies strong Benford's Law.

It is also natural to consider sequences that have different distributions, and in this note we investigate other distributions of leading digits under generalized Zeckendorf expansions as well. In the following paragraphs, we shall explain this approach using base-10 expansion. The results for other expansions are introduced in Section 5 and 6.

Strong Benford's Law for the sequence $\{3^n\}_{n=1}^{\infty}$ under decimal expansion follows from the equidistribution of the fractional part of $\log_{10}(3^n)$ on the interval $(0,1)$. We realized that the function $\log_{10}(x)$ is merely a tool for calculating the leading digits, and that other distributions of leading digits naturally emerge as we modified the function $\log_{10}(x)$.

We noticed that the frequency of leading digits converges when a "continuation" of the sequence $\{10^{n-1}\}_{n=1}^{\infty}$ has convergent behavior over the intervals $[n, n+1]$, and we phrase it more precisely below.

Definition 1.6. Let $\{H_n\}_{n=1}^{\infty}$ be an increasing sequence of positive integers. A continuous function $h:[1,\infty)\to\mathbb{R}$ is called a *uniform continuation of* $\{H_n\}_{n=1}^{\infty}$ if $h(n)=H_n$ for all $n\in\mathbb{N}$, and the following sequence of functions $h_n:[0,1]\to[0,1]$ uniformly converges to an increasing (continuous) function:

$$h_n(p)=\frac{h(n+p)-h(n)}{h(n+1)-h(n)}.$$

If h is a uniform continuation of $\{H_n\}_{n=1}^{\infty}$, let $h_\infty:[0,1]\to[0,1]$ denote the increasing continuous function given by $h_\infty(p)=\lim_{n\to\infty}h_n(p)$.

Theorem 1.8 below is a version specialized for decimal expansion. The proof of this theorem is similar to, and much simpler than the proof of Theorem 5.6 for Zeckendorf expansion, and we leave it to the reader.

Definition 1.7. If $\alpha\in\mathbb{R}$, we denote the fractional part of α by $\operatorname{frc}(\alpha)$. Given a sequence $\{K_n\}_{n=1}^{\infty}$ of real numbers, we say, $\operatorname{frc}(K_n)$ *is equidistributed* if $\operatorname{Prob}\{\, n\in\mathbb{N}:\operatorname{frc}(K_n)\le\beta\,\}=\beta$ for all $\beta\in[0,1]$.

For example, consider the sequence $\{\operatorname{frc}(n\pi)\}_{n=1}^{\infty}$ where $\pi\approx 3.14$ is the irrational number. Then, by Weyl's Equidistribution Theorem, $\operatorname{frc}(n\pi)$ is equidistributed. The sequence $\{\sin^2(n)\}_{n=1}^{\infty}$ is an example of sequences that have $\operatorname{Prob}\{\, n\in\mathbb{N}:\sin^2(n)\le\beta\,\}$ defined for each $\beta\in[0,1]$, and the probability is $\frac{1}{\pi}\cos^{-1}(1-2\beta)$. Thus, it is not equidistributed.

Theorem 1.8. *Let $h:[1,\infty)\to\mathbb{R}$ be a uniform continuation of the sequence $\{10^{k-1}\}_{n=1}^{\infty}$. Then, there is a sequence $\{K_n\}_{n=1}^{\infty}$ of positive integers approaching ∞ (see Theorem 6.19 for the description of K_n) such that $\operatorname{frc}\left(h^{-1}(K_n)\right)$ is equidistributed.*

Let $\{K_n\}_{n=1}^{\infty}$ be a sequence of positive integers approaching ∞ such that $\operatorname{frc}\left(h^{-1}(K_n)\right)$ is equidistributed. Let d be a positive integer of s decimal digits. Then, the probability of the s leading decimal digits of K_n being d is equal to

$${h_\infty}^{-1}\left(\frac{(d+1)-10^{s-1}}{9\cdot 10^{s-1}}\right)-{h_\infty}^{-1}\left(\frac{d-10^{s-1}}{9\cdot 10^{s-1}}\right).$$

Example 1.9. Let $h:[1,\infty)\to\mathbb{R}$ be the function given by $h(x)=10^{x-1}$. Then, h is a uniform continuation of the sequence $\{10^{n-1}\}$, and $h_\infty(p)=\frac{1}{9}(10^p-1)$. By Theorem 6.19, the sequence $\{K_n\}_{n=1}^{\infty}$ with the equidistribution property is given by $K_n=\lfloor 10^{n+\operatorname{frc}(n\pi)}\rfloor$, but there are simpler sequences such as $\{3^n\}_{n=1}^{\infty}$ that have the property.

By Theorem 1.8, the probability of the s leading decimal digits of K_n being d is equal to

$$\log_{10}\frac{d+1}{10^{s-1}} - \log_{10}\frac{d}{10^{s-1}} \;=\; \log_{10}\left(1+\frac{1}{d}\right)$$

where $d \in \mathbb{N}$ has s decimal digits. This distribution is known as strong Benford's Law under base-10 expansion, and we may say that strong Benford's Law under base-10 expansion arises from the exponential continuation of $\{10^{n-1}\}_{n=1}^{\infty}$. For this reason, we call $h(x)$ *a Benford continuation of the base-10 sequence*.

Example 1.10. Let $h : [1,\infty) \to \mathbb{R}$ be the function whose graph is the union of the line segments from $(n, 10^{n-1})$ to $(n+1, 10^n)$ for all $n \in \mathbb{N}$. Let $\{K_n\}_{n=1}^{\infty}$ be the sequence given by $K_n = \lfloor 10^{n+\log_{10}(9\operatorname{frc}(n\pi)+1)} \rfloor$ as described in Theorem 6.19. Then, the fractional part $\operatorname{frc}\left(h^{-1}(K_n)\right)$ is equidistributed. The limit function h_∞ defined in Theorem 1.8 is given by $h_\infty(p) = p$ for $p \in [0,1]$, and given a decimal expansion d of length s, the probability of the s leading decimal digits of K_n being d is (uniformly) equal to $1/(9\cdot 10^{s-1})$ by Theorem 1.8.

The first ten values of K_n are

$$(22, 354, 4823, 60973, 737166, 8646003, 99203371, 219467105,\ 3469004940, 47433388230).$$

For example, if we look at many more terms of K, then the first two digits 22 of K_1 will occur as leading digits with probability $1/90 \approx 0.011$, and the probability for the digits 99 is also 1/90. As in constructing a normal number, it is tricky to construct a sequence of positive integers with this property and prove that it has the property. Let us note here that the s leading decimal digits of the sequence $\{n\}_{n=1}^{\infty}$ has frequency close to $1/(9\cdot 10^{s-1})$, but it oscillates and does not converge as more terms are considered; see Theorem 4.10 for a version under Zeckendorf expansion. In Example 5.4, we demonstrate the "line-segment" continuation of the Fibonacci sequence.

In Example 5.7, we use a more refined "line segment continuation", and demonstrate a uniform continuation that generates the distribution of leading blocks that satisfies strong Benford's Law up to the 4th digits, but does not satisfy the law for the leading blocks of length > 4.

Theorem 1.8 suggests that given a uniform continuation h of the sequence $\{10^{n-1}\}_{n=1}^{\infty}$, we associate certain distributions of leading digits, coming from the equidistribution property. It's natural to consider the converse that given a sequence $\{K_n\}_{n=1}^{\infty}$ with "continuous distribution of leading digits" of arbitrary length, we associate a certain uniform continuation of $\{10^{n-1}\}_{n=1}^{\infty}$. Theorem 1.11 below is a version for base-10 expansion. In Section 5, we introduce results on this topic for the Fibonacci sequence $\{F_n\}_{n=1}^{\infty}$. The proof of Theorem 1.11 is similar to, and simpler than Theorem 5.18 for the Fibonacci expansion, and leave it to the reader.

Theorem 1.11. *Let $\{K_n\}_{n=1}^{\infty}$ be a sequence of positive integers approaching ∞. Let $h_K^* : [0,1] \to [0,1]$ be the function given by $h_K^*(0) = 0$, $h_K^*(1) = 1$, and*

$$h_K^*\left(\tfrac{1}{9}(\beta-1)\right) \;=\; \lim_{s\to\infty} \operatorname{Prob}\left\{\, n \in \mathbb{N} : \text{The } s \text{ leading decimal digits of } K_n \text{ is} \le \lfloor 10^{s-1}\beta \rfloor \,\right\} \qquad (2)$$

where β varies over the real numbers in the interval $[1,10)$ *and we assume that the RHS of (2) is defined for all* $\beta \in [1,10)$. *If* h_K^* *is an increasing continuous function, then there is a uniform continuation* h *of the sequence* $\{10^{n-1}\}_{n=1}^{\infty}$ *such that* ${h_\infty}^{-1} = h_K^*$, *and* $\operatorname{frc}\left(h^{-1}(K_n)\right)$ *is equidistributed.*

If a sequence K of positive integers approaching ∞ satisfies (2) where h_K^* is an increasing continuous function, the sequence is said to *have continuous leading block distribution under base-*10 *expansion*; see Definition 5.16 for Zeckendorf expansion. By Theorem 1.8 and 1.11, we have

Theorem 1.12. *A sequence* $\{K_n\}_{n=1}^{\infty}$ *of positive integers approaching* ∞ *has continuous leading block distribution under base-*10 *expansion if and only if* $\operatorname{frc}\left(h^{-1}(K_n)\right)$ *is equidistributed for some uniform continuation* h *of* $\{10^{n-1}\}_{n=1}^{\infty}$.

Corollary 1.13. *If a sequence* $\{K_n\}_{n=1}^{\infty}$ *satisfies strong Benford's Law under base-*10 *expansion, then* $\operatorname{frc}\left(\log_{10}(K_n)\right)$ *is equidistributed.*

It is interesting that Benford's Law under base-b expansion arises within the Zeckendorf expansion of a positive integer, i.e., if we randomly select an integer n in $[1, F_m)$, then the frequency of the Fibonacci terms with leading decimal digit d among the summands of the Zeckendorf expansion of n is nearly $\log_{10}(1+\frac{1}{d})$ for almost all $n \in [1, F_m)$. This is proved in [3]. In fact, they prove a result for attributes that are far more general than leading digits, and the result holds for all generalized Zeckendorf expansions as well.

Their result immediately applies to a general setup where a generalized Zeckendorf expansion and base-10 expansion are replaced with two arbitrary generalized Zeckendorf expansions. The full statement is found in Theorem 7.1, and below we introduce a specialized version for the binary and Zeckendorf expansions.

Theorem 1.14. *Let* $S = \{n \in \mathbb{N} : \mathrm{LB}_3(2^{n-1}) = (1,0,1)\}$, *and let* $t \in \mathbb{N}$. *Given* $n \in [1, 2^t)$, *let* $n = \sum_{k \in A_n} 2^k$ *be the binary expansion of* n *where* A_n *is a finite subset of* $\mathbb{N}$, *and define* $P_t(n) = \#(A_n \cap S)/\#A_n$.

Then, given a real number $\epsilon > 0$, *the probability of* $n \in [1, 2^t)$ *such that*

$$\left| P_t(n) - \log_\phi \frac{\phi}{1+\phi^{-2}} \right| < \epsilon$$

is equal to $1 + o(1)$ *as a function of* t.

Notice that by Theorem 1.3, $\operatorname{Prob}\left\{ n \in \mathbb{N} : n \in S \right\} = \log_\phi \frac{\phi}{1+\phi^{-2}}$, and hence, we may say that Benford's Law under Zeckendorf expansion arises within the binary expansion of positive integers.

The remainder of this paper is organized as follows. In Section 2, the notations for sequences and coefficient functions are introduced. In Section 3, the distribution of leading blocks of exponential sequences under Zeckendorf expansion is introduced, and Benford's

Law and strong Benford's Law under Zeckendorf expansion are declared. Introduced in Section 4 are the method of calculating the distribution results introduced in Section 3, and also the distribution results for monomial sequences $\{n^a\}_{n=1}^{\infty}$. In Section 5, we introduce a general approach to the distributions of leading blocks under Zeckendorf expansion that are different from that of Benford's Law. The approach establishes the correspondence between the continuations of the Fibonacci sequences and the distributions of leading blocks under Zeckendorf expansion. In Section 6, we introduce definitions and results that generalize the contents of Sections 3, 4, and 5 for generalized Zeckendorf expansions. The absolute Benford's Law mentioned earlier in this section is properly introduced in Section 6 as well. In Section 7, the Benford behavior introduced in Theorem 1.14 is generalized for the setting of two generalized Zeckendorf expansions.

2 Notation and definitions

Notation 2.1. Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and let $\Omega_n := \{k \in \mathbb{N} : k \le n\}$. For simpler notation, let us use a capital letter for a sequence of numbers, and use the infinite tuple notation for listing its values, e.g., $Q = (2,4,6,8,\dots)$. We use the usual subscript notation for individual values, e.g., $Q_3 = 6$.

Definition 2.2. Tuples $(c_1, c_2, \dots, c_t) \in \mathbb{N}_0^t$ where $t \in \mathbb{N}$ are called *coefficient functions of length t* if $c_1 > 0$. If ϵ is a coefficient function of length t, we denote the kth entry by $\epsilon(k)$ (if $k \le t$), and its length t by $\operatorname{len}(\epsilon)$. For a coefficient function ϵ, let $\epsilon * Q$ denote $\sum_{k=1}^{t} \epsilon(k) Q_{t-k+1}$ where $t = \operatorname{len}(\epsilon)$, and let $\epsilon \cdot Q$ denote $\sum_{k=1}^{t} \epsilon(k) Q_k$.

If $\epsilon = (4,1,6,2)$ and Q is a sequence, then $\epsilon * Q = 4Q_4 + Q_3 + 6Q_2 + 2Q_1$, and $\epsilon \cdot Q = 4Q_1 + Q_2 + 6Q_3 + 2Q_4$.

3 Benford's Law for Zeckendorf expansions

Let a and b be two integers > 1 such that $\gcd(a,b) = 1$. The sequence K be the sequence given by $K_n = a^n$ is a standard example of sequences that satisfy Benford's Law under base-b expansion. We shall declare the behavior of the leading digits of the Zeckendorf expansion of a^n to be Benford's Law under Zeckendorf expansion.

Let us begin with formulating Zeckendorf's Theorem in terms of coefficient functions.

Definition 3.1. Let $\mathscr{F}$ be the set of coefficient functions ϵ such that $\epsilon(k) \le 1$ for all $k \le \operatorname{len}(\epsilon)$, and $\epsilon(k)\epsilon(k+1) = 0$ all $k \le \operatorname{len}(\epsilon) - 1$. Let F be the shifted Fibonacci sequence such that $F_{n+2} = F_{n+1} + F_n$ for all $n \in \mathbb{N}$ and $(F_1, F_2) = (1,2)$. Let ϕ be the golden ratio, let $\omega := \phi^{-1}$, and let $\widehat{F} = (1, \omega, \omega^2, \dots)$ be the sequence given by $\widehat{F}_n = \omega^{n-1}$.

Recall the product notation from Definition 2.2.

Theorem 3.2 ([19], Zeckendorf's Theorem)**.** *For each positive integer n, there is a unique coefficient function $\epsilon \in \mathscr{F}$ such that $n = \epsilon * F$.*

Recall the example $3^5 = F_{12} + F_5 + F_2$. If $\epsilon = (1,0,0,0,0,0,0,1,0,0,1,0)$, then $\epsilon \in \mathscr{F}$ and $3^5 = \epsilon * F$.

Definition 3.3. The expression $n = \epsilon * F$ where $n \in \mathbb{N}$ and $\epsilon \in \mathscr{F}$ is called *the $\mathscr{F}$-expansion of n* or *the Zeckendorf expansion of n.*

3.1 Benford's Law

If $\epsilon \in \mathscr{F}$ and $\text{len}(\epsilon) \geq 2$, then $(\epsilon(1), \epsilon(2)) = (1,0)$ is always the case, and hence, the probability of having $(\epsilon(1), \epsilon(2)) = (1,0)$ is 1. For the purpose of demonstration, we consider the first three entries of ϵ.

To denote arbitrarily many *leading blocks of coefficient functions*, which are defined in Definition 3.4 below, we shall use the boldface font and subscripts, e.g., $\mathbf{b}_1$ and $\mathbf{b}_2$, and in particular, $\mathbf{b}_k$ for $k = 1,2$ are not numbers, but tuples. The reader must not be confused with the entries of a sequence Q, e.g., Q_1 and Q_2, which are numbers, and we use the regular font for sequences.

Definition 3.4. A coefficient function of length s is also called *a leading block of length s* in the context of investigating the frequency of leading blocks, and it is denoted with boldface fonts, e.g. $\mathbf{b} = (1,0,0,1) \in \mathscr{F}$, $\mathbf{b}(3) = 0$, and $\mathbf{b}(4) = 1$. Let $\mathscr{F}_3 := \{\mathbf{b}_1, \mathbf{b}_2\}$ where $\mathbf{b}_1 = (1,0,0)$, $\mathbf{b}_2 = (1,0,1)$ are leading blocks of length 3, and the set is called *the set of leading blocks of length 3 under $\mathscr{F}$-expansion.* If $\mathbf{b} \in \mathscr{F}_3$ and $\mathbf{b} = \mathbf{b}_1$, then define $\widetilde{\mathbf{b}} := \mathbf{b}_2$, and and if $\mathbf{b} \in \mathscr{F}_3$ and $\mathbf{b} = \mathbf{b}_2$, then define $\widetilde{\mathbf{b}} := (1,1,0)$.

The block $\widetilde{\mathbf{b}} = (1,1,0)$ is not a member of $\mathscr{F}$, and hence, does not occur as the leading block of an $\mathscr{F}$-expansion, but it's convenient to use for Theorem 3.5, where we rely on the equality $\widetilde{\mathbf{b}} \cdot (1, \omega^1, \omega^2) = \phi$; see Definitions 2.2 and 3.1. The block $\widetilde{\mathbf{b}}$ makes the statements of Definition 3.6 below more aesthetic, and the principle of defining an exclusive block such as $(1,1,0)$ for other generalized Zeckendorf expansions will be explained in Definition 3.7 and Section 6.

The following is a special version of Corollary 4.7, and it is Theorem 1.3 written in terms of the dot product and blocks. Recall the notation LB_s from Definition 1.1, the set $\mathscr{F}_3$ from Definition 3.4, the sequence $\widehat{F}$ from Definition 3.1, and the dot product from Definition 2.2.

Theorem 3.5. *Let K be a sequence given by $K_n = a^n$ where $a > 1$ is an integer. Then, given $\mathbf{b} \in \mathscr{F}_3$,*

$$\text{Prob}\left\{\, n \in \mathbb{N} : \text{LB}_3(K_n) = \mathbf{b} \,\right\} \;=\; \log_\phi \frac{\widetilde{\mathbf{b}} \cdot \widehat{F}}{\mathbf{b} \cdot \widehat{F}}.$$

Motivated from the distribution of these standard sequences, we introduce the following definition.

Definition 3.6. A sequence K of positive integers is said to *satisfy $\mathscr{F}$-Benford's Law* or *satisfy Benford's Law under $\mathscr{F}$-expansion* if given $\mathbf{b} \in \mathscr{F}_3$,

$$\operatorname{Prob}\left\{ n \in \mathbb{N} : \operatorname{LB}_3(K_n) = \mathbf{b} \right\} \;=\; \log_\phi \frac{\widetilde{\mathbf{b}} \cdot \widehat{F}}{\mathbf{b} \cdot \widehat{F}}.$$

Let us demonstrate how the structure of the formulas in Definition 3.6 compares with the one for base-10 expansion. Consider the two leading blocks $\mathbf{c}_1 = (2,1,2)$ and $\mathbf{c}_2 = (2,1,3)$ for base-10 expansion. Let $b = 10$. Then, strong Benford's Law for decimal expansion requires the probability of having the leading block $\mathbf{c}_1$ to be $\log_{10} \frac{213}{212}$, which is equal to

$$\log_b \frac{\mathbf{c}_2 \cdot (1, b^{-1}, b^{-2})}{\mathbf{c}_1 \cdot (1, b^{-1}, b^{-2})} \;=\; \log_b \frac{b^2 \mathbf{c}_2 \cdot (1, b^{-1}, b^{-2})}{b^2 \mathbf{c}_1 \cdot (1, b^{-1}, b^{-2})} \;=\; \log_b \frac{\mathbf{c}_2 \cdot (b^2, b, 1)}{\mathbf{c}_1 \cdot (b^2, b, 1)} \;=\; \log_{10} \frac{213}{212}.$$

The first expression in terms of the negative powers of b is analogous to the ones in Definition 3.6.

3.2 Strong Benford's Law

Under base-b expansion, a sequence K is said to satisfy strong Benford's Law if the probability of the first M leading digits of K_n satisfies a certain logarithmic distribution, and exponential sequences $\{a^n\}_{n=1}^\infty$ where $a > 1$ is an integer are standard examples that satisfy strong Benford's Law under base-b expansion. In Corollary 4.7, we calculate the distribution of leading blocks of arbitrary length of the Zeckendorf expansions of exponential sequence $\{a^n\}_{n=1}^\infty$. We declare this distribution to be *strong Benford's Law under Zeckendorf expansion*. We state the formal definition below.

Recall the convolution $*$ from Definition 2.2.

Definition 3.7. Given an integer $s \geq 2$, let $\mathscr{F}_s := \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_\ell\}$ be the finite set of the leading blocks of length s occurring in the $\mathscr{F}$-expansions of the positive integers such that $1 + \mathbf{b}_k * F = \mathbf{b}_{k+1} * F$ for all $k \leq \ell - 1$. The leading block $\mathbf{b}_\ell$ is called *the largest leading block of length s under $\mathscr{F}$-expansion*.

If s is even, then let $\mathbf{b}_{\ell+1} := (1,0,1,0,\ldots,1,0,1,1)$, and if s is odd, then it is $\mathbf{b}_{\ell+1} := (1,0,1,0,\ldots,1,1,0)$. If $\mathbf{b} = \mathbf{b}_k \in \mathscr{F}_s$, then we denote $\mathbf{b}_{k+1}$ by $\widetilde{\mathbf{b}}$.

Notice that the existence of $\widetilde{\mathbf{b}}$ defined above is guaranteed by Zeckendorf's Theorem. Let us demonstrate examples of $\mathbf{b}$ and $\widetilde{\mathbf{b}}$. Let $\mathbf{b} = (1,0,0,0,1,0) \in \mathscr{F}_6$. Then, $\widetilde{\mathbf{b}} = (1,0,0,1,0,0) \in \mathscr{F}_6$, and $1 + \mathbf{b} * F = \widetilde{\mathbf{b}} * F$. If we list the coefficient functions in $\mathscr{F}_6$ with respect to the lexicographical order, then $\widetilde{\mathbf{b}}$ is the immediate successor of $\mathbf{b}$ if $\mathbf{b} \neq (1,0,1,0,1,0)$.

For each case of s being even or odd, the largest leading block $\mathbf{b}$ of length s satisfies $1 + \mathbf{b} * F = \widetilde{\mathbf{b}} * F$. If $\mathbf{b}' = (1,0,1,0,1,0)$, then $\widetilde{\mathbf{b}}' = (1,0,1,0,1,1)$, and below we shall demonstrate that the equality $\widetilde{\mathbf{b}}' \cdot \widehat{F} = \sum_{k=0}^{2} \omega^{2k} + \omega^5 = \phi$ makes the sum of the probabilities in Theorem 3.8 and Definition 3.9 be 1.

Let us compare this setup with the case of base-10 expansion. Let $\mathbf{c} = (4,5,6,7,8,9)$ be the leading block of length 6 for base-10 expansion, and let the sequence H given by $H_n = 10^{n-1}$ be the "base" sequence. Then, $1+\mathbf{c}*H = \widetilde{\mathbf{c}}*H$ where $\widetilde{\mathbf{c}} = (4,5,6,7,9,0)$. If we list all the coefficient functions of length 6, with respect to the lexicographical order, that are legal for base-10 expansion, then $\widetilde{\mathbf{c}}$ is the immediate successor of $\mathbf{c}$. If $\mathbf{c}' = (9,10,9,9,9,9)$, then we let $\widetilde{\mathbf{c}}' = (9,10,0,0,0,0)$, and $\sum_{n=1}^{6}\widetilde{\mathbf{c}}'(n)10^{n-1} = 1+\mathbf{c}'*H = 10^6$. If strong Benford's Law under base-10 expansion is satisfied, the probability of having the leading block $\mathbf{c}'$ under base-10 expansion is

$$\log_{10}\frac{\widetilde{\mathbf{c}}'*H}{\mathbf{c}'*H} \;=\; \log_{10}\frac{\widetilde{\mathbf{c}}'\cdot\widehat{H}}{\mathbf{c}'\cdot\widehat{H}} \;=\; 1-\log_{10}\mathbf{c}'\cdot\widehat{H}$$

where $\widehat{H}$ is the sequence given by $\widehat{H}_n = 10^{-(n-1)}$.

Recall the sequence $\widehat{F}$ from Definition 3.1.

Theorem 3.8. *Let K be a sequence of positive integers given by $K_n = ab^n(1+o(1))$ where a and b are positive real numbers such that $\log_\phi b$ is irrational. Then, given $\mathbf{b}\in\mathscr{F}_s$ where $s\ge 2$,*

$$\operatorname{Prob}\{\,n\in\mathbb{N} : \mathrm{LB}_s(K_n)=\mathbf{b}\,\} \;=\; \log_\phi\frac{\widetilde{\mathbf{b}}\cdot\widehat{F}}{\mathbf{b}\cdot\widehat{F}}.$$

Proof. It follows immediately from Corollary 4.7. □

Let us demonstrate below that the probabilities add up to 1 for $s=6$, but the argument is sufficiently general to be extended for all cases of s. Let $\mathscr{F}_6 = \{\mathbf{b}_1,\dots,\mathbf{b}_\ell\}$ such that $\mathbf{b}_{k+1} = \widetilde{\mathbf{b}}_k$ for all $1\le k\le \ell$. Then, $\mathbf{b}_1 = (1,0,0,0,0,0)$ and $\mathbf{b}_\ell = (1,0,1,0,1,0)$. Then, $\mathbf{b}_{\ell+1} = (1,1,0,0,0,0)$, and

$$\sum_{k=1}^{\ell}\log_\phi\frac{\widetilde{\mathbf{b}}_k\cdot\widehat{F}}{\mathbf{b}_k\cdot\widehat{F}} \;=\; \sum_{k=1}^{\ell}\log_\phi(\mathbf{b}_{k+1}\cdot\widehat{F})-\log_\phi(\mathbf{b}_k\cdot\widehat{F}) \;=\; \log_\phi(\mathbf{b}_{\ell+1}\cdot\widehat{F})-\log_\phi 1 \;=\; 1.$$

Definition 3.9. Let K be a sequence of positive integers approaching ∞. Then, K is said to *satisfy strong Benford's Law under $\mathscr{F}$-expansion* if given $\mathbf{b}\in\mathscr{F}_s$ where $s\ge 2$,

$$\operatorname{Prob}\{\,n\in\mathbb{N} : \mathrm{LB}_s(K_n)=\mathbf{b}\,\} \;=\; \log_\phi\frac{\widetilde{\mathbf{b}}\cdot\widehat{F}}{\mathbf{b}\cdot\widehat{F}}.$$

Example 3.10. Let K be a sequence satisfying strong Benford's Law under $\mathscr{F}$-expansion, e.g., $\{2^n\}_{n=1}^{\infty}$; see Theorem 3.8. Let $\mathbf{b} = (1,0,0,0,1,0)$, so $\widetilde{\mathbf{b}} = (1,0,0,1,0,0)$. Then,

$$\operatorname{Prob}\{\,n\in\mathbb{N} : \mathrm{LB}_6(K_n)=\mathbf{b}\,\} \;=\; \log_\phi\frac{1+\omega^3}{1+\omega^4}\approx 0.157.$$

4 Calculations

Notice that $\log_b(x)$ makes it convenient to calculate the distribution of the leading digits of exponential sequences $\{a^n\}_{n=1}^{\infty}$ under base-b expansion where $b>1$ is an integer. In this section, we introduce an analogue of $\log_b(x)$ for Zeckendorf expansion in Section 4.1, and use it for various calculations.

As mentioned in the introduction, these functions are merely a tool for calculating the leading digits, and in Section 5, we consider other continuations, and demonstrate their connections to different distributions of leading digits.

4.1 An analytic continuation of the Fibonacci sequence

Below we introduce an analytic continuation of the Fibonacci sequence.

Definition 4.1. Let $\alpha = \frac{\phi}{\sqrt{5}}$, and define $\mathfrak{F} : \mathbb{R} \to \mathbb{R}$ be the function given by

$$\mathfrak{F}(x) = \alpha(\phi^x + \phi^{-x}\cos(\pi x)\phi^{-2}).$$

We call the function *a Benford continuation of the Fibonacci sequence*.

Notice that $F_n = \frac{1}{\sqrt{5}}(\phi^{n+1} - (-1/\phi)^{n+1}) = \frac{\phi}{\sqrt{5}}(\phi^n + (-1)^n\phi^{-(n+2)})$. Thus, $\mathfrak{F}$ is a real analytic continuation of F_n, so $\mathfrak{F}(n) = F_n$ for all $n \in \mathbb{N}$. It is an increasing function on $[1,\infty)$. Let $\mathfrak{F}^{-1}$ denote the inverse function of $\mathfrak{F} : [1,\infty) \to \mathbb{R}$. Comparing it with the case of base-10 expansion, we find that 10^{x-1} is an analytic continuation of the sequence $\{10^{n-1}\}_{n=1}^{\infty}$, and its inverse is $1 + \log_{10}(x)$, which is the main object for the equidistribution for Benford's Law under base-10 expansion. The equidistribution property described in Theorem 4.5 is associated with strong Benford's Law under $\mathscr{F}$-expansion, and the name of the function is due to this connection.

Lemma 4.2. *For real numbers $x \geq 1$, we have $\mathfrak{F}(x) = \alpha\phi^x + O(\phi^{-x})$, and*

$$\mathfrak{F}^{-1}(x) \ = \ \log_\phi(x) - \log_\phi(\alpha) + O(1/x^2).$$

Proof. Let $y = \alpha\phi^x + \alpha\phi^{-x}\cos(\pi x)\phi^{-2}$ and let $w = \alpha\phi^{-x}\cos(\pi x)\phi^{-2} = O(\phi^{-x})$. Since $y = \alpha\phi^x + o(1)$, we have $w = O(1/y)$. Then, $y = \alpha\phi^x + w$ implies

$$\begin{aligned} x \ &= \ \log_\phi(y-w) - \log_\phi \alpha \ = \ \log_\phi(y) - \log_\phi \alpha + \log_\phi(1 - w/y) \\ &= \ \log_\phi(y) - \log_\phi \alpha + O(|w|/y) \ = \ \log_\phi(y) - \log_\phi \alpha + O(1/y^2). \end{aligned}$$

□

4.2 Equidistribution

Recall the set $\mathscr{F}_s$ of leading blocks from Definition 3.7. In this section, having a leading block $\mathbf{b} \in \mathscr{F}_s$ is interpreted in terms of the fractional part of the values of $\mathfrak{F}^{-1}$.

Definition 4.3. Given $\epsilon \in \mathbb{N}_0^t$ and an integer $s \leq t$, let $\epsilon|s := (\epsilon(1), \ldots, \epsilon(s))$.

Recall $\widehat{F}$ from Definition 3.1 and the product notation from Definition 2.2.

Lemma 4.4. *Let K be a sequence of positive real numbers approaching ∞, and let s be an integer ≥ 2. Let $\mathbf{b} \in \mathscr{F}_s$, and let $A_{\mathbf{b}} := \{n \in \mathbb{N} : \mathrm{LB}_s(K_n) = \mathbf{b}\}$. Then, there are real numbers $\gamma_n = o(1)$ and $\widetilde{\gamma}_n = o(1)$ such that $n \in A_{\mathbf{b}}$ if and only if*

$$\log_\phi \mathbf{b} \cdot \widehat{F} + \gamma_n \;\leq\; \mathrm{frc}\big(\mathfrak{F}^{-1}(K_n)\big) \;<\; \log_\phi \widetilde{\mathbf{b}} \cdot \widehat{F} + \widetilde{\gamma}_n \tag{3}$$

where $\widetilde{\gamma}_n = 0$ if $\mathbf{b}$ is the largest block of length s.

Proof. Suppose that $n \in \mathbb{N}$ is sufficiently large, so that $\mathbf{b}' := \mathrm{LB}_s(K_n)$ exists. By Zeckendorf's Theorem, there is $\mu \in \mathscr{F}$ such that $K_n = \mu * F$, so $m := \mathrm{len}(\mu) \geq s$, and $\mathbf{b}' = \mu|s$. There are $\epsilon \in \mathscr{F}$ of length m and a coefficient function $\check{\epsilon}$ of length m such that $\epsilon|s = \mathbf{b}'$, $\check{\epsilon}|s = \widetilde{\mathbf{b}}'$, $\epsilon(k) = \check{\epsilon}(k) = 0$ for all $k > s$, so $\epsilon * F \leq K_n < \check{\epsilon} * F$. Recall α from Definition 4.1. Then,

$$\epsilon * F \;=\; \alpha \sum_{k=1}^{s} \epsilon(k)\phi^{m-k+1} + O(1) \;=\; \alpha\phi^m(1+o(1)) \sum_{k=1}^{s} \epsilon(k)\omega^{k-1} \;=\; \alpha\phi^m(1+o(1))\mathbf{b}' \cdot \widehat{F}.$$

By Lemma 4.2,

$$\mathfrak{F}^{-1}(\epsilon * F) \;=\; m + \log_\phi(\mathbf{b}' \cdot \widehat{F}) + \gamma_n, \quad \gamma_n \;=\; o(1).$$

Similarly, we have $\mathfrak{F}^{-1}(\check{\epsilon} * F) = m + \log_\phi(\widetilde{\mathbf{b}}' \cdot \widehat{F}) + \widetilde{\gamma}_n$ where $\widetilde{\gamma}_n = o(1)$. If $\mathbf{b}'$ is the largest block of length s, then $\check{\epsilon} * F = F_{m+1}$, and hence, $\mathfrak{F}^{-1}(\check{\epsilon} * F) = m+1$, which implies $\widetilde{\gamma}_n = 0$. In general, $\check{\epsilon} * F \leq F_{m+1}$, so $\mathfrak{F}^{-1}(\check{\epsilon} * F) \leq m+1$.

Thus, if $n \in A_{\mathbf{b}}$, then $\mathbf{b}' = \mathbf{b}$, and

$$\begin{aligned} \epsilon * F \leq K_n \;<\; \check{\epsilon} * F &\Rightarrow \mathfrak{F}^{-1}(\epsilon * F) \leq \mathfrak{F}^{-1}(K_n) \;<\; \mathfrak{F}^{-1}(\check{\epsilon} * F) \\ &\Rightarrow \log_\phi \mathbf{b} \cdot \widehat{F} + \gamma_n \;\leq\; \mathrm{frc}\big(\mathfrak{F}^{-1}(K_n)\big) \;<\; \log_\phi \widetilde{\mathbf{b}} \cdot \widehat{F} + \widetilde{\gamma}_n. \end{aligned}$$

There is no difficulty in reversing this argument, and we leave the proof of the converse to the reader. □

Theorem 4.5. *Let K be an increasing sequence of positive integers such that $\mathrm{frc}\big(\mathfrak{F}^{-1}(K_n)\big)$ is equidistributed. Then, K satisfies strong Benford's Law under the $\mathscr{F}$-expansion.*

Proof. Notice that $\mathrm{Prob}\{\, n \in \mathbb{N} : \mathrm{LB}_s(K_n) = \mathbf{b} \,\}$ where $s \geq 2$ is equal to the probability of n satisfying (3). Let $t \in \mathbb{N}$. Then, there is an integer M_t such that $|\gamma_n|$ and $|\widetilde{\gamma}_n|$ are $< 1/t$ for all $n \geq M_t$. Thus, by Lemma 4.4,

$$\begin{aligned} &\mathrm{Prob}\{\, k \in \Omega_n : \mathrm{LB}_s(K_n) = B \,\} + o(1) \\ &\quad\leq\; \mathrm{Prob}\Big\{ k \in \Omega_n : \log_\phi \mathbf{b} \cdot \widehat{F} - \tfrac{1}{t} \;\leq\; \mathrm{frc}\big(\mathfrak{F}^{-1}(K_n)\big) \;<\; \log_\phi \widetilde{\mathbf{b}} \cdot \widehat{F} + \tfrac{1}{t} \Big\} + o(1) \\ &\qquad\Rightarrow\; \limsup_n \mathrm{Prob}\{\, k \in \Omega_n : \mathrm{LB}_s(K_n) = \mathbf{b} \,\} \;\leq\; \log_\phi \frac{\widetilde{\mathbf{b}} \cdot \widehat{F}}{\mathbf{b} \cdot \widehat{F}} + \frac{2}{t}. \\ &\mathrm{Prob}\{\, k \in \Omega_n : \mathrm{LB}_s(K_n) = \mathbf{b} \,\} + o(1) \\ &\quad\geq\; \mathrm{Prob}\Big\{\, k \in \Omega_n : \log_\phi \mathbf{b} \cdot \widehat{F} + \tfrac{1}{t} \leq \mathrm{frc}\big(\mathfrak{F}^{-1}(K_n)\big) \;<\; \log_\phi \widetilde{\mathbf{b}} \cdot \widehat{F} - \tfrac{1}{t} \,\Big\} + o(1) \\ &\qquad\Rightarrow\; \liminf_n \mathrm{Prob}\{\, k \in \Omega_n : \mathrm{LB}_s(K_n) = \mathbf{b} \,\} \;\geq\; \log_\phi \frac{\widetilde{\mathbf{b}} \cdot \widehat{F}}{\mathbf{b} \cdot \widehat{F}} - \frac{2}{t}. \end{aligned}$$

Since liminf and limsup are independent of t, we prove that $\operatorname{Prob}\{\, n \in \mathbb{N} : \mathrm{LB}_s(K_n) = \mathbf{b} \,\} = \log_\phi \frac{\widetilde{\mathbf{b}}\cdot\widehat{F}}{\mathbf{b}\cdot\widehat{F}}$.

□

The converse of Theorem 4.5 is true as well, i.e., if K satisfies strong Benford's Law under $\mathscr{F}$-expansion, then $\operatorname{frc}\big(\mathfrak{F}^{-1}(K_n)\big)$ is equidistributed. We shall prove it in Section 5.

The following lemma is useful, and it is probably known.

Lemma 4.6. *Let $h : \mathbb{N} \to \mathbb{R}$ be a function such that $\operatorname{frc}(h(n))$ is equidistributed, and let $E : \mathbb{N} \to \mathbb{R}$ be a function such that $E(n) \to 0$ as $n \to \infty$. Then, $\operatorname{frc}(h(n) + E(n))$ is equidistributed.*

Corollary 4.7. *Let K be a sequence of positive integers given by $K_n = ab^n(1 + o(1))$ where a and b are positive real numbers such that $\log_\phi b$ is irrational. Then, $\operatorname{frc}\big(\mathfrak{F}^{-1}(K_n)\big)$ is equidistributed, and hence, given $\mathbf{b} \in \mathscr{F}_s$ where $s \ge 2$,*

$$\operatorname{Prob}\{\, n \in \mathbb{N} : \mathrm{LB}_s(K_n) = \mathbf{b} \,\} \;=\; \log_\phi \frac{\widetilde{\mathbf{b}}\cdot\widehat{F}}{\mathbf{b}\cdot\widehat{F}}.$$

Proof. By Lemma 4.2,

$$\mathfrak{F}^{-1}(K_n) \;=\; n\log_\phi b - \log_\phi(a/\alpha) + \log_\phi(1 + o(1)) + o(1).$$

Since $\log_\phi b$ is irrational, by Weyl's Equidistribution Theorem, $\operatorname{frc}\Big(n\log_\phi b\Big)$ is equidistributed, and by the lemma, $\operatorname{frc}\Big(n\log_\phi b + o(1)\Big)$ is equidistributed. Shifting it by a constant $-\log_\phi(a/\alpha)$ does not change the equidistribution property, and this concludes the proof. □

For example, if K is a sequence given by $K_n = \sum_{k=1}^{N} a_k b_k^n$ where $a_k, b_k \in \mathbb{Z}$, $a_1 > 0$, and $b_1 > |b_k|$ for all $k \ge 2$, then $K_n = a_1 b_1^n(1 + o(1))$, and $\operatorname{frc}\big(\mathfrak{F}^{-1}(K_n)\big)$ is equidistributed. Many increasing sequences K of positive integers given by a linear recurrence with constant positive integer coefficients satisfy $K_n = ab^n(1 + o(1))$ where $\log_\phi(b)$ is irrational, and hence, $\operatorname{frc}\big(\mathfrak{F}^{-1}(K_n)\big)$ is equidistributed.

4.3 The leading blocks of integer powers

Let a be a positive integer, and let K be the sequence given by $K_n = n^a$. Then, K does not satisfy Benford's Law under the base-10 expansion, but it has a close relationship with Benford's Law [14]. In this section, we show that both statements are true under $\mathscr{F}$-expansion as well. Recall Ω_n from Notation 2.1 and $\mathscr{F}_3$ from Definition 3.4, and let $\mathbf{b}_1 := (1,0,0) \in \mathscr{F}_3$. We also introduce the oscillating behavior of $\operatorname{Prob}\{\, k \in \Omega_n : \mathrm{LB}_3(K_k) = \mathbf{b}_1 \,\}$ as $n \to \infty$, and hence, $\operatorname{Prob}\{\, n \in \mathbb{N} : \mathrm{LB}_3(K_n) = \mathbf{b}_1 \,\}$ does not exist.

Example 4.8. Let K be the sequence given by $K_n = n$, and let $t > 0$ be a large integer. Given a sufficiently large positive random integer $n < F_{t+1}$, let $n = \mu * F$ be the $\mathscr{F}$-expansion, and

$M := \operatorname{len}(\mu)$. Notice that $\mathrm{LB}_3(n) = \mathbf{b}_1$ if and only if $n = F_M + m$ where $0 \le m < F_{M-2}$. Thus, there are F_{M-2} integers n in $[1, F_{t+1})$ such that $F_M \le n < F_{M+1}$ and $\mathrm{LB}_3(n) = \mathbf{b}_1$. Thus,

$$\begin{aligned}\operatorname{Prob}\left\{ n \in \Omega_{F_{t+1}} : \mathrm{LB}_3(n) = \mathbf{b}_1 \right\} &= \left(\frac{1}{F_{t+1}} \sum_{M=3}^{t} F_{M-2} \right) + o(1) = \left(\frac{1}{F_{t+1}} \sum_{M=3}^{t} \alpha\phi^{M-2} + o(1) \right) + o(1) \\ &= \frac{1}{\alpha\phi^{t+1} + o(1)} \frac{\alpha\phi^{t-1}}{\phi - 1} + o(1) = \frac{1}{\phi^2(\phi-1)} + o(1) = \phi - 1 + o(1)\end{aligned}$$

as function of t. However, by Theorem 4.10, we have

$$\begin{aligned}\limsup_n \operatorname{Prob}\left\{ k \in \Omega_n : \mathrm{LB}_3(k) = \mathbf{b}_1 \right\} &= \frac{\phi+1}{\phi+2} \approx .724, \\ \liminf_n \operatorname{Prob}\left\{ k \in \Omega_n : \mathrm{LB}_3(k) = \mathbf{b}_1 \right\} &= \phi - 1 \approx .618.\end{aligned}$$

Thus, $\operatorname{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_3(n) = \mathbf{b}_1 \right\}$ does not exist.

Recall $\mathfrak{F}$ from Definition 4.1, and its inverse $\mathfrak{F}^{-1}$. We use the function $\mathfrak{F}$ to more generally handle the distribution of the leading blocks of $\{n^a\}_{n=1}^{\infty}$ with any length. Given a positive integer m, let $A_m = \{n \in \mathbb{N} : n < F_m^{1/a}\}$.

Lemma 4.9. *If* $\beta \in [0,1]$, *then*

$$\operatorname{Prob}\left\{ n \in A_m : \operatorname{frc}\left(\mathfrak{F}^{-1}(n^a)\right) \le \beta \right\} = \frac{\phi^{\beta/a} - 1}{\phi^{1/a} - 1} + O(m\phi^{-m/a}).$$

Proof. Let $m \in \mathbb{N}$, and let $n \in A'_{m+1} := A_{m+1} \setminus A_m$, so that $F_m \le n^a < F_{m+1}$ and $m \le \mathfrak{F}^{-1}(n^a) < m+1$. Thus, given a real number $\beta \in [0,1]$,

$$\begin{aligned}\left\{n \in A'_{m+1} : \operatorname{frc}\left(\mathfrak{F}^{-1}(n^a)\right) \le \beta\right\} &= \left\{n \in A'_{m+1} : m \le \mathfrak{F}^{-1}(n^a) \le m + \beta\right\} \\ &= \left\{n \in A'_{m+1} : \mathfrak{F}(m)^{1/a} \le n \le \mathfrak{F}(m+\beta)^{1/a}\right\} \\ \Rightarrow \#\left\{n \in A'_{m+1} : \operatorname{frc}\left(\mathfrak{F}^{-1}(n^a)\right) \le \beta\right\} &= \mathfrak{F}(m+\beta)^{1/a} - \mathfrak{F}(m)^{1/a} + O(1) \\ &= \alpha^{1/a}\phi^{(m+\beta)/a} - \alpha^{1/a}\phi^{m/a} + O(1) \\ \Rightarrow \#\left\{n \in A_{m+1} : \operatorname{frc}\left(\mathfrak{F}^{-1}(n^a)\right) \le \beta\right\} &= \sum_{k=1}^{m} \alpha^{1/a}\phi^{(k+\beta)/a} - \alpha^{1/a}\phi^{k/a} + O(1) \\ &= \alpha^{1/a}\phi^{(m+\beta)/a}\gamma - \alpha^{1/a}\phi^{m/a}\gamma + O(m), \quad \gamma = \frac{\phi^{1/a}}{\phi^{1/a} - 1}.\end{aligned}$$

This proves that

$$\begin{aligned}\operatorname{Prob}\left\{ n \in A_{m+1} : \operatorname{frc}\left(\mathfrak{F}^{-1}(n^a)\right) \le \beta \right\} &= \frac{\alpha^{1/a}\phi^{(m+\beta)/a}\gamma - \alpha^{1/a}\phi^{m/a}\gamma + O(m)}{F_{m+1}^{1/a} + O(1)} \\ &= \frac{\phi^{\beta/a}\gamma - \gamma + O(m\phi^{-m/a})}{\phi^{1/a} + O(\phi^{-m/a})} = \frac{\phi^{\beta/a} - 1}{\phi^{1/a} - 1} + O(m\phi^{-m/a}).\end{aligned}$$

□

Recall from Lemma 4.4 that

$$\operatorname{Prob}\left\{\, n\in A_m : \mathrm{LB}_3(n^a)=\mathbf{b}_1 \,\right\} = \operatorname{Prob}\left\{\, n\in A_m : \operatorname{frc}\left(\mathfrak{F}^{-1}(n^a)\right) \le \delta_1 + o(1) \,\right\}$$

where $\delta_1 := \log_\phi \frac{\widetilde{\mathbf{b}}_1\cdot\widehat{F}}{\mathbf{b}_1\cdot\widehat{F}}$. Thus, as $m\to\infty$, by Lemma 4.9,

$$\operatorname{Prob}\left\{\, n\in A_m : \mathrm{LB}_3(n^a)=\mathbf{b}_1 \,\right\} \;\to\; \frac{\phi^{\delta_1/a}-1}{\phi^{1/a}-1} \;=\; \frac{(1+\omega^2)^{1/a}-1}{\phi^{1/a}-1}$$

where $\omega=\phi^{-1}$. Let us show that

$$\operatorname{Prob}\left\{\, n\in A_m : \mathrm{LB}_3(n^a)=\mathbf{b}_1 \,\right\} \;\not\to\; \delta_1$$

as $m\to\infty$. We claim that the ratio $\frac{(1+\omega^2)^{1/a}-1}{\phi^{1/a}-1}$ is not equal to $\delta_1=\log_\phi(1+\omega^2)$. Since $a\in\mathbb{N}$, the ratio is an algebraic number over $\mathbb{Q}$. However, by the Gelfand-Schneider Theorem, $\log_\phi(1+\omega^2)$ is a transcendental number. Thus, K does not satisfy Benford's Law under the $\mathscr{F}$-expansion.

However, as noted in [14] for base-b expansions, we have

$$\lim_{a\to\infty}\lim_{m\to\infty}\operatorname{Prob}\left\{\, n\in A_m : \mathrm{LB}_3(n^a)=\mathbf{b}_1 \,\right\} \;=\; \lim_{a\to\infty}\frac{\phi^{\delta_1/a}-1}{\phi^{1/a}-1} \;=\; \delta_1 \;=\; \log_\phi(1+\omega^2).$$

Even though the leading blocks of K_n do not satisfy Benford's Law under $\mathscr{F}$-expansion, the limiting behavior of high power sequences for special values of n resembles Benford's Law.

Recall Ω_n from Definition 2.1. Let us use Lemma 4.9 to prove that $\operatorname{Prob}\left\{\, k\in\Omega_n : \operatorname{frc}\left(\mathfrak{F}^{-1}(K_k)\right)\le\beta \,\right\}$ oscillates, and does not converge.

Theorem 4.10. *Let β be a real number in $[0,1]$, and let $r := (\phi^{\beta/a}-1)/(\phi^{1/a}-1)$. Given an integer $n>1$, let $\mathfrak{F}^{-1}(n^a)=m+p$ where $p=\operatorname{frc}\left(\mathfrak{F}^{-1}(n^a)\right)$ and $m\in\mathbb{N}$. Then,*

$$P_n := \operatorname{Prob}\left\{\, k\in\Omega_n : \operatorname{frc}\left(\mathfrak{F}^{-1}(K_k)\right)\le\beta \,\right\} \;=\; \begin{cases} \frac{r+\phi^{p/a}-1}{\phi^{p/a}} + O(m\phi^{-m/a}) & \text{if } 0\le p\le\beta \\ \frac{r+\phi^{\beta/a}-1}{\phi^{p/a}} + O(m\phi^{-m/a}) & \text{if } \beta<p<1 \end{cases}.$$

In particular,

$$\limsup P_n = r\phi^{1/a-\beta/a} = \beta + O(1/a), \quad \text{and} \quad \liminf P_n = r = \beta + O(1/a).$$

Proof. Let m be a sufficiently large positive integer, and let $n\in A_{m+1}\setminus A_m$. Let $n=\mathfrak{F}(m+p)^{1/a}$ for $p\in[0,1)$. If $p\le\beta$, then, $\operatorname{frc}\left(\mathfrak{F}^{-1}(n^a)\right)=\operatorname{frc}\left(\mathfrak{F}^{-1}\mathfrak{F}(m+p)\right)=\operatorname{frc}(m+p)=p\le\beta$, and if $p>\beta$, then, $\operatorname{frc}\left(\mathfrak{F}^{-1}(n^a)\right)=p>\beta$. Thus,

$$\left\{n\in A_{m+1}\setminus A_m : \operatorname{frc}\left(\mathfrak{F}^{-1}(n^a)\right)\le\beta\right\} \;=\; \left\{n\in A_{m+1}\setminus A_m : n\le\mathfrak{F}(m+\beta)^{1/a}\right\}.$$

If $n \le \mathfrak{F}(m+\beta)^{1/a}$, i.e., $p \le \beta$, then by Lemma 4.9

$$\begin{aligned}
P_n &= \frac{1}{n}\Big(\text{Prob}\{\, k \in A_m : \text{frc}\big(\mathfrak{F}^{-1}(k^a)\big) \le \beta \,\}\ \#A_m + n - \mathfrak{F}(m)^{1/a} + O(1)\Big) \\
&= \frac{r\mathfrak{F}(m)^{1/a} + O(m) + \mathfrak{F}(m+p)^{1/a} - \mathfrak{F}(m)^{1/a}}{\mathfrak{F}(m+p)^{1/a} + O(1)} \\
&= \frac{r + O(m\phi^{-m/a}) + \phi^{p/a} - 1}{\phi^{p/a} + O(\phi^{-m/a})} \ = \ \frac{r + \phi^{p/a} - 1}{\phi^{p/a}} + O(m\phi^{-m/a})
\end{aligned}$$

If $n > \mathfrak{F}(m+\beta)^{1/a}$, i.e., $p > \beta$, then

$$P_n = \frac{r + \phi^{\beta/a} - 1}{\phi^{p/a}} + O(m\phi^{-m/a}) \ = \ \frac{r\phi^{1/a}}{\phi^{p/a}} + O(m\phi^{-m/a}).$$

$$\text{Thus, } \limsup P_n = \frac{r + \phi^{\beta/a} - 1}{\phi^{\beta/a}} = \frac{r\phi^{1/a}}{\phi^{\beta/a}}, \quad \liminf P_n = \frac{r\phi^{1/a}}{\phi^{1/a}} = r.$$

□

Thus, $\text{Prob}\{\, n \in \mathbb{N} : \text{frc}\big(\mathfrak{F}^{-1}(K_n)\big) \le \beta \,\}$ does not converge, but $\text{frc}\big(\mathfrak{F}^{-1}(K_n)\big)$ is almost equidistributed for large values of a.

Example 4.11. Let $\mathbf{b}$ and $\widetilde{\mathbf{b}}$ be the blocks defined in Example 3.10, and let K be the sequence given by $K_n = n^2$. By Lemma 4.4, if $D := \{n \in \mathbb{N} : \text{LB}_6(K_n) = \mathbf{b}\}$, then for $n \in D$,

$$\log_\phi(1+\omega^4) + o(1) \ < \ \text{frc}\big(\mathfrak{F}^{-1}(K_n)\big) \ < \ \log_\phi(1+\omega^3) + o(1)$$

where the upper and lower bounds are functions of $n \in D$. Let $\beta = \log_\phi(1+\omega^4)$ and $\widetilde{\beta} = \log_\phi(1+\omega^3)$. Recall Ω_n from Definition 2.1. Then,

$$\begin{aligned}
&\text{Prob}\{\, k \in \Omega_n : \text{LB}_6(K_k) = \mathbf{b} \,\} = \\
&\qquad \text{Prob}\{\, k \in \Omega_n : \text{frc}\big(\mathfrak{F}^{-1}(K_n)\big) < \widetilde{\beta} \,\} \ - \ \text{Prob}\{\, k \in \Omega_n : \text{frc}\big(\mathfrak{F}^{-1}(K_n)\big) < \beta \,\} \ + \ o(1).
\end{aligned}$$

Let $r = (\phi^{\beta/2} - 1)/(\phi^{1/2} - 1)$ and $\widetilde{r} = (\phi^{\widetilde{\beta}/2} - 1)/(\phi^{1/2} - 1)$, and let $n = \mathfrak{F}(m+p)^{1/a}$ where $p = \text{frc}\big(\mathfrak{F}^{-1}(n^a)\big) \in [0,1)$. Then, by Theorem 4.10, we have

$$\text{Prob}\{\, k \in \Omega_n : \text{LB}_6(K_k) = \mathbf{b} \,\} \ = \ \begin{cases} \frac{\widetilde{r}+\phi^{p/2}-1}{\phi^{p/2}} - \frac{r+\phi^{p/2}-1}{\phi^{p/2}} + o(1) & \text{if } p \le \beta\ , \\ \frac{\widetilde{r}+\phi^{p/2}-1}{\phi^{p/2}} - \frac{r+\phi^{\beta/2}-1}{\phi^{p/2}} + o(1) & \text{if } \beta < p \le \widetilde{\beta} \\ \frac{\widetilde{r}+\phi^{\widetilde{\beta}/2}-1}{\phi^{p/2}} - \frac{r+\phi^{\beta/2}-1}{\phi^{p/2}} + o(1) & \text{if } p > \widetilde{\beta}\ . \end{cases}$$

$$\begin{aligned}
\Rightarrow \limsup_n \text{Prob}\{\, k \in \Omega_n : \text{LB}_6(K_k) = \mathbf{b} \,\} \ &= \ \frac{\widetilde{r} + \phi^{\widetilde{\beta}/2} - 1}{\phi^{\widetilde{\beta}/2}} - \frac{r + \phi^{\beta/2} - 1}{\phi^{\widetilde{\beta}/2}} \approx 0.1737 \\
\liminf_n \text{Prob}\{\, k \in \Omega_n : \text{LB}_6(K_k) = \mathbf{b} \,\} \ &= \ \frac{\widetilde{r} + \phi^{\beta/2} - 1}{\phi^{\beta/2}} - \frac{r + \phi^{\beta/2} - 1}{\phi^{\beta/2}} \approx 0.1419.
\end{aligned}$$

5 Other continuations

Reflecting upon Lemma 4.4 and Theorem 4.5, we realized that we could consider different continuations of the Fibonacci sequence F, and ask which sequence satisfies the equidistribution property, and which distributions its leading blocks follow. Let us demonstrate the idea in Example 5.4. The claims in this example can be proved using Theorem 5.6. Recall the Benford continuation $\mathfrak{F}$ from Definition 4.1.

Definition 5.1. Given $n \in \mathbb{N}$, let $\mathfrak{F}_n : [0,1] \to [0,1]$ be the increasing function given by

$$\mathfrak{F}_n(p) := \frac{\mathfrak{F}(n+p) - \mathfrak{F}(n)}{\mathfrak{F}(n+1) - \mathfrak{F}(n)} = \frac{\mathfrak{F}(n+p) - \mathfrak{F}(n)}{F_{n-1}} = \phi(\phi^p - 1) + o(1)$$

where $F_0 := 1$. Let $\mathfrak{F}_\infty : [0,1] \to [0,1]$ be the increasing function given by $\mathfrak{F}_\infty(p) = \phi(\phi^p - 1)$.

Recall uniform continuations of sequences from Definition 1.6.

Lemma 5.2. *The function $\mathfrak{F}$ is a uniform continuation of F.*

Proof. Notice that $\mathfrak{F}_n(p) = \phi(\phi^p - 1) + \gamma(n,p)$ where $\left|\gamma(n,p)\right| < C/\phi^n$ where C is independent of p and n. Thus, it uniformly converges to $\phi(\phi^p - 1)$. □

Lemma 5.3. *Let $p \in [0,1]$ be a real number. Then, $\mathfrak{F}(n + \mathfrak{F}_n{}^{-1}(p)) = F_n + (F_{n+1} - F_n)p$.*

Proof. Let $p' = \mathfrak{F}_n{}^{-1}(p)$. Then, $\mathfrak{F}_n(p') = p$, and hence, $\frac{\mathfrak{F}(n+p') - \mathfrak{F}(n)}{F_{n+1} - F_n} = p$. The assertion follows from the last equality. □

Example 5.4. Let $f : [1,\infty) \to \mathbb{R}$ be the increasing continuous function whose graph is the union of the line segments from (n, F_n) to $(n+1, F_{n+1})$ for $n \in \mathbb{N}$. Then, $f_\infty(p) = p$ for all $p \in [0,1]$. Let K be the sequence given by $K_n = \left\lfloor \mathfrak{F}(n + \mathfrak{F}_n{}^{-1}(\operatorname{frc}(n\pi))) \right\rfloor$. Then, by Lemma 5.3,

$$f^{-1}\left(\mathfrak{F}(n + \mathfrak{F}_n{}^{-1}(\operatorname{frc}(n\pi))\right) = n + \operatorname{frc}(n\pi) \;\Rightarrow\; \operatorname{frc}\left(f^{-1}(K_n)\right) = \operatorname{frc}(n\pi) + o(1),$$

which is equidistributed.

Recall $\mathscr{F}_s$ from Definition 3.7 where $s \geq 2$, and let $\mathbf{b} \in \mathscr{F}_s$. Recall $\widehat{F}$ from Definition 3.1 and the product notation from Definition 2.2. Then, by Theorem 5.6,

$$\operatorname{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_s(K_n) = \mathbf{b} \right\} = \phi(\widetilde{\mathbf{b}} \cdot \widehat{F} - \mathbf{b} \cdot \widehat{F}) = \phi^{-s+2}(\widetilde{\mathbf{b}} * \overline{F} - \mathbf{b} * \overline{F})$$

where $\overline{F}$ is the sequence given by $\overline{F}_n = \phi^{n-1}$. If $\mathbf{b}(s) = 0$, then $\omega^{s-2}(\widetilde{\mathbf{b}} * \overline{F} - \mathbf{b} * \overline{F}) = \omega^{s-2}$, and if $\mathbf{b}(s) = 1$, then $\omega^{s-2}(\widetilde{\mathbf{b}} * \overline{F} - \mathbf{b} * \overline{F}) = \omega^{s-1}$. For example, if $s = 6$, then

$$\operatorname{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_6(K_n) = (1,0,0,1,0,1) \right\} = \omega^5$$
$$\operatorname{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_6(K_n) = (1,0,1,0,1,0) \right\} = \omega^4.$$

It's nearly a uniform distribution.

Let us show that the probabilities add up to 1. Notice that $\#\mathscr{F}_s = F_{s-1}$, $\#\{\mathbf{b} \in \mathscr{F}_s : \mathbf{b}(s) = 0\} = F_{s-2}$, and $\#\{\mathbf{b} \in \mathscr{F}_s : \mathbf{b}(s) = 1\} = F_{s-3}$. Then, by Binet's Formula, the following sum is equal to 1:

$$\sum_{\mathbf{b} \in \mathscr{F}_s} \omega^{s-2}(\widetilde{\mathbf{b}} * \overline{F} - \mathbf{b} * \overline{F}) \;=\; \frac{F_{s-2}}{\phi^{s-2}} + \frac{F_{s-3}}{\phi^{s-1}} = 1.$$

By Lemma 5.3, we have $K_n = \lfloor F_n + (F_{n+1} - F_n)\operatorname{frc}(n\pi) \rfloor$ for $n \in \mathbb{N}$, and the following are the first ten values of K_n:

$$(1,2,3,6,11,19,33,36,64,111).$$

Let us introduce and prove the main results on continuations.

Lemma 5.5. *Let f be a uniform continuation of F, and let K be a sequence of positive real numbers approaching ∞. Then,* $\operatorname{frc}\big(f^{-1}(\lfloor K_n \rfloor)\big) = \operatorname{frc}\big(f^{-1}(K_n)\big) + o(1)$.

Proof. Let $n \in \mathbb{N}$. Then, $F_m \le \lfloor K_n \rfloor \le K_n < F_{m+1}$ for $m \in \mathbb{N}$ depending on n. Let $K_n = f(m+p)$ and $\lfloor K_n \rfloor = f(m+p')$ where $p, p' \in [0,1]$ are real numbers, which depend on n. Then, $F_m + f_m(p')(F_{m+1} - F_m) + O(1) = F_m + f_m(p)(F_{m+1} - F_m)$, and hence, $f_m(p') + o(1) = f_m(p)$. Thus,

$$f^{-1}(K_n) \;=\; m + p = m + f_m{}^{-1}\big(f_m(p') + o(1)\big) \;=\; m + f_m{}^{-1}\big(f_\infty(p') + o(1)\big).$$

By the uniform convergence,

$$=\; m + f_\infty{}^{-1}\big(f_\infty(p') + o(1)\big) + o(1) \;=\; m + f_\infty{}^{-1}\big(f_\infty(p')\big) + o(1) \;=\; m + p' + o(1).$$

Therefore, $\operatorname{frc}\big(f^{-1}(K_n)\big) = \operatorname{frc}\big(f^{-1}(\lfloor K_n \rfloor)\big) + o(1)$.

□

Theorem 5.6. *Let $f : [1,\infty) \to \mathbb{R}$ be a uniform continuation of F. Then there is a sequence K of positive integers approaching ∞, e.g., $K_n = \big\lfloor \mathfrak{F}\big(n + \mathfrak{F}_n{}^{-1} \circ f_n(\operatorname{frc}(n\pi)\big) \big\rfloor$, such that $\operatorname{frc}\big(f^{-1}(K_n)\big)$ is equidistributed.*

Let K be a sequence of positive integers approaching ∞ such that $\operatorname{frc}\big(f^{-1}(K_n)\big)$ is equidistributed. Let $\mathbf{b} \in \mathscr{F}_s$ where $s \ge 2$. Then,

$$\begin{aligned} \operatorname{Prob}\big\{\, n \in \mathbb{N} : \operatorname{LB}_s(K_n) = \mathbf{b} \,\big\} \;&=\; f_\infty{}^{-1} \circ \mathfrak{F}_\infty(\log_\phi \widetilde{\mathbf{b}} \cdot \widehat{F}) - f_\infty{}^{-1} \circ \mathfrak{F}_\infty(\log_\phi \mathbf{b} \cdot \widehat{F}) \\ &=\; f_\infty{}^{-1}\big(\phi(\widetilde{\mathbf{b}} \cdot \widehat{F} - 1)\big) - f_\infty{}^{-1}\big(\phi(\mathbf{b} \cdot \widehat{F} - 1)\big). \end{aligned} \tag{4}$$

Proof. Let $x \ge 1$ be a real number, and let $F_n \le x < F_{n+1}$ for $n \in \mathbb{N}$. Since $\mathfrak{F}$ and f are increasing continuations of F, there are two unique real numbers p and p' in $[0,1]$ such that $x = \mathfrak{F}(n+p) = f(n+p')$. We claim that

$$f^{-1}(x) = n + f_n{}^{-1}(\mathfrak{F}_n(p)), \tag{5}$$

and $\mathfrak{F}^{-1}(x) = n + \mathfrak{F}_n{}^{-1}(f_n(p'))$. To prove the claim, note

$$\begin{aligned}\mathfrak{F}(n+p) = f(n+p') \;\Rightarrow\; F_n + \mathfrak{F}_n(p)(F_{n+1}-F_n) \;&=\; F_n + f_n(p')(F_{n+1}-F_n)\\ \Rightarrow\; p' = f_n{}^{-1}(\mathfrak{F}_n(p)),\; p &= \mathfrak{F}_n{}^{-1}(f_n(p')).\end{aligned}$$

Then $f(n+p') = x$ and $\mathfrak{F}(n+p) = x$ imply the claim.

Let $\overline{K}$ and K be the sequences given by $\overline{K}_n = \mathfrak{F}\left(n + \mathfrak{F}_n{}^{-1} \circ f_n(\mathrm{frc}(n\pi))\right)$ and $K_n = \left\lfloor \overline{K}_n \right\rfloor$. Given $n \in \mathbb{N}$, let $p_n = \mathfrak{F}_n{}^{-1} \circ f_n(\mathrm{frc}(n\pi))$. Then,

$$f^{-1}(\overline{K}_n) \;=\; n + f_n{}^{-1}\big(\mathfrak{F}_n(p_n)\big) \;=\; n + \mathrm{frc}(n\pi).$$

Thus, $\mathrm{frc}\Big(f^{-1}(\overline{K}_n)\Big)$ is equidistributed. If we further assume that f is a uniform continuation, then, by Lemmas 4.6 and 5.5, $\mathrm{frc}\Big(f^{-1}(\left\lfloor \overline{K}_n \right\rfloor)\Big) = \mathrm{frc}\left(f^{-1}(K_n)\right)$ is equidistributed as well.

Let K be a sequence of positive integers approaching ∞ such that $\mathrm{frc}\left(f^{-1}(K_n)\right)$ is equidistributed. Let $\mathbf{b} \in \mathscr{F}_s$, and let $A_{\mathbf{b}} := \{n \in \mathbb{N} : \mathrm{LB}_s(K_n) = \mathbf{b}\}$. Let $n \in A_{\mathbf{b}}$, and $F_m \le K_n < F_{m+1}$ for $m \in \mathbb{N}$ depending on n. Let $K_n = \mathfrak{F}(m+p) = f(m+p')$ where p and p' are real numbers in $[0,1]$ depending on n.

Then, by Lemma 4.4,

$$\begin{aligned}
&\log_\phi \mathbf{b}\cdot\widehat{F} + o(1) \;<\; \mathrm{frc}\left(\mathfrak{F}^{-1}(K_n)\right) \;<\; \log_\phi \widetilde{\mathbf{b}}\cdot\widehat{F} + o(1)\\
\Rightarrow\;& \log_\phi \mathbf{b}\cdot\widehat{F} + o(1) \;<\; \mathrm{frc}\left(m + \mathfrak{F}_n{}^{-1}(f_n(p'))\right) \;<\; \log_\phi \widetilde{\mathbf{b}}\cdot\widehat{F} + o(1)\\
\Rightarrow\;& \log_\phi \mathbf{b}\cdot\widehat{F} + o(1) \;<\; \mathfrak{F}_n{}^{-1}(f_n(p')) \;<\; \log_\phi \widetilde{\mathbf{b}}\cdot\widehat{F} + o(1)\\
\Rightarrow\;& f_n{}^{-1}\circ\mathfrak{F}_n(\log_\phi \mathbf{b}\cdot\widehat{F} + o(1)) \;<\; p' \;<\; f_n{}^{-1}\circ\mathfrak{F}_n(\log_\phi \widetilde{\mathbf{b}}\cdot\widehat{F} + o(1))\\
\Rightarrow\;& f_\infty{}^{-1}\circ\mathfrak{F}_\infty(\log_\phi \mathbf{b}\cdot\widehat{F}) + o(1) \;<\; \mathrm{frc}\left(f^{-1}(K_n)\right) \;<\; f_\infty{}^{-1}\circ\mathfrak{F}_\infty(\log_\phi \widetilde{\mathbf{b}}\cdot\widehat{F}) + o(1).
\end{aligned}$$

Since $\mathrm{frc}\left(f^{-1}(K_n)\right)$ is equidistributed, the above inequalities imply the assertion (4). □

Let us demonstrate a continuation for which the distribution of leading blocks of length 4 coincides with that of strong Benford's Law, but the distribution does not coincide for higher length blocks.

Example 5.7. Consider $\mathscr{F}_4 = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$, i.e.,

$$\mathbf{b}_1 = (1,0,0,0),\; \mathbf{b}_2 = (1,0,0,1),\; \mathbf{b}_3 = (1,0,1,0).$$

Let $p_k = \log_\phi(\mathbf{b}_k\cdot\widehat{F}) < 1$ for $k = 1,2,3$, and let $p_0 = 0$ and $p_4 = 1$. For each $n \in \mathbb{N}$, define $f_n : [0,1] \to [0,1]$ to be the function whose graph is the union of line segments from $(p_k, \mathfrak{F}_\infty(p_k))$ to $(p_{k+1}, \mathfrak{F}_\infty(p_{k+1}))$ for $k = 0,1,2,3$. Notice that f_n is defined independently of n, and that it defines a uniform continuation $f : [1,\infty) \to [1,\infty)$ such that $f_\infty = f_n$ for all $n \in \mathbb{N}$ as follows: Given $x \in [1,\infty)$, find $n \in \mathbb{N}$ such that $n \le x < n+1$, and define $f(x) = F_n + f_n(x-n)(F_{n+1} - F_n)$.

Note that $f_\infty(p_k) = \mathfrak{F}_\infty(p_k)$, i.e., $f_\infty{}^{-1}(\mathfrak{F}_\infty(p_k)) = p_k$ for $k = 0,1,2,3$. By Theorem 5.6, if $\mathrm{frc}\left(f^{-1}(K_n)\right)$ is equidistributed, we have

$$\mathrm{Prob}\left\{\, n \in \mathbb{N} : \mathrm{LB}_4(K_n) = \mathbf{b}_k \,\right\} \;=\; p_{k+1} - p_k \;=\; \log_\phi \frac{\widetilde{\mathbf{b}}_k\cdot\widehat{F}}{\mathbf{b}_k\cdot\widehat{F}}$$

where $\widetilde{\mathbf{b}}_3 = (1,0,1,1)$ as defined in Definition 3.7. However, the leading blocks of length > 4 do not satisfy Benford's Law under $\mathscr{F}$-expansion.

The following is an example where f_∞ is analytic.

Example 5.8. Let $f : [1,\infty) \to \mathbb{R}$ be the function given by $f(n+p) = F_n + (F_{n+1} - F_n)p^2$ where $n \in \mathbb{N}$ and $p \in [0,1)$. Then, $f_\infty(p) = p^2$.

Let K be the sequence given by $K_n = \lfloor \mathfrak{F}(n + \mathfrak{F}_n{}^{-1}(p^2)) \rfloor$, and let $\mathbf{b} \in \mathscr{F}_s$. Then, by Theorem 5.6,

$$\operatorname{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_s(K_n) = \mathbf{b} \right\} = \sqrt{\phi(\widetilde{\mathbf{b}} \cdot \widehat{F} - 1)} - \sqrt{\phi(\mathbf{b} \cdot \widehat{F} - 1)}.$$

Converse

Let's consider the converse of Theorem 5.6, i.e., given a sequence K of positive integers approaching ∞, let us construct a uniform continuation f, if possible, such that $\operatorname{frc}\left(f^{-1}(K_n)\right)$ is equidistributed. Recall the set $\mathscr{F}_s$ from Definition 3.7.

Definition 5.9. A sequence K of positive integers approaching ∞ is said to have *strong leading block distribution under $\mathscr{F}$-expansion* if $\operatorname{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_s(K_n) = \mathbf{b} \right\}$ exists for each integer $s \geq 2$ and each $\mathbf{b} \in \mathscr{F}_s$.

Example 5.10. Let K be the Lucas sequence, i.e., $K = (2,1,3,4,\ldots)$ and $K_{n+2} = K_{n+1} + K_n$. Recall that $F_n = \frac{1}{10}(5+\sqrt{5})\phi^n(1+o(1))$ and $K_n = \frac{1}{2}(\sqrt{5}-1)\phi^n(1+o(1))$, and let $\alpha = \frac{1}{10}(5+\sqrt{5})$ and $a = \frac{1}{2}(\sqrt{5}-1)$. Then, by Lemma 4.2,

$$\operatorname{frc}\left(\mathfrak{F}^{-1}(K_n)\right) = -\log_\phi(a/\alpha) + o(1) \approx .328 + o(1).$$

By Lemma 4.4, the leading block of K_n being $\mathbf{b}_1 = (1,0,0)$ is determined by whether $0 \leq \operatorname{frc}\left(\mathfrak{F}^{-1}(K_n)\right) < \log_\phi(1+\omega^2) \approx .67$. Thus, $\operatorname{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_3(K_n) = \mathbf{b}_1 \right\} = 1$, and $\operatorname{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_3(K_n) = \mathbf{b}_2 \right\} = 0$.

In fact, the sequence K has strong leading block distribution. Recall $\widehat{F}$ from Definition 3.1, and let us claim that $\mathbf{b} \cdot \widehat{F} \neq \frac{\alpha}{a} = \frac{1}{10}(5+3\sqrt{5})$ for all $s \in \mathbb{N}$ and $\mathbf{b} \in \mathscr{F}_s$. Notice that

$$\frac{\alpha}{a} - 1 = \sum_{k=1}^{\infty} \omega^{4k}. \tag{6}$$

The equality (6) is called *the Zeckendorf expansion of a real number in* $(0,1)$ since it is a power series expansion in ω where no consecutive powers are used; a formal definition is given in Definition 5.11 below. By the uniqueness of Zeckendorf expansions of the real numbers in $(0,1)$, the above infinite sum in (6) is not equal to any finite sum $\mathbf{b} \cdot \widehat{F} - 1$ where $\mathbf{b} \in \mathscr{F}_s$; see Theorem 5.13.

Let s be an integer ≥ 2, and let $\mathscr{F}_s = \{\mathbf{b}_1, \ldots, \mathbf{b}_\ell\}$. Then, there is $k \in \mathbb{N}$ such that $\mathbf{b}_k \cdot \widehat{F} < \frac{\alpha}{a} < \mathbf{b}_{k+1} \cdot \widehat{F}$. This implies that

$$\log_\phi(\mathbf{b}_k \cdot \widehat{F}) < \log_\phi(\tfrac{\alpha}{a}) < \log_\phi(\mathbf{b}_{k+1} \cdot \widehat{F}).$$

Since $\operatorname{frc}\left(\mathfrak{F}^{-1}(K_n)\right) = \log_\phi(\alpha/a)+o(1)$ for all $n \in \mathbb{N}$, by Lemma 4.4, we have $\operatorname{Prob}\{\, n \in \mathbb{N} : \mathrm{LB}_s(K_n) = \mathbf{b}_k \,\} = 1$. For example, consider the case of $s = 9$, and notice that $\omega^4 + \omega^8 < \frac{\alpha}{a} - 1 < \omega^4 + \omega^7$ by (6). Then, we have $\mathbf{b}\cdot\widehat{F} < \frac{\alpha}{a} < \widetilde{\mathbf{b}}\cdot\widehat{F}$ where

$$\mathbf{b} = (1,0,0,0,1,0,0,0,1) \text{ and } \widetilde{\mathbf{b}} = (1,0,0,0,1,0,0,1,0),$$

and the probability of having the leading block $\mathbf{b}$ in the values of the Lucas sequence is 1.

Recall uniform continuations from Definition 1.6. Since the distribution of the leading blocks of the Lucas sequence K is concentrated on one particular block in $\mathscr{F}_s$ for each s, there does not exist a uniform continuation f, described in Theorem 5.6, whose equidistribution is associated with the leading block distributions of the Lucas sequence K. For a uniform continuation to exist, the values of the leading block distributions must be put together into a continuous function, and below we formulate the requirement more precisely.

Definition 5.11. Let $\mathbf{I}$ denote the interval $(0,1)$ of real numbers. An infinite tuple $\mu \in \prod_{k=1}^{\infty} \mathbb{N}_0$ is called a *Zeckendorf expression for* $\mathbf{I}$ if $\mu(k) \le 1$, $\mu(k)\mu(k+1) = 0$, and for all $j \in \mathbb{N}_0$, the sequence $\{\mu(j+n)\}_{n=1}^{\infty}$ is not equal to the sequence $\{1+(-1)^{n+1})/2\}_{n=1}^{\infty} = (1,0,1,0,\dots)$. Let $\mathscr{F}^*$ be the set of Zeckendorf expressions for $\mathbf{I}$.

Given $s \in \mathbb{N}$ and $\mu \in \mathscr{F}^*$, let $\mu|s := (\mu(1),\dots,\mu(s))$. Given $s \in \mathbb{N}$ and $\{\mu,\tau\} \subset \mathscr{F}^*$, we declare $\mu|s < \tau|s$ if $\mu|s\cdot\widehat{F} < \tau|s\cdot\widehat{F}$, which coincides with the lexicographical order on $\mathscr{F}$.

Notation 5.12. Given a sequence Q of real numbers, and $\mu \in \prod_{k=1}^{\infty}\mathbb{N}_0$, we define $\mu\cdot Q := \sum_{k=1}^{\infty}\mu(k)Q_k$, which may or may not be a convergent series.

Theorem 5.13 ([10], Zeckendorf Theorem for $\mathbf{I}$)**.** *Given a real number $\beta \in \mathbf{I}$, there is a unique $\mu \in \mathscr{F}^*$ such that $\beta = \sum_{k=1}^{\infty}\mu(k)\omega^k = (\mu\cdot\widehat{F})\omega$.*

For the uniqueness of μ in the theorem, we require the infinite tuples such as $(0,1,0,1,0,\dots)$ to be not a member of $\mathscr{F}^*$ since $\sum_{k=1}^{\infty}\omega^{2k} = \omega$, which is analogous to $0.0999\ldots = 0.1$ in decimal expansion.

Proposition 5.14 ([10])**.** *Let $\{\mu,\tau\} \subset \mathscr{F}^*$. Then, $\mu\cdot\widehat{F} < \tau\cdot\widehat{F}$ if and only if $\mu|s < \tau|s$ for some $s \in \mathbb{N}$.*

Given a sequence with strong leading block distribution, we shall construct a function on $\mathbf{I}$ in Definition 5.16 below, and it is well defined by Lemma 5.15.

Lemma 5.15. *Given a real number $\beta \in \mathbf{I}$, there is a unique $\mu \in \mathscr{F}^*$ such that $\mu(1) = 1$ and $\phi(\mu\cdot\widehat{F} - 1) = \beta$.*

Proof. Let $\widehat{F}^*$ be the sequence defined by $\widehat{F}^*_n = \omega^n$. Given a real number $\beta \in \mathbf{I}$, we have $0 < \omega + \beta\omega^2 < 1$. By Theorem 5.13, there is $\mu \in \mathscr{F}^*$ such that $(\mu\cdot\widehat{F})\omega = \mu\cdot\widehat{F}^* = \omega + \beta\omega^2$, which implies $\phi(\mu\cdot\widehat{F} - 1) = \beta$. We claim that $\mu(1) = 1$. If $\mu(1) = 0$, then by Proposition 5.14, $\omega + \beta\omega^2 = \mu\cdot\widehat{F}^* = (0,\dots)\cdot\widehat{F}^* < \omega = (1,0,0,\dots)\cdot\widehat{F}^*$, which implies a false statement $\beta\omega^2 < 0$. Thus, $\mu(1) = 1$. □

Recall from Definition 5.11 the definition of inequalities on tuples.

Definition 5.16. Let K be a sequence of positive integers with strong leading block distribution under $\mathscr{F}$-expansion such that given $\mu \in \mathscr{F}^*$ and an integer $s \geq 2$ such that $\mu(1) = 1$, the following limit exists:

$$\lim_{s\to\infty} \operatorname{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_s(K_n) \leq \mu|s \right\} \tag{7}$$

where $\mu|s$ is identified in $\mathscr{F}_s$.

Let $f_K^* : [0,1] \to [0,1]$ be the function given by $f_K^*(0) = 0$, $f_K^*(1) = 1$, and $f_K^*(\phi(\mu \cdot \widehat{F} - 1))$ is equal to the value in (7). If f_K^* is continuous and increasing, then K is said to *have continuous leading block distribution under $\mathscr{F}$-expansion*.

Lemma 5.17. *Let K be a sequence with continuous leading block distribution under $\mathscr{F}$-expansion, and let f_K^* be the function defined in Definition 5.16. Let $\mu \in \mathscr{F}^*$ such that there is $t \in \mathbb{N}$ such that $\mu(1) = 1$ and $\mu(k) = 0$ for all $k > t$. Then, $f_K^*(\phi(\mu|t \cdot \widehat{F} - 1)) \leq \operatorname{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_t(K_n) \leq \mu|t \right\}$.*

Proof. Let $s > t$, and let us show that $A_s := \{n \in \mathbb{N} : \mathrm{LB}_s(K_n) \leq \mu|s\} \subset A_t := \{n \in \mathbb{N} : \mathrm{LB}_t(K_n) \leq \mu|t\}$. Let $n \in A_s$, and $\tau := \mathrm{LB}_s(K_n)$, so that either $\tau = \mu|s$, or $\tau < \mu|s$, i.e., there is a smallest index $m \leq s$ such that $\tau(m) < \mu(m)$ and $\tau(k) = \mu(k)$ for all $k < m$. For the case of $\tau < \mu|s$, notice that $\mu(k) = 0$ for all $k > t$ and $\tau(m) < \mu(m)$ imply that $m \leq t$, and hence, $\tau|t = \mathrm{LB}_t(K_n) < \mu|t$. If $\tau = \mu|s$, then clearly $\tau|t = \mu|t$. Thus, for both cases, we have $n \in A_t$.

From $A_s \subset A_t$, it follows that $\operatorname{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_s(K_n) \leq \mu|s \right\} \leq \operatorname{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_t(K_n) \leq \mu|t \right\}$, and from the definition of f_K^*, it follows that

$$\lim_{s\to\infty} \operatorname{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_s(K_n) \leq \mu|s \right\} = f_K^*(\phi(\mu \cdot \widehat{F} - 1)) \leq \operatorname{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_t(K_n) \leq \mu|t \right\}.$$

Since $\mu|t \cdot \widehat{F} = \mu \cdot \widehat{F}$, we prove $f_K^*(\phi(\mu|t \cdot \widehat{F} - 1)) \leq \operatorname{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_t(K_n) \leq \mu|t \right\}$. □

Recall uniform continuations from Definition 1.6.

Theorem 5.18. *Let K be a sequence with continuous leading block distribution under $\mathscr{F}$-expansion. Let f_K^* be the function defined in Definition 5.16. Then, there is a uniform continuation f of F such that ${f_\infty}^{-1} = f_K^*$ and $\operatorname{frc}\left(f^{-1}(K_n)\right)$ is equidistributed.*

Proof. Let $f : [1,\infty) \to \mathbb{R}$ be the function given by $f(x) = F_n + (F_{n+1} - F_n)(f_K^*)^{-1}(p)$ where $x = n + p$ and $p = \operatorname{frc}(x)$. Then, f is a uniform continuation of F_n since $(f_K^*)^{-1}$ is independent of n. Then, $f_\infty = (f_K^*)^{-1}$, i.e., ${f_\infty}^{-1} = f_K^*$.

Let $\beta \in (0,1)$ be a real number, and below we show that $\operatorname{Prob}\left\{ n \in \mathbb{N} : \operatorname{frc}\left(f^{-1}(K_n)\right) \leq \beta \right\}$ exists, and it is equal to β. Recall $\mathfrak{F}$ from Definition 4.1 and $\mathfrak{F}_n$ from Definition 5.1. Let $n \in \mathbb{N}$, and let $m \in \mathbb{N}$ such that $F_m \leq K_n < F_{m+1}$. Then, $K_n = f(m + p_n') = \mathfrak{F}(m + p_n)$ where

$p_n, p'_n \in [0,1]$, i.e., $f_\infty(p'_n) = \mathfrak{F}_m(p_n)$. By Theorem 5.13 and Lemma 5.15, there is a unique $\mu \in \mathscr{F}^*$ such that $f_\infty(\beta) = \phi(\mu\cdot\widehat{F} - 1)$ and $\mu(1) = 1$. Recall $\mathfrak{F}_\infty$ from Definition 5.1. Notice that

$$\begin{aligned} &\operatorname{frc}\left(f^{-1}(K_n)\right) \;=\; p'_n \;\le\; \beta \Rightarrow {f_\infty}^{-1}(\mathfrak{F}_m(p_n)) \;\le\; \beta \;\Rightarrow\; p_n \le {\mathfrak{F}_m}^{-1}(f_\infty(\beta)) \\ \Rightarrow\;& \operatorname{frc}\left(\mathfrak{F}^{-1}(K_n))\right) \;\le\; {\mathfrak{F}_m}^{-1}(f_\infty(\beta)) \;=\; {\mathfrak{F}_\infty}^{-1}(f_\infty(\beta)) + o(1) \;=\; \log_\phi(\mu\cdot\widehat{F}) + o(1). \end{aligned}$$

Fix an integer $t \ge 2$. By Proposition 5.14, we have $\mu\cdot\widehat{F} = \mu|t\cdot\widehat{F} + \gamma_t < \widetilde{\mu|t}\cdot\widehat{F}$ where $\gamma_t \ge 0$ and $\widetilde{\mu|t} \in \mathscr{F}_t$ is as defined Definition 3.7. Since $\log_\phi(\widetilde{\mu|t}\cdot\widehat{F}) - \log_\phi(\mu\cdot\widehat{F}) > 0$, there is $M_t \in \mathbb{N}$ such that for all $n \ge M_t$,

$$\operatorname{frc}\left(\mathfrak{F}^{-1}(K_n))\right) \;\le\; \log_\phi(\mu\cdot\widehat{F}) + o(1) \;<\; \log_\phi(\widetilde{\mu|t}\cdot\widehat{F}).$$

By Lemma 4.4, this implies $\mathrm{LB}_t(K_n) \le \mu|t$. Recall $\Omega_n = \{k \in \mathbb{N} : k \le n\}$;

$$\begin{aligned} &\operatorname{Prob}\left\{\, k \in \Omega_n : \operatorname{frc}\left(f^{-1}(K_k)\right) \le \beta \,\right\} + o(1) \;\le\; \operatorname{Prob}\left\{\, k \in \Omega_n : \mathrm{LB}_t(K_k) \le \mu|t \,\right\} + o(1) \\ \Rightarrow\;& \limsup_n \operatorname{Prob}\left\{\, k \in \Omega_n : \operatorname{frc}\left(f^{-1}(K_k)\right) \le \beta \,\right\} \;\le\; \operatorname{Prob}\left\{\, n \in \mathbb{N} : \mathrm{LB}_t(K_n) \le \mu|t \,\right\}. \end{aligned}$$

Let us work on the lim inf of the probability. Since $\beta \ne 0$, there is $t_0 > 1$ such that $\mu(t_0) > 0$. Thus, if $t > t_0$ is sufficiently large, then there are at least two entries 1 in $\mu|t$, and $\mu|t$ has more entries after the second entry of 1 from the left. Recall the product $*$ from Definition 2.2. This choice of t allows us to have the unique coefficient functions $\check{\mu}$ and $\widehat{\mu}$ in $\mathscr{F}_t$ such that $1 + \check{\mu} * F = \widehat{\mu} * F$ and $1 + \widehat{\mu} * F = \mu|t * F$. Then, by Lemma 4.4,

$$\begin{aligned} &\mathrm{LB}_t(K_n) \le \check{\mu} \;\Rightarrow\; \operatorname{frc}\left(\mathfrak{F}^{-1}(K_n)\right) \;<\; \log_\phi(\widehat{\mu}\cdot\widehat{F}) + o(1) \\ \Rightarrow\;& p_n \;<\; {\mathfrak{F}_m}^{-1}(\phi(\widehat{\mu}\cdot\widehat{F} - 1)) + o(1) \\ \Rightarrow\;& \mathfrak{F}_m(p_n) \;=\; f_\infty(p'_n) \;<\; \phi(\widehat{\mu}\cdot\widehat{F} - 1) + o(1) \\ \Rightarrow\;& p'_n \;=\; \operatorname{frc}\left(f^{-1}(K_n)\right) \;<\; {f_\infty}^{-1}(\phi(\widehat{\mu}\cdot\widehat{F} - 1)) + o(1) \\ &\qquad\qquad\qquad\qquad\quad\; < \; {f_\infty}^{-1}(\phi(\mu|t\cdot\widehat{F} - 1)) \quad \text{by Proposition 5.14,} \\ &\qquad\qquad\qquad\qquad\quad\; \le \; {f_\infty}^{-1}(\phi(\mu\cdot\widehat{F} - 1)) \;=\; \beta \\ \Rightarrow\;& \operatorname{Prob}\left\{\, k \in \Omega_n : \mathrm{LB}_t(K_k) \le \check{\mu} \,\right\} + o(1) \;\le\; \operatorname{Prob}\left\{\, k \in \Omega_n : \operatorname{frc}\left(f^{-1}(K_k)\right) \le \beta \,\right\} + o(1) \\ \Rightarrow\;& \operatorname{Prob}\left\{\, n \in \mathbb{N} : \mathrm{LB}_t(K_n) \le \check{\mu} \,\right\} \;\le\; \liminf_n \operatorname{Prob}\left\{\, k \in \Omega_n : \operatorname{frc}\left(f^{-1}(K_k)\right) \le \beta \,\right\} \end{aligned}$$

By Lemma 5.17,

$${f_\infty}^{-1}(\phi(\check{\mu}\cdot\widehat{F} - 1)) \;\le\; \liminf_n \operatorname{Prob}\left\{\, k \in \Omega_n : \operatorname{frc}\left(f^{-1}(K_k)\right) \le \beta \,\right\}.$$

It is given that $\operatorname{Prob}\left\{\, n \in \mathbb{N} : \mathrm{LB}_t(K_n) \le \mu|t \,\right\} \to {f_\infty}^{-1}(\phi(\mu\cdot\widehat{F} - 1))$ as $t \to \infty$. Let us calculate the other bound;

$$\begin{aligned}
2+\check{\mu}*F \;=\; \mu|t*F \;\Rightarrow\; & 2+\sum_{k=1}^{t}\check{\mu}(k)F_{t-k+1} = \sum_{k=1}^{t}\mu(k)F_{t-k+1} \\
\Rightarrow\; & 2+\sum_{k=1}^{t}\check{\mu}(k)\Big(\alpha\phi^{t-k+1}+O(\phi^{-t+k-1})\Big) \;=\; \sum_{k=1}^{t}\mu(k)\Big(\alpha\phi^{t-k+1}+O(\phi^{-t+k-1})\Big) \\
\Rightarrow\; & O(1)+\alpha\sum_{k=1}^{t}\check{\mu}(k)\phi^{t-k+1} \;=\; \alpha\sum_{k=1}^{t}\mu(k)\phi^{t-k+1} \\
\Rightarrow\; & O(\phi^{-t})+\sum_{k=1}^{t}\check{\mu}(k)\omega^{k-1} \;=\; \sum_{k=1}^{t}\mu(k)\omega^{k-1} \\
\Rightarrow\; & o(1)+\check{\mu}\cdot\widehat{F} \;=\; \mu|t\cdot\widehat{F} \;\Rightarrow\; \check{\mu}\cdot\widehat{F}\to\mu\cdot\widehat{F} \\
\Rightarrow\; & {f_\infty}^{-1}(\phi(\check{\mu}\cdot\widehat{F}-1))\to {f_\infty}^{-1}(\phi(\mu\cdot\widehat{F}-1)) \;=\; \beta.
\end{aligned}$$

□

It is clear that if f is a uniform continuation of F, and K is a sequence of positive integers approaching ∞ such that $\operatorname{frc}\big(f^{-1}(K_n)\big)$ is equidistributed, then, by Lemma 4.4, K has continuous leading block distribution under $\mathscr{F}$-expansion. Therefore, we have the following.

Theorem 5.19. *Let K be a sequence of positive integers approaching ∞. Then, K has continuous leading block distribution under $\mathscr{F}$-expansion if and only if there is a uniform continuation f of F such that* $\operatorname{frc}\big(f^{-1}(K_n)\big)$ *is equidistributed.*

6 Benford's Law under generalized Zeckendorf expansion

The contents in Sections 3, 4, and 5 are for Zeckendorf expansion, but the arguments of the proofs apply to the setup for generalized Zeckendorf expansion without difficulties. In this section, we introduce definitions and results for generalized Zeckendorf expansion without proofs, but only refer to the corresponding theorems for Zeckendorf expansion proved in the earlier sections.

6.1 Generalized Zeckendorf expansion

Let us review the generalized Zeckendorf expansion. Recall $\mathbb{N}_0$ from Definition 2.1.

Definition 6.1. Given a tuple $L=(a_1,a_2,\dots,a_N)\in\mathbb{N}_0^N$ where $N\ge 2$ and $a_1>0$, let Θ be the following infinite tuple in $\prod_{k=1}^{\infty}\mathbb{N}_0$:

$$(a_1,a_2,\dots,a_{N-1},a_N,a_1,a_2,\dots,a_{N-1},a_N,\dots)$$

where the finite tuple $(a_1, a_2, \dots, a_{N-1}, a_N)$ repeats. Let $\Theta(k)$ denote the kth entry of Θ, and let $\Theta|s = (\Theta(1), \dots, \Theta(s))$ for $s \in \mathbb{N}$.

Recall len from Definition 2.2. Let $\mathscr{H}^\circ$ be the recursively-defined set of tuples ϵ with arbitrary finite length such that $\epsilon \in \mathscr{H}^\circ$ if and only if there is smallest $s \in \mathbb{N}_0$ such that $\epsilon|s = \Theta|s$, $\epsilon(s+1) < \Theta(s+1)$, and $(\epsilon(s+2), \dots, \epsilon(n)) \in \mathscr{H}^\circ$ where $n = \text{len}(\epsilon)$ and s is allowed to be $\text{len}(\epsilon)$. Let $\mathscr{H} := \{\epsilon \in \mathscr{H}^\circ : \epsilon(1) > 0\}$. The set $\mathscr{H}$ is called a *periodic Zeckendorf collection of coefficient functions for positive integers*, and L is called *a principal maximal block of the periodic Zeckendorf collection $\mathscr{H}$.*

Notice that if $L = (1,0,1,0)$ is a principal maximal block of the periodic Zeckendorf collection $\mathscr{H}$, then $L' = (1,0)$ is a principal maximal block of $\mathscr{H}$ as well. For this reason, the indefinite article was used in the statement of the definition of principal maximal blocks.

Example 6.2. Let $\mathscr{H}$ be the (periodic) Zeckendorf collection determined by the principal maximal block $L = (3,2,1)$. Then, $\Theta = (3,2,1,3,2,1,\dots)$, and (0) and $(3,2,1)$ are members of $\mathscr{H}^\circ$. For $(0) \in \mathscr{H}^\circ$, we set $s = 0$ in Definition 6.1, and for $(3,2,1) \in \mathscr{H}^\circ$, we set $s = 3$.

Let $\epsilon = (3,2,0)$ and $\mu = (3,1,3,2,0)$. For ϵ, if $s = 2$, by the definition, we have $\epsilon \in \mathscr{H}$. For μ, if $s = 1$, then $\mu|1 = \Theta|1$, $\mu(2) < \Theta(2)$, and $(\mu(3), \dots, \mu(5)) = \epsilon \in \mathscr{H}^\circ$. Listed below are more examples of members of $\mathscr{H}$:

$$(3,2,1,3,2,1),\ (3,0,0,3),\ (1,2,3,1,0,3),\ (1,2,3,1,1,0).$$

Recall the product notation from Definition 2.2.

Definition 6.3. Let $\mathscr{H}$ be a set of coefficient functions, and let H be an increasing sequence of positive integers. If given $n \in \mathbb{N}$, there is a unique $\epsilon \in \mathscr{H}$ such that $\epsilon * H = n$, then H is called a *fundamental sequence* of $\mathscr{H}$, and the expression $\epsilon * H$ is called an *$\mathscr{H}$-expansion*.

If $\mathscr{H}$ is a periodic Zeckendorf collection for positive integers, then, by Theorem 6.4 below, there is a unique fundamental sequence of $\mathscr{H}$.

Theorem 6.4 ([10, 17])**.** *Let $\mathscr{H}$ be a periodic Zeckendorf collection, and let $L = (a_1, \dots, a_N)$ be its principal maximal block. Then, there is a unique fundamental sequence H of $\mathscr{H}$, and it is given by the following recursion:*

$$H_{n+N} = a_1 H_{n+N-1} + \cdots + a_{N-1} H_{n+1} + (1 + a_N) H_n \ \text{ for all } n \in \mathbb{N}, \text{ and} \tag{8}$$

$$H_n = 1 + \sum_{k=1}^{n-1} a_k H_{n-k} \ \text{ for all } 1 \le n \le N+1.$$

If $L = (1,0)$, then its periodic Zeckendorf collection is $\mathscr{F}$ defined in Definition 3.1, and its fundamental sequence is the Fibonacci sequence. If $L = (9,9)$, then the fundamental sequence H is given by $H_n = 10^{n-1}$, and $\epsilon * H$ for $\epsilon \in \mathscr{H}$ are base-10 expansions.

Definition 6.5. Let $L = (a_1, \dots, a_N)$ be the list defined in Definition 6.1. Let $\psi = \psi_{\mathscr{H}} = \psi_L$ be the dominant real zero of the polynomial $g = g_{\mathscr{H}} = g_L(x) := x^N - \sum_{k=1}^{N-1} a_k x^{N-k} - (1 + a_N)$, and $\theta := \psi^{-1}$. Let $\widehat{H}$ be the sequence given by $\widehat{H}_n = \theta^{n-1}$.

By (8), the sequence $\widehat{H}$ in Definition 6.5 satisfies

$$\widehat{H}_n \;=\; a_1\widehat{H}_{n+1}+\cdots+a_{N-1}\widehat{H}_{n+N-1}+(1+a_N)\widehat{H}_{n+N} \quad \text{for all } n\in\mathbb{N}. \tag{9}$$

The following proposition is proved in [10, Lemma 43] and [16, Lemma 2.1].

Proposition 6.6. *Let $L=(a_1,\dots,a_N)$ be the list defined in Definition 6.1, and let $g=x^N-\sum_{k=1}^{N-1}a_kx^{N-k}-(1+a_N)$ be the polynomial. Then, g has one and only one positive real zero ψ, it is a simple zero, and there are no other complex zeros z such that $|z|\ge\psi$.*

Theorem 6.7. *Let $\mathscr{H}$ be a periodic Zeckendorf collection with a principal maximal block $L=(a_1,\dots,a_N)$, and let H be the fundamental sequence of $\mathscr{H}$. Then $H_n=\delta\psi^n+O(\psi^{rn})$ for $n\in\mathbb{N}$ where δ and r are positive (real) constants, $r<1$, and ψ is the dominant zero defined in Definition 6.5.*

Proof. Let g be the characteristic polynomial of degree N defined in Definition 6.5, and let $\{\lambda_1,\dots,\lambda_m\}$ be the set of m distinct (complex) zeros of g where $m\le N$ and $\lambda_1=\psi$. Then, by Proposition 6.6, we have $|\lambda_k|<\psi$ for $2\le k\le m$. Since ψ is a simple zero, by the generalized version of Binet's formula [15], there are polynomials h_k for $2\le k\le m$ and a constant δ such that $H_n=\delta\psi^n+\sum_{k=2}^m h_k(n)\lambda_k^n$ for $n\in\mathbb{N}$. Thus, there is a positive real number $r<1$ such that $H_n=\delta\psi^n+O(\psi^{rn})$ for $n\in\mathbb{N}$.

Notice that $\lim_{n\to\infty}H_n/\psi^n=\delta$, and let us show that δ is a positive real number, and in particular, it is non-zero. By [11, Theorem 5.1],

$$\delta \;=\; \lim_{n\to\infty}\frac{H_n}{\psi^n} \;=\; \frac{1}{\psi g'(\psi)}\sum_{k=1}^{N}\frac{H_k}{(k-1)!}\left[\frac{d^{k-1}}{dx^{k-1}}\frac{g(x)}{x-\psi}\right]_{x=0}. \tag{10}$$

By the product rule, we have

$$\left[\frac{d^{k-1}}{dx^{k-1}}\frac{g(x)}{x-\psi}\right]_{x=0} \;=\; \left[\sum_{j=0}^{k-1}\binom{k-1}{j}g^{(j)}(x)(x-\psi)^{-1-j}\prod_{t=1}^{j}(-t)\right]_{x=0}.$$

Notice that if $1\le j\le N-1$, then $g^{(j)}(0)=-a_{N-j}j!\le 0$, and if $g(0)=-(1+a_N)<0$. The inequality $(-\psi)^{-1-j}\prod_{t=1}^{j}(-t)<0$ for all $0\le j\le k-1$ follows immediately from considering the cases of j being even or odd. Thus, the summands in (10) are non-negative, and some are positive. This concludes the proof of δ being a positive real number. ☐

6.2 Strong Benford's Law

Let $\mathscr{H}$, H, and ψ be as defined in Definition 6.1, 6.3, and 6.5, and we begin with definitions related to leading blocks under $\mathscr{H}$-expansion.

Definition 6.8. Let $n = \epsilon * H$ for $n \in \mathbb{N}$ and $\epsilon \in \mathscr{H}$. If $s \le \text{len}(\epsilon)$, then $(\epsilon(1),\ldots,\epsilon(s)) \in \mathscr{H}$ is called *the leading block of n with length s under $\mathscr{H}$-expansion*. Recall that $N = \text{len}(L)$. If $N \le s \le \text{len}(\epsilon)$, let $\text{LB}_s^{\mathscr{H}}(n)$, or simply $\text{LB}_s(n)$ if the context is clear, denote the leading block of length s, and if $s \le \text{len}(\epsilon)$ and $s < N$, then let $\text{LB}_s^{\mathscr{H}}(n)$ or simply $\text{LB}_s(n)$ denote $(\epsilon(1),\ldots,\epsilon(s),0,\ldots,0) \in \mathbb{N}_0^N$. If $s > \text{len}(\epsilon)$, $\text{LB}_s(n)$ is declared to be undefined.

Recall the product $*$ from Definition 2.2. Given an integer $s \ge N$, let $\mathscr{H}_s := \{\mathbf{b}_1,\mathbf{b}_2,\ldots,\mathbf{b}_\ell\}$ be the finite set of the leading blocks of length s occurring in the $\mathscr{H}$-expansions of $\mathbb{N}$ such that $1+\mathbf{b}_k * H = \mathbf{b}_{k+1} * H$ for all $k \le \ell - 1$. Recall the truncation notation from Definition 4.3. If $1 \le s < N$, then let $\mathscr{H}_s := \{\mathbf{b}_1,\mathbf{b}_2,\ldots,\mathbf{b}_\ell\}$ be the finite set of the leading blocks of length N occurring in the $\mathscr{H}$-expansions of $\mathbb{N}$ such that $\mathbf{b}_k(j) = 0$ for all $1 \le k \le \ell$ and $j > s$ and $1+\mathbf{b}_k|s * H = \mathbf{b}_{k+1}|s * H$ for all $k \le \ell - 1$. The leading block $\mathbf{b}_\ell$ is called *the largest leading block in $\mathscr{H}_s$*.

The exclusive block $\mathbf{b}_{\ell+1}$ is a coefficient function of length s defined as follows. If $s \ge N$, $s \equiv p \pmod N$, and $0 \le p < N$, then

$$\mathbf{b}_{\ell+1} := (a_1,\ldots,a_{N-1},a_N,\ldots,a_1,\ldots,a_{N-1},1+a_N,c_1,\ldots,c_p)$$

where $c_k = 0$ for all k. If $1 \le s < N$, then $\mathbf{b}_{\ell+1} := (a_1,\ldots,a_{N-1},1+a_N)$. If $\mathbf{b}$ is a leading block $\mathbf{b}_k \in \mathscr{H}_s$, then we denote $\mathbf{b}_{k+1}$ by $\widetilde{\mathbf{b}}$.

If $s < N$, then the leading blocks $\mathbf{b}$ in $\mathscr{H}_s$ has lengths N with $N-s$ last entries of 0, and this case is treated as above in order to make $\mathbf{b}$ and $\widetilde{\mathbf{b}}$ in the statement and proof of Lemma 4.4 fit into the case of periodic Zeckendorf collections; see Lemma 6.13.

By [10, Definition 2 & Lemma 3] and Theorem 6.4, the subscript numbering of $\mathbf{b}_k \in \mathscr{H}_s$ for $1 \le k \le \ell$ coincides with the lexicographical order on the coefficient functions. If $\mathbf{b}$ is the largest leading block in $\mathscr{H}_s$ where $s \ge N$, then

$$\mathbf{b} = (\ldots,a_1,\ldots,a_N,a_1,\ldots,a_p) \text{ if } s \equiv p \pmod N \text{ and } 0 \le p < N,$$

and $1+\mathbf{b} * H = \widetilde{\mathbf{b}} * H = (\ldots,a_1,\ldots,1+a_N,0,\ldots,0) * H = H_{s+1}$ where the last p entries of $\widetilde{\mathbf{b}}$ are zeros. If $s \equiv 0 \pmod N$ and $\mathbf{b}$ is the largest leading block in $\mathscr{H}_s$, then

$$\widetilde{\mathbf{b}} = (a_1,\ldots,a_{N-1},a_N,\ldots,a_1,\ldots,a_{N-1},1+a_N).$$

If $s < N$ and $\mathbf{b}$ is the largest leading block in $\mathscr{H}_s$, then $\widetilde{\mathbf{b}} = (a_1,\ldots,a_{N-1},1+a_N)$. Recall $\widehat{H}$ from Definition 6.5. For all cases, if $\mathbf{b}$ is the largest leading block in $\mathscr{F}_s$, then $\widetilde{\mathbf{b}} \cdot \widehat{H} = \psi$.

The proof of Theorem 6.9 below follows immediately from Lemma 6.12 and Theorem 6.14.

Theorem 6.9. *Let K be a sequence of positive integers such that $K_n = ab^n(1+o(1))$ where a and b are positive real numbers such that $\log_\psi b$ is irrational. Then, given $\mathbf{b} \in \mathscr{H}_s$,*

$$\text{Prob}\left\{\, n \in \mathbb{N} : \text{LB}_s(K_n) = \mathbf{b} \,\right\} \;=\; \log_\psi \frac{\widetilde{\mathbf{b}} \cdot \widehat{H}}{\mathbf{b} \cdot \widehat{H}}.$$

Motivated from the leading block distributions of the exponential sequences considered in Theorem 6.9, we declare strong Benford's Law under $\mathscr{H}$-expansion as follows.

Definition 6.10. A sequence K of positive integers is said to *satisfy strong Benford's Law under $\mathscr{H}$-expansion* if given $\mathbf{b} \in \mathscr{H}_s$,

$$\operatorname{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_s(K_n) = \mathbf{b} \right\} \;=\; \log_\psi \frac{\widetilde{\mathbf{b}} \cdot \widehat{H}}{\mathbf{b} \cdot \widehat{H}}.$$

6.3 Benford continuation of H

Let $\mathscr{H}$, H, and ψ be as defined in Definition 6.1, 6.3, and 6.5. Recall that we used a real analytic continuation of the Fibonacci sequence for Zeckendorf expansion, but as demonstrated in the earlier sections, the leading block distributions are determined by its limit $\mathfrak{F}_\infty$. Thus, rather than using a real analytic continuation of H, we may use the limit version directly, which is far more convenient. By Theorem 6.7, $H_n = \delta\psi^n + O(\psi^{rn}) = \delta\psi^n(1 + o(1))$ where δ and $r < 1$ are positive real constants, and we define the following:

Definition 6.11. Let $\mathfrak{H} : [1, \infty) \to \mathbb{R}$ be the function given by

$$\mathfrak{H}(x) = H_n + (H_{n+1} - H_n)\frac{\psi^p - 1}{\psi - 1}$$

where $x = n + p$ and $p = \operatorname{frc}(x)$, and it is called *a Benford continuation of H*.

Recall Definition 1.6. Then, $\mathfrak{H}$ is a uniform continuation of H, and $\mathfrak{H}_\infty(p) = \frac{\psi^p - 1}{\psi - 1}$ for all $p \in [0, 1]$. We leave the proof of the following to the reader.

Lemma 6.12. *For real numbers $x \in [1, \infty)$, we have $\mathfrak{H}(x) = \delta\psi^x(1 + o(1))$, and $\mathfrak{H}^{-1}(x) = \log_\psi(x) - \log_\psi \delta + o(1)$.*

Recall $\mathscr{H}_s$ from Definition 6.8 and $\widehat{H}$ from Definition 6.5.

Lemma 6.13. *Let K be a sequence of positive real numbers approaching ∞. Let $\mathbf{b} \in \mathscr{H}_s$, and let $A_{\mathbf{b}} := \{n \in \mathbb{N} : \mathrm{LB}_s(K_n) = \mathbf{b}\}$. Then, there are real numbers $\gamma_n = o(1)$ and $\widetilde{\gamma}_n = o(1)$ such that $n \in A_{\mathbf{b}}$ if and only if*

$$\log_\psi \mathbf{b} \cdot \widehat{H} + \gamma_n \;\le\; \operatorname{frc}\left(\mathfrak{H}^{-1}(K_n)\right) \;<\; \log_\psi \widetilde{\mathbf{b}} \cdot \widehat{H} + \widetilde{\gamma}_n, \tag{11}$$

where $\widetilde{\gamma}_n = 0$ when $\mathbf{b}$ is the largest leading block of length s.

There is no difficulty in applying the arguments of the proof of Lemma 4.4 to Lemma 6.13, and we leave the proof to the reader.

Recall Definition 6.10.

Theorem 6.14. *Let K be an increasing sequence of positive integers such that $\operatorname{frc}\left(\mathfrak{H}^{-1}(K_n)\right)$ is equidistributed. Then, K satisfies strong Benford's Law under the $\mathscr{H}$-expansion.*

There is no difficulty in applying the arguments of the proof of Theorem 4.5 to Theorem 6.14, and we leave the proof to the reader.

6.4 Absolute Benford's Law

Let $\mathscr{H}$, H, and ψ be as defined in Definition 6.1, 6.3, and 6.5. Introduced in [10] is a full generalization of Zeckendorf expressions, which is based on the very principle of how Zeckendorf expressions are constructed in terms of lexicographical order. In this most general sense, the collection $\mathscr{H}$ is called a periodic Zeckendorf collection of coefficient functions. We believe that a property concerning all periodic Zeckendorf collections may be noteworthy, and as in the notion of normal numbers, we introduce the following definition.

Definition 6.15. A sequence K of positive integers is said to *satisfy absolute Benford's Law* if K satisfies strong $\mathscr{H}$-Benford's Law for each periodic Zeckendorf collection $\mathscr{H}$.

Recall the Lucas sequence $K = (2, 1, 3, 4, \dots)$ from Example 5.10. It satisfies strong Benford's Law under all base-b expansions, but it does not satisfy strong Benford's Law under Zeckendorf expansion. Thus, the Lucas sequence does not satisfy absolute Benford's Law.

Theorem 6.16. *Let γ be a positive real number such that γ is not equal to ψ^r for any $r \in \mathbb{Q}$ and any dominant real zero ψ of $g_{\mathscr{H}}$ where $\mathscr{H}$ is as defined in Definition 6.5. Let K be the sequence given by $K_n = \lfloor \gamma^n \rfloor$. Then, K satisfies absolute Benford's Law.*

Proof. Let H and ψ be as defined in Definitions 6.3 and 6.5, and let $\mathfrak{H}$ be the Benford continuation defined in Definition 6.11. Note that ψ is algebraic. Notice that $\lfloor \gamma^n \rfloor = \gamma^{n+o(1)}$, and $\log_\psi(\gamma)$ is irrational. Thus, by Lemma 6.12,

$$\mathfrak{H}^{-1}(K_n) \;=\; (n + o(1))\log_\psi(\gamma) - \log_\psi(\delta) + o(1) \;=\; n\log_\psi(\gamma) - \log_\psi(\delta) + o(1).$$

By Weyl's Equidistribution Theorem,

$$\Rightarrow\; \operatorname{Prob}\left\{\, n \in \mathbb{N} : \operatorname{frc}\left(\mathfrak{H}^{-1}(K_n)\right) \le \beta \,\right\} \;=\; \operatorname{Prob}\left\{\, n \in \mathbb{N} : \operatorname{frc}\left(n\log_\psi(\gamma)\right) \le \beta \,\right\} \;=\; \beta.$$

By Theorem 6.14, K satisfies Benford's Law under $\mathscr{H}$-expansion. □

Corollary 6.17. *Let $\gamma > 1$ be a real number that is not an algebraic integer. Then, the sequence K given by $K_n = \lfloor \gamma^n \rfloor$ satisfies absolute Benford's Law.*

Proof. The dominant real zero ψ defined in Definition 6.5 is an algebraic integer, and so is ψ^r for all $r \in \mathbb{Q}$. Thus, if $\gamma \in \mathbb{R}$ is not an algebraic integer, then by Theorem 6.16, K satisfies absolute Benford's Law. □

Example 6.18. Let K be the sequence given by $K_n = \left\lfloor \frac{\phi}{\sqrt{5}} (\frac{89}{55})^n \right\rfloor$, which is considered in the introduction. Since $\frac{89}{55}$ is not an algebraic integer, by Corollary 6.17, the sequence K satisfies absolute Benford's Law.

6.5 Other Continuations

Let $\mathscr{H}$, H, and ψ be as defined in Definition 6.1, 6.3, and 6.5, and recall Definition 1.6. As in Section 5, we relate other continuations of H to the distributions of leading blocks under $\mathscr{H}$-expansion.

Recall the Benford continuation $\mathfrak{H}$ from Definition 6.11, uniform continuations h and h_∞ from Definition 1.6, and the definition of $\widetilde{\mathbf{b}}$ from Definition 6.8.

Theorem 6.19. *Let $h : [1,\infty) \to \mathbb{R}$ be a uniform continuation of H. Then, there is a sequence K of positive integers approaching ∞, e.g., $K_n = \lfloor \mathfrak{H}(n + \mathfrak{H}_n{}^{-1} \circ h_n(\operatorname{frc}(n\pi))) \rfloor$, such that* $\operatorname{frc}(h^{-1}(K_n))$ *is equidistributed.*

Let K be a sequence of of positive integers approaching ∞ such that $\operatorname{frc}(h^{-1}(K_n))$ *is equidistributed. Let $\mathbf{b} \in \mathscr{H}_s$. Then,*

$$\begin{aligned}\operatorname{Prob}\{\, n \in \mathbb{N} : \operatorname{LB}_s(K_n) = \mathbf{b} \,\} &= h_\infty{}^{-1} \circ \mathfrak{H}_\infty(\log_\psi \widetilde{\mathbf{b}} \cdot \widehat{H}) - h_\infty{}^{-1} \circ \mathfrak{H}_\infty(\log_\psi \mathbf{b} \cdot \widehat{H}) \\ &= h_\infty{}^{-1}\left(\frac{\widetilde{\mathbf{b}} \cdot \widehat{H} - 1}{\psi - 1}\right) - h_\infty{}^{-1}\left(\frac{\mathbf{b} \cdot \widehat{H} - 1}{\psi - 1}\right).\end{aligned}$$

There is no difficulty in applying the arguments of the proof of Theorem 5.6 to Theorem 6.19, and we leave the proof to the reader.

Recall that $\mathbf{I} = (0,1)$. As in Definition 5.11, we introduce expressions for $\mathbf{I}$ that are associated with $\mathscr{H}$. Recall also the infinite tuple Θ, θ, and $\widehat{H}$, from Definitions 6.1 and 6.5.

Definition 6.20. An infinite tuple $\mu \in \prod_{k=1}^{\infty} \mathbb{N}_0$ is called an *$\mathscr{H}$-expression for* $\mathbf{I}$ if there is a smallest $i \in \mathbb{N}$ such that $\mu(i) > 0$, $(\mu(i), \dots, \mu(k)) \in \mathscr{H}$ for all $k \geq i$, and for all $j \in \mathbb{N}_0$, the sequence $\{\mu(j+n)\}_{n=1}^{\infty}$ is not equal to the sequence $\{\Theta(n)\}_{n=1}^{\infty}$. Let $\mathscr{H}^*$ be the set of $\mathscr{H}$-expressions for $\mathbf{I}$.

Given $s \in \mathbb{N}$ and $\{\mu, \tau\} \subset \mathscr{H}^*$, we declare $\mu|s < \tau|s$ if $\mu|s \cdot \widehat{H} < \tau|s \cdot \widehat{H}$, which coincides with the lexicographical order on $\mathbb{N}_0^s$. We define $\mu \cdot \widehat{H} := \sum_{k=1}^{\infty} \mu(k)\theta^{k-1}$, which is a convergent series.

Theorem 6.21 and Proposition 6.22 below are proved in [10].

Theorem 6.21 (Zeckendorf Theorem for $\mathbf{I}$). *Given a real number $\beta \in \mathbf{I}$, there is a unique $\mu \in \mathscr{H}^*$ such that $\beta = \sum_{k=1}^{\infty} \mu(k)\theta^k = (\mu \cdot \widehat{H})\theta$.*

Proposition 6.22. *Let $\{\mu, \tau\} \subset \mathscr{H}^*$. Then, $\mu \cdot \widehat{H} < \tau \cdot \widehat{H}$ if and only if $\mu|s < \tau|s$ for some $s \in \mathbb{N}$.*

By Theorem 6.21, Proposition 6.22 and (9), the function from $\{\mu \in \mathscr{F}^* : \mu(1) = 1\}$ to $[0,1)$ given by the following is bijective:

$$\mu \mapsto \frac{\mu \cdot \widehat{H} - 1}{\psi - 1},$$

and hence, h_K^* defined in Definition 6.23 is well defined.

Definition 6.23. Let K be a sequence of positive integers approaching ∞ such that given $\mu \in \mathscr{H}^*$ such that $\mu(1) = 1$, the following limit exists:

$$\lim_{s\to\infty} \operatorname{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_s(K_n) \le \mu|s \right\}. \tag{12}$$

Let $h_K^* : [0,1] \to [0,1]$ be the function given by $h_K^*(0) = 0$, $h_K^*(1) = 1$, and $h_K^*\left(\frac{\mu\cdot\widehat{H}-1}{\psi-1}\right)$ is equal to the value in (12). If h_K^* is continuous and increasing, then K is said to *have continuous leading block distribution under $\mathscr{H}$-expansion*.

Theorem 6.24. *Let K be a sequence with continuous leading block distribution under $\mathscr{H}$-expansion. Let h_K^* be the function defined in Definition 6.23. Then, there is a uniform continuation h of H_n such that ${h_\infty}^{-1} = h_K^*$ and* $\operatorname{frc}\left(h^{-1}(K_n)\right)$ *is equidistributed.*

There is no difficulty in applying the arguments of the proof of Theorem 5.18 to Theorem 6.24, and we leave the proof to the reader.

7 Benford behavior within expansions

As mentioned in the introduction, Benford's Law under base-b expansion arises with Zeckendorf expansion, and let us review this result, which is available in [4].

Let $\mathscr{K}$ be a periodic Zeckendorf collection defined in Definition 6.1, and let K be the fundamental sequence of $\mathscr{K}$, defined in Definition 6.3. Let S be an infinite subset of $\{K_n : n \in \mathbb{N}\}$ such that $q(S) := \operatorname{Prob}\left\{ n \in \mathbb{N} : K_n \in S \right\}$ exists. Recall the product $*$ from Definition 2.2. For a randomly selected integer $n \in [1, K_{t+1})$, let $\mu * K$ be the $\mathscr{K}$-expansion of n, let $M = \operatorname{len}(\mu)$, and define

$$P_t(n) := \frac{\sum_{k=1}^{M} \mu(k)\chi_S(K_k)}{\sum_{k=1}^{M} \mu(k)} \tag{13}$$

where χ_S is the characteristic function on $\{K_k : k \in \mathbb{N}\}$, i.e., $\chi_S(K_k) = 1$ if $K_k \in S$ and $\chi_S(K_k) = 0$, otherwise. Proved in [3] is that given a real number $\epsilon > 0$, the probability of $n \in [1, K_{t+1})$ such that $|P_t(n) - q(S)| < \epsilon$ is equal to $1 + o(1)$ as a function of t. For Benford behavior, we let S be the set of K_n that have (fixed) leading decimal digit d. Then, $q(S) = \log_{10}(1 + \frac{1}{d})$, and the probability of having a summand K_n with leading digit d within the $\mathscr{K}$-expansion is nearly $q(S)$ most of the times.

This result immediately applies to our setup. Let $\mathscr{H}$ and H be as defined in Definition 6.1 different from $\mathscr{K}$ and K. For example, let $\mathscr{H}$ be the base-b expressions, and let $\mathscr{K}$ be the Zeckendorf expressions. Then, H is the sequence given by $H_n = b^{n-1}$ and $K = F$ is the Fibonacci sequence. Recall from Definition 6.8 that $\mathscr{H}_s$ is a set of leading blocks under $\mathscr{H}$-expansion, and that $\mathrm{LB}_s^{\mathscr{H}}(n)$ denotes the leading block of n in $\mathscr{H}_s$ under $\mathscr{H}$-expansion. By Corollary 4.7, the sequence K satisfies (strong) Benford's Law under $\mathscr{H}$-expansion, i.e.,

$$\operatorname{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_s^{\mathscr{H}}(K_n) = \mathbf{b} \right\} \;=\; \log_\psi \frac{\widetilde{\mathbf{b}} \cdot \widehat{H}}{\mathbf{b} \cdot \widehat{H}}$$

where $\mathbf{b} \in \mathscr{H}_s$ and $\psi = b$, and this is Benford's Law under base-b expansion. The case considered in the introduction is that $\mathscr{H}$ is the Zeckendorf expansion and $\mathscr{K}$ is the binary expansion. The following is a corollary of [4, Theorem 1.1]. Recall Definition 6.5.

Theorem 7.1. *Let $\mathscr{H}$ and H be as defined in Definition 6.1, and Let K be the fundamental sequence of a periodic Zeckendorf collection $\mathscr{K}$ such that $\psi^r_{\mathscr{H}} \neq \psi_{\mathscr{K}}$ for all $r \in \mathbb{Q}$ where $\psi_{\mathscr{H}}$ and $\psi_{\mathscr{K}}$ are the dominant real zeros of $g_{\mathscr{H}}$ and $g_{\mathscr{K}}$, respectively. Given $\mathbf{b} \in \mathscr{H}_s$, let $S_{\mathbf{b}} := \{K_n : \mathrm{LB}_s^{\mathscr{H}}(K_n) = \mathbf{b},\ n \in \mathbb{N}\}$. For a randomly selected integer $n \in [1, K_{t+1})$, let $P_t(n)$ be the proportion defined in (13) with respect to $S = S_{\mathbf{b}}$. Then, given a real number $\epsilon > 0$, the probability of $n \in [1, K_{t+1})$ such that*

$$\left| P_t(n) - \log_{\psi_{\mathscr{H}}} \frac{\widetilde{\mathbf{b}} \cdot \widehat{H}}{\mathbf{b} \cdot \widehat{H}} \right| < \epsilon$$

is equal to $1 + o(1)$ as a function of t.

8 Future work

Instead of the leading digit, one can look at the distribution of the digit in the second, third, or generally any location. For a sequence that is strong Benford, the further to the right we move in location, the more uniform is the distribution of digits. A natural question is to ask whether or not a similar phenomenon happens with Zeckendorf decompositions, especially as there is a natural furthest to the right one can move.

We can also look at signed Zeckendorf decompositions. Alpert [1] proved that every integer can be written uniquely as a sum of Fibonacci numbers and their additive inverses where if two consecutive summands have the same sign then their indices differ by at least 4 and if they are of opposite sign then their indices differ by at least 3. We now have more possibilities for the leading block, and one can ask about the various probabilities. More generally, one can consider the f-decompositions introduced in [13], or the non-periodic Zeckendorf collections introduced in [10].

Additionally, one can explore sequences where there is no longer a unique decomposition, see for example [5, 6, 7, 8, 9], and ask what is the distribution of possible leading blocks. There are many ways we can formulate this question. We could look at all legal decompositions, we could look at what happens for specific numbers, we could look at what happens for specific types of decompositions, such as those arising from the greedy algorithm or those that use the fewest or most summands.

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