

# ON SMOOTHING SINGULARITIES OF ELLIPTIC ORBITAL INTEGRALS ON $GL(n)$ AND BEYOND ENDOSCOPY

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ABSTRACT. Recent work of Altuğ completes the preliminary analysis of Langlands' Beyond Endoscopy proposal for  $GL(2)$  and the standard representation. We show that Altuğ's method of smoothing the real elliptic orbital integrals using an approximate functional equation extends to  $GL(n)$ . We also discuss the case of an arbitrary reductive group, and obstructions for generalizing the analysis of the  $p$ -adic orbital integrals.

## CONTENTS

1. Introduction	1
2. Preliminaries	4
3. Approximate Functional Equation	7
4. Smoothing of the real orbital integral	10
5. Further directions	12
Appendix A. The $p$ -adic orbital integrals	13
References	14

## 1. INTRODUCTION

**1.1. Overview of Beyond Endoscopy.** One of the key conjectures of the Langlands Program is the Functoriality Conjecture: given two reductive groups  $G'$  and  $G$ , and an  $L$ -homomorphism of the associated  $L$ -group  ${}^L G'$  to  ${}^L G$ , one expects a transfer of automorphic forms on  $G'$  to automorphic forms on  $G$ . Most cases of functoriality known today fall under the banner of endoscopy, that is, where  $G'$  is an endoscopic group of  $G$ . The problem of endoscopy is addressed by the stable trace formula, recently made unconditional by Ngô's solution of the Fundamental Lemma [Ngô]. Anticipating this, Langlands proposed a new strategy to attack the general case, referred to as Beyond Endoscopy [Lan1].

If an automorphic form  $\pi$  on  $G$  is a functorial transfer from a smaller  $G'$ , then one expects the  $L$ -function  $L(s, \pi, r)$  to have a pole at  $s = 1$  for some representation  $r$  of  ${}^L G$ . In particular, the order of  $L(s, \pi, r)$  at  $s = 1$ , which we denote by  $m_r(\pi)$ ,

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should be nonzero if and only if  $\pi$  is a transfer. Langlands' idea then is to weight the spectral terms in the stable trace formula by  $m_r(\pi)$ , resulting in a trace formula whose spectral side detects only  $\pi$  for which the  $m_r(\pi)$  is nonzero.

Since in general  $L(s, \pi, r)$  is not a priori defined at  $s = 1$ , we account for the weight factor by taking the residue at  $s = 1$  of the logarithmic derivative of  $L(s, \pi, r)$ .<sup>1</sup> This should lead to an  $r$ -trace formula,

$$S_{\text{cusp}}^r(f) = \lim_{N \rightarrow \infty} \frac{1}{|V_N|} \sum_{v \in V_N} \log(q_v) S_{\text{cusp}}^1(f_v^r), \quad (1.1)$$

where  $S_{\text{cusp}}^1$  represents the usual stable trace formula,  $V_N$  is a finite set of valuations of the global field  $F$ , and  $q_v$  the order of the residue field of  $F_v$  is less than  $N$  (see [Art, §2] for details).

Following Arthur [Art], the stable distribution should have a decomposition

$$S_{\text{cusp}}^r(f) = \sum_{G'} \iota(r, G') \hat{P}_{\text{cusp}}^{\tilde{G}'}(f'), \quad (1.2)$$

where  $\hat{P}_{\text{cusp}}^{\tilde{G}'}(f')$  are called primitive stable distributions on elliptic ‘beyond endoscopic’ groups  $G'$ , and by primitive one means the spectral contribution to the stable trace formula of tempered, cuspidal automorphic representations that are not functorial images from some smaller group. These primitive distributions, giving a new *primitive* trace formula, are to be defined inductively, and one hopes to establish these from the  $r$ -trace formula.

As is usual with trace formulae, one would like both the  $r$ -trace formula and the primitive trace formula to be an identity of spectral and geometric sides. But since one only wants tempered automorphic representations to contribute to (1.1), one has to first remove the contribution of the nontempered representations. Inspired by work of Ngô, a suggestion was put forth in [FLN] to apply Poisson summation to the elliptic contribution, over a linear space called the Steinberg-Hitchin base. There it was shown that the dominant term of the Poisson summation canceled with the contribution of the trivial representation, the most nontempered term. One of the key issues that arise is the singularities of orbital integrals, which prevent the use of the Poisson formula, and is discussed in [Lan2] for the group  $SL(2)$ .

In related work, the recent thesis of Altuğ completes the preliminary analysis carried out in [Lan1, Part II], for  $GL(2)$  and the standard representation ([Alt1], see also [Alt2]). Working over  $\mathbb{Q}$ , and restricting ramification to the infinite prime, Altuğ smooths the singularities of the archimedean orbital integral by expressing the volume factors as values of Hecke  $L$ -functions and a strategic application of the approximate functional equation. By a detailed analysis, Altuğ shows that not only does the trivial representation contribute to the dominant term of the Poisson summation, but also the continuous spectral term associated to the nontrivial Weyl element of  $GL(2)$ . Based on this analysis, Arthur outlines in [Art] a list of problems to be addressed in order to establish the primitive trace formula. Finally, we mention recent work of Mok [Mok] that establishes a weak form of the  $r$ -trace formula for odd orthogonal groups with  $r$  being the standard or second fundamental representation, using the endoscopic classification of orthogonal groups and known properties of the relevant  $L$ -functions.

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<sup>1</sup>Another possibility is to take the residue of  $L(s, \pi, r)$  itself at  $s = 1$ , in which case  $m_r(\pi)$  is more complicated.

**1.2. Main result.** In this paper, we study to what extent Altuğ's method generalizes to a more general reductive group, focusing on Problem III in [Art]. Relying on Shelstad's characterization of real orbital integrals [Sh], we show that Altuğ's use of the approximate functional equation to smooth the singularities of real orbital integrals can be generalized to  $GL(n)$ .

In particular, we apply Altuğ's method [Alt1] to study the elliptic part of the trace formula of  $G = GL(n)$ ,

$$\sum_{\gamma \text{ ell}} \text{meas}(\gamma) \int_{G_\gamma(\mathbb{A}) \backslash G(\mathbb{A})} f(g^{-1}\gamma g) dg, \quad (1.3)$$

the sum is taken over elliptic conjugacy classes of  $G(\mathbb{Q})$ . Choosing test functions as in §2.1, we rewrite it in (2.20) as

$$\sum_{\pm p^k \text{ tr}(\gamma), \dots, \text{tr}(\gamma^{n-1})} \sum_{\frac{1}{|s_\gamma|} L(1, \sigma_E) \theta_\infty^\pm(\gamma) \prod_q \text{Orb}(f_q; \gamma)}, \quad (1.4)$$

where  $E$  is the extension of  $\mathbb{Q}$  defined by the elliptic element  $\gamma$ ,  $\sigma_E$  the Galois representation appearing in the factorization  $\zeta_E(s) = \zeta_{\mathbb{Q}}(s)L(s, \sigma_E)$ , and the product is taken over all primes  $q$  of  $\mathbb{Q}$ . (See Section 2 for precise definitions.) Here the  $L$ -value represents the global volume term  $\text{meas}(\gamma)$  before.

We show that the approximate functional equation can be used again to smooth the archimedean orbital integral. By using Shelstad's characterization of real orbital integral ([Sh]), we have:

**Theorem 1.1.** *Let  $\theta_\infty^\pm(\gamma)$  be defined as in (2.19),  $\phi$  any Schwartz function on  $\mathbb{R}$ , and  $\alpha > 0$ . Then the function defined by*

$$f(x_1, \dots, x_{n-1}) = \theta_\infty^\pm(x_1, \dots, x_{n-1}) \phi(|D(x_1, \dots, x_{n-1})|^{-\alpha}) \quad (1.5)$$

*is smooth.*

In particular, we take  $\phi$  to be the cutoff functions  $V_s$  and  $V_{1-s}$  in the approximate functional equation (cf. Theorem 3.1), we obtain the main result:

**Corollary 1.2.** *Assume Artin's conjecture for  $n > 3$ .*

$$L(1, \sigma_E) \theta_\infty^\pm(\gamma) \quad (1.6)$$

*is smooth. This result is unconditional for  $n = 2, 3$ .*

This result represents a first step in towards establishing (1.1) for general groups. In particular, in order to apply Poisson summation over the Steinberg-Hitchin base one must address the singularities of the orbital integrals. As a side note, in establishing the invariant trace formula Arthur circumvents this by applying Fourier transform to the spectral terms instead.

We should point out that our application of the approximate functional equation is not completely similar to Altuğ's for reasons related to the  $p$ -adic orbital integrals. The obstruction to directly applying Altuğ's analysis is the following: in [Lan1, Lemma1], the  $p$ -adic orbital integrals are expressed in terms of the quadratic residue symbol  $(\frac{D}{p})$ , so that in [Alt1] they are combined with the volume terms  $L(1, (\frac{D}{p}))$  into an auxiliary Dirichlet series. The approximate functional equation is then applied to this Dirichlet series. In the general case, it is not clear that the  $p$ -adic orbital integrals can be related to the Artin representation obtained. More importantly, for general groups one does not have a closed formula for evaluating

orbital integrals, with certain exceptions like  $GL(3)$  due to [Kot1] (see the discussion in Appendix A). For this reason, we expect that new methods will be required to address the general case.

**1.3. Outline.** This paper is organized as follows: In Section 2, we introduce the necessary definitions and notation, and using the class number formula arrive at (1.4). In Section 3, we introduce the approximate functional equation for Artin  $L$ -functions. In Section 4, we describe the characterization of orbital integrals on real reductive groups, and prove Theorem 1.1. Finally, in Section 5 we give indications on how our analysis can be generalized to general reductive groups, using work of Ono and Shyr on Tamagawa numbers of algebraic tori. In Appendix A, we state the  $p$ -adic orbital integrals for  $GL(3)$ , and briefly discuss the problems in higher rank.

## 2. PRELIMINARIES

**2.1. Notation.** We follow closely the setting of [Lan1] and [Alt1]. Let  $G = GL(n)$  and  $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$  be the ring of adèles of  $\mathbb{Q}$ . Denote by  $v$  any valuation of  $\mathbb{Q}$ ,  $q$  any finite prime, and  $p$  a fixed prime.

An element  $\gamma \in G(\mathbb{Q})$  is said to be *elliptic* if its characteristic polynomial is irreducible over  $\mathbb{Q}$ . Let  $Z_+$  be the set of all matrices in the center of  $G(\mathbb{R})$  with positive entries, and  $G_\gamma$  be the centralizer of  $\gamma$  in  $G$ . The discriminant of  $\gamma$  is given by

$$D_\gamma = \prod_{i < j} (\gamma_i - \gamma_j)^2, \quad (2.1)$$

where the  $\gamma_i$ 's are the distinct eigenvalues of  $\gamma$ . An elliptic element  $\gamma$  defines, by its characteristic polynomial, a degree  $n$  extension  $E$  of  $\mathbb{Q}$ , such that

$$D_\gamma = s_\gamma^2 D_E. \quad (2.2)$$

for some integer  $s_\gamma$ , and  $D_E$  is the discriminant of  $E$ .

Now let  $f = \prod_v f_v$  be a function in  $C_c^\infty(G(\mathbb{A}))$ . Define the global volume term

$$\text{meas}(\gamma) = \text{meas}(Z_+ G_\gamma(\mathbb{Q}) \backslash G_\gamma(\mathbb{A})) \quad (2.3)$$

and the orbital integral

$$\text{Orb}(f_v; \gamma) = \int_{G_\gamma(\mathbb{Q}_v) \backslash G(\mathbb{Q}_v)} f_v(g^{-1} \gamma g) dg_v. \quad (2.4)$$

The elliptic part of the Arthur-Selberg trace formula refers to

$$\sum_{\gamma \text{ ell}} \text{meas}(\gamma) \int_{G_\gamma(\mathbb{A}) \backslash G(\mathbb{A})} f(g^{-1} \gamma g) dg = \sum_{\gamma \text{ ell}} \text{meas}(\gamma) \prod_v \text{Orb}(f_v; \gamma), \quad (2.5)$$

where the sum is understood to be over representatives  $\gamma$  of elliptic conjugacy classes in  $G(\mathbb{Q})$ . Note that since  $G = GL(n)$ , these orbital integrals are in fact stable distributions.

The measures and test functions in (2.3) and (2.4) are to be chosen analogously as in [Alt1]. We mention the choices once again here. We first describe the choice of measure on  $G$ . At any finite prime  $q$ , we choose any Haar measure on  $G(\mathbb{Q}_q)$  giving measure 1 on  $G(\mathbb{Z}_q)$ , and the same with  $G_\gamma(\mathbb{Q}_q)$ ; at infinity, we choose any Haar measure on  $G(\mathbb{R})$ . In keeping with [Alt1, p.1797], we note that there are more

natural ways to normalize measures, but we make this choice so as to remain in consistent with the hypotheses of [Lan1, Alt1].

For a finite prime  $q$  and a nonnegative integer  $k$  let us first define  $f_q^{(r)} \in C_c(\mathbb{Q}_q)$  to be the characteristic function of the set

$$\{X \in \text{Mat}_n(\mathbb{Z}_q) : |\det(X)|_q = q^{-r}\}, \quad (2.6)$$

where  $\text{Mat}_n(\mathbb{Z}_q)$  denotes the set of  $n$  by  $n$  matrices with entries in  $\mathbb{Z}_q$ , and  $|\cdot|_q$  the  $q$ -adic absolute value on  $\mathbb{Q}$ .

The local functions  $f_v \in C_c^\infty(\mathbb{Q}_v)$  are defined as follows.

- At  $q \neq p$ , choose  $f_q^{(0)}$ .
- At  $p$ , choose  $f_p^{p,k} := p^{-k/2} f_p^{(k)}$  for a fixed  $k > 0$ .
- At  $\infty$ , choose  $f_\infty \in C_c^\infty(Z_+ \backslash G(\mathbb{R}))$  such that its orbital integrals are compactly supported.

Note that the choice of  $f_p$  relates to  $L(s, \pi, \text{Sym}^k)$ , and the rest meaning we only allow ramification at infinity. For further discussion of these choices, see [Lan1, §2].

Finally, define  $f^{p,k}$  by

$$f^{p,k} := f_\infty \cdot f_p^{p,k} \cdot \prod_{q \neq p} f_q^{p,k}. \quad (2.7)$$

This is the test function we shall use throughout.

**2.2. Class number formula and measures.** Denote by  $\mathbb{A}_E$  the adèle ring of  $E$ , and  $I_E = \mathbb{A}_E^\times$  the ideles of  $E$ . Let  $|\cdot|_v$  be the normalized absolute value on the completion  $E_v$  and  $|\cdot|_{\mathbb{A}_E} : I_E \rightarrow \mathbb{R}^\times$  be the absolute value defined by

$$|x|_{\mathbb{A}_E} = \prod_v |x_v|_v, \quad (2.8)$$

where  $x = (x_v)$ . Here  $|\cdot|_{\mathbb{A}_E}$  is a group homomorphism and we define the norm-one idele group to be its kernel, denoted  $I_E^1$ , and  $E^\times \backslash I_E^1$  the norm-one idele class group.

Let  $\gamma$  be an elliptic element in  $G(\mathbb{Q})$ . Recall that it defines a degree  $n$  extension  $E$  of  $\mathbb{Q}$ . It was observed by Langlands (c.f. Equation (19) in [Lan1]) that

$$Z_+ G_\gamma(\mathbb{Q}) \backslash G_\gamma(\mathbb{A}) = Z_+ E^\times \backslash I_E = E^\times \backslash I_E^1. \quad (2.9)$$

Having chosen the measures on  $Z_+ G_\gamma(\mathbb{Q}) \backslash G_\gamma(\mathbb{A})$  as above, we then require the measure on  $E^\times \backslash I_E^1$  to be such that

$$\text{meas}(Z_+ G_\gamma(\mathbb{Q}) \backslash G_\gamma(\mathbb{A})) = \text{meas}(E^\times \backslash I_E^1). \quad (2.10)$$

With this choice of measure on  $E^\times \backslash I_E^1$ , by Tate's thesis [T, Theorem 4.3.2], we have

$$\text{meas}(E^\times \backslash I_E^1) = \frac{2^{r_1} (2\pi)^{r_2} h_E R_E}{w_E}, \quad (2.11)$$

where  $h_E$  is the class number of  $E$ ,  $R_E$  is the regulator of  $E$ ,  $r_1$  is the number of real embeddings of  $E$  and  $r_2$  is the number of pairs of complex embeddings of  $E$ , and  $w_E$  is the number of roots of unity in  $E$ .

**Definition 2.1.** Let  $E$  be a number field. The Dedekind zeta function of  $E$  is defined by

$$\zeta_E(s) = \sum_{\mathfrak{a}} \frac{1}{N_{E/\mathbb{Q}}(\mathfrak{a})^s} = \prod_{\mathfrak{p}} \frac{1}{1 - N_{E/\mathbb{Q}}(\mathfrak{p})^{-s}} \quad (2.12)$$

for  $\Re s > 1$ , where the sum is over all non-zero integral ideals of  $E$  and the product is over all prime ideals of  $E$ .

Hecke showed that Dedekind zeta function  $\zeta_E(s)$  can be analytically continued to the whole complex plane except having a simple pole at  $s = 1$ . Recall the classical Dirichlet class number formula

$$\operatorname{Res}_{s=1} \zeta_E(s) = \frac{2^{r_1} (2\pi)^{r_2} h_E R_E}{w_E \sqrt{|D_E|}}, \quad (2.13)$$

which will be related to the idele class group via (2.11).

**Definition 2.2.** Let  $E$  be a number field. Let  $\rho : \operatorname{Gal}(E/\mathbb{Q}) \rightarrow \operatorname{GL}(d, \mathbb{C})$  be a finite dimensional Galois representation. The Artin  $L$ -function associated to  $\rho$  is given in terms of the Euler product:

$$L(s, \rho) = \prod_p \det(I - \rho(\operatorname{Fr}_p) p^{-s})^{-1}, \quad (2.14)$$

where  $\operatorname{Fr}_p$  is the Frobenius element in  $\operatorname{Gal}(E/\mathbb{Q})$ .

Denote by  $\sigma_E$  the Artin representation obtained from the factorization  $\zeta_E(s) = \zeta_{\mathbb{Q}}(s) L(s, \sigma_E)$ , where  $\zeta_{\mathbb{Q}}$  is the Riemann zeta function. Note that  $\sigma_E$  may be reducible.

By equations (2.11) and (2.13), we have

$$\operatorname{meas}(E^\times \backslash I_E^1) = \sqrt{|D_E|} L(1, \sigma_E), \quad (2.15)$$

hence we have

$$\operatorname{meas}(Z_+ G_\gamma(\mathbb{Q}) \backslash G_\gamma(\mathbb{A})) = \sqrt{|D_E|} L(1, \sigma_E). \quad (2.16)$$

**2.3. Rewriting the elliptic term.** By the choice of test function  $f^{p,k}$  in (2.7), the right hand side of (2.5) is non-vanishing if and only if  $|\det(\gamma)|_q = 1$  for any finite prime  $q \neq p$  and  $|\det(\gamma)|_p = p^{-k}$ . Therefore  $\det(\gamma) \in \mathbb{Z}_q$  for any finite prime  $q$  and  $\det(\gamma) \in \mathbb{Z}$ . Moreover,  $|\det(\gamma)| = p^k$ .

Also  $\gamma$  is elliptic if and only if  $D_\gamma$  is not a perfect square. We parametrize  $\gamma \in G(\mathbb{Q})$  by the coefficients of its characteristic polynomial,

$$X^n - a_1 X^{n-1} + \cdots + (-1)^n a_n \quad (2.17)$$

where

$$(a_1, \dots, a_n) = (\operatorname{tr}(\gamma), \dots, \operatorname{tr}(\gamma^{n-1}), \det(\gamma)). \quad (2.18)$$

By (2.18) and  $\det(\gamma) = \pm p^k$ , we write

$$\theta_\infty^\pm(\gamma) = \theta_\infty^\pm(\operatorname{tr}(\gamma), \dots, \operatorname{tr}(\gamma^{n-1}), \pm p^k), \quad (2.19)$$

where  $\theta_\infty^\pm(\gamma) = |D_\gamma|^{1/2} \operatorname{Orb}(f_\infty; \gamma)$ , since  $\theta_\infty$  is invariant under conjugation, therefore we can consider  $\theta_\infty^\pm(\gamma)$  as a function on  $\mathbb{R}^{n-1}$ .

The right hand side of (2.5) then becomes

$$\sum_{\pm p^k} \sum' \frac{1}{|s_\gamma|} L(1, \sigma_E) \theta_\infty^\pm(\gamma) \prod_q \operatorname{Orb}(f_q^{p,k}; \gamma), \quad (2.20)$$

where the inner sum is taken over  $n$ -tuples  $(\operatorname{tr}(\gamma), \dots, \operatorname{tr}(\gamma^{n-1}))$  in  $\mathbb{Z}^{n-1}$ , with  $D_\gamma$  non-square.

*Remark 2.3.* The discriminant that appears in the class number formula is that of the number field, whereas the discriminant that appears in the orbital integral is that of the characteristic polynomial. Since  $s_\gamma \neq 0$ , the factor  $1/|s_\gamma|$  does not affect our proof of smoothness.

In order to establish the  $r$ -trace formula, one would like to be able to apply the Poisson summation formula to (2.20), so as to remove the contribution of the nontempered spectrum. However as noted in [Alt1, p.1799], one of the issues is the smoothness of  $\theta_\infty^\pm(\gamma)$ , which may have singularities on the variety  $D_\gamma = 0$ , described in Section 4. We shall use the approximate functional equation of Artin  $L$ -function to smooth out the singularities of  $\theta_\infty^\pm(\gamma)$  in the next two sections.

### 3. APPROXIMATE FUNCTIONAL EQUATION

In this section, we follow the exposition of [IK, pp. 94–95, 98, 125–126, 141–143] and show that the cutoff functions  $V_s$  and  $V_{1-s}$  in Theorem 3.1 are Schwartz. This will be used in the next section to show  $L(1, \sigma_E)\theta_\infty^\pm(\gamma)$  is smooth.

First, we recall the approximate functional equation for Artin  $L$ -functions.

**Theorem 3.1.** *Let  $L(s, \rho)$  be the Artin  $L$ -function associated to a Galois representation  $\rho$ , and assume the Artin Conjecture. Let  $G(u)$  be any function which is holomorphic and bounded in the strip  $-4 < \Re u < 4$ , even, and normalized by  $G(0) = 1$ . Let  $X > 0$ . Then for  $s$  in the strip  $0 \leq \Re s \leq 1$ , we have*

$$L(s, \rho) = \sum_n \frac{\lambda_\rho(n)}{n^s} V_s \left( \frac{n}{X\sqrt{q}} \right) + \epsilon(s, \rho) \sum_n \frac{\bar{\lambda}_\rho(n)}{n^{1-s}} V_{1-s} \left( \frac{nX}{\sqrt{q}} \right), \quad (3.1)$$

where

$$\begin{aligned} L(s, \rho) &= \sum_{n=1}^{\infty} \frac{\lambda_\rho(n)}{n^s} \\ V_s(y) &= \frac{1}{2\pi i} \int_{(3)} y^{-u} G(u) \frac{\gamma(s+u, \rho)}{\gamma(s, \rho)} \frac{du}{u} \\ \epsilon(s, \rho) &= \epsilon(\rho) q^{\frac{1}{2}-s} \frac{\gamma(1-s, \rho)}{\gamma(s, \rho)}; \end{aligned} \quad (3.2)$$

$\epsilon(\rho)$  is the root number of  $L(s, \rho)$  and is a complex number with modulus 1.

*Proof.* See Theorem 5.4 of [IK].  $\square$

Note that the formula is valid even without assuming the Artin conjecture, in which case there would be additional terms accounting for possible contributions of poles of  $L(s, \rho)$  along the lines  $\Re s = 0, 1$ .

**3.1. Gamma factors of Artin  $L$ -functions.** The gamma factor of the Artin  $L$ -function is a product of local gamma factors  $\gamma_v(s, \rho)$  over infinite places  $v$  of  $E$ . Let  $r_1$  and  $r_2$  be as before, so that  $r_1 + 2r_2 = n$ , where  $n$  is the degree of the number field  $E$ . Let  $\sigma_v$  be the Frobenius conjugacy class associated to the completion  $E_v$ . Then  $\sigma_v$  is of order 2 if  $v$  is a real place that extends to two complex places of  $L$ , and equals 1 otherwise. Hence, we have

$$\gamma_v(s, \rho) = \begin{cases} \pi^{-ds/2} \Gamma\left(\frac{s}{2}\right)^d \Gamma\left(\frac{s+1}{2}\right)^d & \text{if } v \text{ is a complex place} \\ \pi^{-ds/2} \Gamma\left(\frac{s}{2}\right)^{d_v^+} \Gamma\left(\frac{s+1}{2}\right)^{d_v^-} & \text{if } v \text{ is a real place,} \end{cases} \quad (3.3)$$

where  $d = \deg(\rho)$  is the dimension of  $\rho$  and  $d_v^+$ ,  $d_v^-$  are the multiplicities, respectively, of the eigenvalue  $+1$ ,  $-1$  for  $\rho(\sigma_v)$

If  $E$  is a degree  $n$  number field, then the degree of the representation  $\rho$  is  $n - 1$ . If  $\rho$  is a reducible representation, then  $L(\rho, s)$  further factorizes into a product of  $L$ -functions of non-trivial irreducible representations of  $\text{Gal}(E/\mathbb{Q})$ .

**Example 3.2.** Suppose  $E$  is a cubic field. We have the factorization  $\zeta_E(s) = \zeta_{\mathbb{Q}}(s)L(s, \sigma_E)$ , and the degree of the Galois representation  $\sigma_E$  is 2. Either  $(r_1, r_2) = (1, 1)$  or  $(r_1, r_2) = (3, 0)$ . We consider both cases.

- (1) Case (1):  $(r_1, r_2) = (1, 1)$ . The cubic extension  $E/\mathbb{Q}$  is not Galois, and  $\sigma_E$  is an irreducible representation.
- (2) Case (2):  $(r_1, r_2) = (3, 0)$ . The cubic extension is Galois, and  $\sigma_E$  is a reducible representation, decomposing as  $\chi$  and  $\chi^{-1}$ , where  $\chi$  is the cubic character. The conjugacy class is order two and our character is order three, so we only get  $d^+ = 2$  and  $d^- = 0$ . Hence, we have the following factorization:

$$\zeta_E(s) = \zeta_{\mathbb{Q}}(s)L(s, \chi)L(s, \chi^{-1}),$$

where  $L(s, \chi)$  is the Hecke  $L$ -function associated to the cubic character  $\chi$  defined by  $E$ .

One knows that the irreducible two-dimensional representation of  $\text{Gal}(E/\mathbb{Q}) = S_3$  corresponds to a modular form, hence  $L(s, \sigma_E)$  is entire in Case (1). Case (2) is simpler, and is known by class field theory.

*Remark 3.3.* The case of  $n = 2$  has been completed in [Alt1]. For  $n > 3$ , there could be Artin  $L$ -functions associated to irreducible representations of degree greater than 2. It was conjectured by Artin that such Artin  $L$  functions are indeed entire. However, this conjecture remains wide open. Hence for  $n > 3$ , we assume Artin's conjecture.

**3.2. The cutoff functions.** We now examine the cutoff functions in the approximate functional equation of Artin  $L$  functions for general Galois representation  $\rho$  with degree  $d$ .

**Theorem 3.4.** *The functions  $V_s$  and  $V_{1-s}$  are Schwartz, where  $s$  is any complex number such that  $0 \leq \Re s \leq 1$ .*

*Proof.* Let  $u = \sigma + it$ . Fix any  $s \in \mathbb{C}$  such that  $0 \leq \Re s \leq 1$ . We use the following form of Stirling's approximation:

$$|\Gamma(u)| = (2\pi)^{1/2} |t|^{\sigma-1/2} e^{-\pi|t|/2} (1 + O(|t|^{-1})). \quad (3.4)$$



Through shifting contours we show that  $V_s(y)$  is bounded by  $y^{-m}$  for any  $m$ ; below  $d^+, d^-$  denote the sum over complex  $v$  of  $d_v^+$  and  $d_v^-$ , respectively:

$$\begin{aligned}
 \gamma(s+u, \rho) &= \prod_{v|\infty} \gamma_v(s+u, \rho) \\
 &= \prod_{v \text{ real}} \left( \pi^{-d(s+u)/2} \Gamma\left(\frac{s+u}{2}\right)^{d_v^+} \Gamma\left(\frac{s+u+1}{2}\right)^{d_v^-} \right) \\
 &\quad \times \prod_{v \text{ complex}} \left( \pi^{-d(s+u)/2} \Gamma\left(\frac{s+u}{2}\right)^d \Gamma\left(\frac{s+u+1}{2}\right)^d \right) \\
 &= \pi^{-d(s+u)r_1/2} \Gamma\left(\frac{s+u}{2}\right)^{d^+} \Gamma\left(\frac{s+u+1}{2}\right)^{d^-} \\
 &\quad \times \pi^{-d(s+u)r_2} \Gamma\left(\frac{s+u}{2}\right)^{2dr_2} \Gamma\left(\frac{s+u+1}{2}\right)^{2dr_2} \\
 &= \pi^{-d(s+u)n/2} \Gamma\left(\frac{s+u}{2}\right)^{d^++2dr_2} \Gamma\left(\frac{s+u+1}{2}\right)^{d^-+2dr_2}. \tag{3.5}
 \end{aligned}$$

Recall that

$$V_s(y) = \frac{1}{2\pi i} \int_{(3)} y^{-u} G(u) \frac{\gamma(s+u, \rho)}{\gamma(s, \rho)} \frac{du}{u}. \tag{3.6}$$

Since  $G(u)$  is bounded in this region, we may without loss of generality assume that  $G(u)$  is identical to 1. Then

$$V_s(y) = \frac{1}{2\pi i} \frac{1}{\gamma(s, \rho)} \int_{(3)} y^{-u} \frac{\Gamma\left(\frac{s+u}{2}\right)^a \Gamma\left(\frac{s+u+1}{2}\right)^b}{u \pi^{d(s+u)n/2}} du, \tag{3.7}$$

where  $a := d^+ + 2dr_2$  and  $b := d^- + 2dr_2$ .

We now shift contours to  $(m)$ . We note that the integrand defining  $V_s$  has no poles in the region  $\Re s > 3$ , so it suffices to show that for the rectangle of height  $2T$  that is symmetric about the  $x$ -axis with vertical sides at  $(3)$  and  $(m)$ ,<sup>2</sup> the contour integrals along the upper and lower edges go to 0 as  $T \rightarrow \infty$ . In what follows we absorb all constant factors as  $C = C_s$ . First we treat the integral along the top

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<sup>2</sup>Unless  $m = 3$ , in which case no shift is necessary.

contour. Let  $a' := dn/2$ .

$$\begin{aligned}
& C \int_3^m y^{-(\sigma+iT)} \Gamma\left(\frac{s+\sigma+iT}{2}\right)^a \Gamma\left(\frac{s+\sigma+iT+1}{2}\right)^b \pi^{-a'(\sigma+iT)} \frac{d\sigma}{\sigma+iT} \\
& \ll \int_3^m y^{-\sigma} \Gamma\left(\frac{s+\sigma}{2} + i\frac{T}{2}\right)^a \Gamma\left(\frac{s+\sigma+1}{2} + i\frac{T}{2}\right)^b \pi^{-a'\sigma} \frac{d\sigma}{\sqrt{\sigma^2+T^2}} \\
& \ll \int_3^m y^{-\sigma} \left(\sqrt{2\pi} \left(\frac{T}{2}\right)^{(\sigma-1)/2} e^{-\pi T/4}\right)^a (1+O(T^{-a})) \\
& \quad \times \left(\sqrt{2\pi} \left(\frac{T}{2}\right)^{(\sigma-1)/2} e^{-\pi T/4}\right)^b (1+O(T^{-b})) \pi^{-a'\sigma} \frac{d\sigma}{\sqrt{\sigma^2+T^2}}, \tag{3.8}
\end{aligned}$$

where the last estimate follows from (3.4). We see that this is majorized by

$$\begin{aligned}
& \frac{e^{-a\pi T/4} e^{-b\pi T/4}}{T} \int_3^m \frac{\left(\frac{T}{2}\right)^{(a+b)\sigma/2}}{(\pi^{a'}y)^\sigma} d\sigma \\
& \ll \frac{e^{-(a+b)\pi T/4}}{T} \int_3^m \frac{T^{(a+b)\sigma/2}}{(2^{(a+b)/2}\pi^{a'}y)^\sigma} d\sigma \\
& \ll \frac{e^{-(a+b)\pi T/4}}{T} \int_3^m \left(\frac{T^{(a+b)/2}}{2^{(a+b)/2}\pi^{a'}y}\right)^m d\sigma \\
& \ll_m e^{-(a+b)\pi T/4} T^{(a+b)m/2-1}, \tag{3.9}
\end{aligned}$$

which tends to 0 as  $T \rightarrow \infty$ .

The calculation for the bottom contour follows similarly. Therefore we may shift contours to obtain

$$\begin{aligned}
& \int_{(3)} y^{-u} \frac{\Gamma\left(\frac{s+u}{2}\right)^a \Gamma\left(\frac{s+1+u}{2}\right)^b}{u\pi^{a'(s+u)}} du = \int_{(m)} y^{-u} \frac{\Gamma\left(\frac{s+u}{2}\right)^a \Gamma\left(\frac{s+u+1}{2}\right)^b}{u\pi^{a'(s+u)}} du \\
& = \int_{-\infty}^{\infty} y^{-(m+it)} \frac{\Gamma\left(\frac{s+(m+iT)}{2}\right)^a \Gamma\left(\frac{s+1+(m+iT)}{2}\right)^b}{\pi^{a'(s+(m+iT))}} \frac{idt}{m+it} \\
& = \frac{C}{y^m} \int_{-\infty}^{\infty} y^{-it} \Gamma\left(\frac{s+(m+iT)}{2}\right)^a \Gamma\left(\frac{s+1+(m+iT)}{2}\right)^b \pi^{-a' iT} \frac{dt}{m+it}. \tag{3.10}
\end{aligned}$$

The integral converges, therefore

$$V_s(y) \ll_m y^{-m} \tag{3.11}$$

for any  $n$ . Hence  $V_s(y)$  is Schwartz. Since  $s$  was chosen arbitrarily in the strip  $0 < \Re s < 1$ ,  $V_{1-s}(y)$  is also Schwartz.  $\square$

#### 4. SMOOTHING OF THE REAL ORBITAL INTEGRAL

We now prove our main theorem. Consider the real orbital integral  $\text{Orb}(f_\infty; \gamma)$ . Let  $\gamma$  be a regular element in  $G$ , which is an  $n$  by  $n$  matrix with distinct eigenvalues, and  $T_{\text{reg}}$  be the set of all regular elements in  $T = G_\gamma$ . Also let  $f_\infty \in C_c^\infty(Z_+ \backslash G(\mathbb{R}))$ .

We have the following characterization which can be deduced from Shelstad [Sh], describing the smoothness of the real orbital integral.

Consider a real reductive group  $G(\mathbb{R})$ , for example,  $GL(n, \mathbb{R})$ , and  $f$  a rapidly decreasing (Schwartz) function on this group, in the sense of Harish-Chandra. Define the orbital integral

$$\Phi_f^T(\gamma) = \int_{G/T} f(g\gamma g^{-1}) \frac{dg}{dt}$$

where  $T$  is a Cartan subgroup of  $G$ , the measures  $dg, dt$  are defined by fixing an invariant form  $\omega_G$  on  $G(\mathbb{R})$ , and  $\gamma$  is a regular element in  $T$ , which to us will be a  $n$  by  $n$  matrix with distinct eigenvalues.

**Theorem 4.1** (Shelstad [Sh]). *The orbital integral  $\Phi_f^T$  is a well-defined, smooth function on  $T_{\text{reg}}$ , invariant under the Weyl group and such that  $|D_\gamma|^{\frac{1}{2}}\Phi_f^T$  is rapidly decreasing in  $T_{\text{reg}}$ .*

*Moreover let  $Z = T - T_{\text{reg}}$ . Then behavior of  $\Phi_f^T$  near  $z \in Z$  is as follows: There exist a neighborhood  $N_z$  of  $z$  in  $T$  and smooth functions  $g_1$  and  $g_2$  on  $N_z$  such that for  $\gamma \in N_z \cap T_{\text{reg}}$ ,*

$$\Phi_f^T(\gamma) = g_1(\gamma) + |D_\gamma|^{-1/2}g_2(\gamma)$$

where  $D_\gamma$  is the discriminant function of  $\gamma$ . It satisfies the following properties:

- (1)  $g_1(\gamma) \equiv 0$  when  $T$  is split (for us, it means the eigenvalues of  $\gamma$  are in  $\mathbb{Q}$ ), and
- (2) for each  $X$  in the center of the universal enveloping algebra of  $\mathfrak{g}(\mathbb{C})$  the restriction of  $X_T g_2$  to  $Z \cap N_z$  is independent of  $T$  (this is a condition on the derivatives of all orders).

*Proof.* The proof of this statement as given in [Sh] follows from Theorem 17.1 in [HC1, p.145], and in particular Lemma 17.4 of [HC1, p.147] and Lemma 40 of [HC2, p.491].  $\square$

We apply this result easily to our setting.

**Corollary 4.2.** *The orbital integral  $\text{Orb}(f_\infty; \gamma)$  is a smooth function on  $T_{\text{reg}}$  and  $\theta_\infty^\pm(\gamma) := |D_\gamma|^{1/2}\text{Orb}(f_\infty; \gamma)$  is Schwartz on  $T_{\text{reg}}$ . Moreover for  $z \notin T_{\text{reg}}$ , there exists a Weyl group invariant neighborhood  $N_z$  of  $z$  and smooth functions  $g_1, g_2$  on  $N_z$  such that for any  $\gamma \in N_z \cap T_{\text{reg}}$ ,*

$$\text{Orb}(f_\infty; \gamma) = g_1(\gamma) + |D_\gamma|^{-1/2}g_2(\gamma), \quad (4.1)$$

satisfying the properties in Theorem 4.1.

We remark that  $z \in T_{\text{reg}}$  if and only if  $D_\gamma$  is nonzero. In other words,  $\text{Orb}(f_\infty; \gamma)$  may have singularities on  $T \setminus T_{\text{reg}}$ . Note also that for  $p$ -adic orbital integrals, a similar behavior may also occur, but we do not consider this here.

**Theorem 4.3.** *Let  $\phi$  be any Schwartz function on  $\mathbb{R}$  and  $\alpha > 0$ . Then the function defined by*

$$f(x_1, \dots, x_{n-1}) = \theta_\infty^\pm(x_1, \dots, x_{n-1})\phi(|D(x_1, \dots, x_{n-1})|^{-\alpha}) \quad (4.2)$$

is smooth.

*Proof.* By Theorem 4.2,  $f$  is smooth on  $T_{\text{reg}}$ . Now consider  $(a_1, \dots, a_{n-1}) \notin T_{\text{reg}}$ , i.e.,  $D(a_1, \dots, a_{n-1}) = 0$ .

Since  $g_i$  is smooth on a neighborhood of  $(a_1, \dots, a_{n-1})$  and  $\phi$  is Schwartz, we have

$$\lim_{(x_1, \dots, x_{n-1}) \rightarrow (a_1, \dots, a_{n-1})} f(x_1, \dots, x_{n-1}) = 0. \quad (4.3)$$

So, we can redefine  $f(a_1, \dots, a_{n-1}) = 0$ . Choose  $M > 0$  such that  $M\alpha > 1$ . Since  $\phi$  is Schwartz,

$$\phi(x) \ll_M |x|^{-M} \quad (4.4)$$

for  $x \neq 0$ .

Let  $h \neq 0$ . By the differentiability of  $D(x, y)$ ,

$$\lim_{h \rightarrow 0} \frac{D(a_1 + h, \dots, a_{n-1})}{h} = \frac{\partial D}{\partial x_1}(a_1, \dots, a_{n-1}) \quad (4.5)$$

$$\frac{1}{h} \phi(|D(a_1 + h, \dots, a_{n-1})|^{-\alpha}) \ll_M \left| \frac{D(a_1 + h, \dots, a_{n-1})}{h} \right| |D(a_1 + h, \dots, a_{n-1})|^{M\alpha - 1}, \quad (4.6)$$

which tends to 0 as  $h$  tends to 0.

We have  $\theta_{\infty}^{\pm}(x, y)$  is Schwartz on  $T_{\text{reg}} = \{(x_1, \dots, x_{n-1}) : D(x_1, \dots, x_{n-1}) \neq 0\}$ . In particular it is bounded on  $T_{\text{reg}}$ . We have

$$\begin{aligned} \frac{\partial f}{\partial x_1}(a_1, \dots, a_{n-1}) &= \lim_{h \rightarrow 0} \frac{f(a_1 + h, \dots, a_{n-1}) - f(a_1, \dots, a_{n-1})}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \theta_{\infty}^{\pm}(a_1 + h, \dots, a_{n-1}) \phi(|D(a_1 + h, \dots, a_{n-1})|^{-\alpha}) = 0. \end{aligned} \quad (4.7)$$

Similarly,  $\frac{\partial f}{\partial x_i}(a_1, \dots, a_{n-1}) = 0$  for  $1 \leq i \leq n-1$ . Inductively and noting that (4.4) indeed holds for all  $M > 0$ , we have all of the partial derivatives at  $(a_1, \dots, a_{n-1})$  exists. Thus  $f$  is a smooth function on  $\mathbb{R}^{n-1}$  and this completes the proof.  $\square$

## 5. FURTHER DIRECTIONS

Now let  $G$  be a reductive group over  $\mathbb{Q}$ . In this section we indicate how the preceding analysis can be extended to general  $G$ , though for general  $G$  we only consider unstable elliptic orbital integrals. As before,  $\gamma$  will be an elliptic element of  $G(\mathbb{Q})$ , so that  $G_{\gamma}(\mathbb{Q})$  is a torus. Based on work of Ono [Ono], Shyr deduced a class number relation for tori. We briefly describe this, and refer to [Shy, §3] for details.

Consider  $T = G_{\gamma}$  as an algebraic torus over  $\mathbb{Q}$ , and let  $\hat{T} = \text{Hom}(T, \mathbb{G}_m)$  be the  $\mathbb{Z}$ -module of rational characters of  $T$ . The torus  $T$  splits over a finite separable extension  $K$  of  $\mathbb{Q}$ , and  $\Gamma = \text{Gal}(K/\mathbb{Q})$  acts on  $\hat{T}$ . Then  $\hat{T}$  becomes a free  $\Gamma$ -module with rank  $r = \dim(T)$ , and denote by  $\chi_T$  the character of the  $\Gamma$ -module. The character decomposes into

$$\chi_T = \sum_{i=0}^h m_i \chi_i \quad (5.1)$$

for some integer  $h$ . Here  $\chi_i$  are irreducible characters of  $G$  with  $\chi_0$  the principal character, and  $m_i$  the multiplicity, whereby  $m_0 = r$ . It follows then that the Artin

$L$ -function factorizes as

$$L(s, \chi_T) = \zeta(s)^r \prod_{i=1}^h L(s, \chi_i)^{m_i}. \quad (5.2)$$

Moreover, for  $i \geq 2$ ,  $L(1, \chi_i)$  is nonzero, so that the value

$$\rho_T := \lim_{s \rightarrow 1} (s-1)^r L(s, \chi_T) = \prod_{i=1}^h L(1, \chi_i) \quad (5.3)$$

is finite and nonzero, and is called the quasi-residue of  $T$  over  $\mathbb{Q}$ . By [Ono], it is independent of choice of splitting field.

Now, choosing canonical Haar measures related to the Tamagawa numbers, Shyr obtains the relation

$$\rho_T = \frac{h_T R_T}{\tau_T w_T D_T^{1/2}} \quad (5.4)$$

where  $\tau_T$  is the Tamagawa number of  $T$ , and the other  $h_T, R_T, w_T$ , and  $D_T$  are arithmetic invariants of  $T$  defined analogous to those appearing in Dirichlet's class number formula (2.13).

Then one may proceed as in Section 2, and in particular, using (2.16) to write the volume term as the value at 1 of an Artin  $L$ -function, and apply the approximate functional equation. Then by similar estimates in Section 4 the real orbital integral may be smoothed.

*Remark 5.1.* In the case of  $G = GL(n)$  the element  $\gamma$  defines a degree  $n$  extension  $E$  over  $\mathbb{Q}$ , and the torus is simply the Weil restriction  $\text{Res}_{E/\mathbb{Q}}(\mathbb{G}_m)$ , split by  $K$ . By the remark following [Shy, Theorem 1], one indeed recovers

$$\rho_T = \text{Res}_{s=1} \zeta_E(s), \quad (5.5)$$

recovering the original case, and in particular the analytic class number formula.

#### APPENDIX A. THE $p$ -ADIC ORBITAL INTEGRALS

In this appendix we discuss the obstructions to carrying out the analysis for  $p$ -adic orbital integrals. By Equation (59) in [Lan1, §2.5], the product of the  $p$ -adic orbital integrals for  $GL(2)$  can be expressed using the Kronecker symbol,

$$\prod_q \int_{G_\gamma(\mathbb{Q}_q) \backslash G(\mathbb{Q}_q)} f_q(g^{-1} \gamma g) d\bar{g}_q = \sum_{f|s} f \prod_{q|f} \left(1 - \left(\frac{D}{q}\right) \frac{1}{q}\right). \quad (A.1)$$

Then by [Alt1, §2.2.2], this can be combined with global volume factor by a change of variables to give

$$\sqrt{|D|} L\left(1, \left(\frac{D}{\cdot}\right)\right) \sum_{f|s} f \prod_{q|f} \left(1 - \left(\frac{D}{q}\right) \frac{1}{q}\right) = \sum_{f|s} \frac{1}{f} L\left(1, \left(\frac{(m^2 - N)/f^2}{\cdot}\right)\right), \quad (A.2)$$

where  $s^2 D = m^2 - N$  and  $D$  a fundamental discriminant of a quadratic number field.

In order to generalize this to other groups, one would need an expression for the  $p$ -adic orbital integrals related to the Artin representation  $\chi_T$  as in Section 5. Unfortunately, for general groups we do not know of a closed formula for these

integrals. Though it is interesting to note that by [Ngô], and not to mention [Hal], one knows that evaluating such  $p$ -adic orbital integrals are closely related to counting points on varieties over finite fields.

Finally, for the readers' interest we point out that in the case of  $GL(3)$  the  $p$ -adic orbital integrals for the unit element of the Hecke algebra have in fact been computed explicitly by Kottwitz [Kot1, p.661]. If the elliptic element  $\gamma = \alpha + \pi^n\beta$  generates an unramified cubic extension, one has

$$\frac{p^{3n+1}(p+1)(p^2+p+1) - 3p^{2n}(p^2+p+1) + 3}{(p-1)^2(p+1)}, \quad (\text{A.3})$$

while for a ramified cubic extension one has

$$\frac{p^{3n+1}(p+1)p^{1+\text{val}(\beta)} - p^{2n}(p^2 + (p+1)p^{2\text{val}(\beta)}) + 1}{(p-1)^2(p+1)}, \quad (\text{A.4})$$

where  $\text{val}(\beta) = 1$  or  $2$ , but it does not seem clear what the analogous identity for (A.1) might be.

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