

COMBINATORIAL AND ADDITIVE NUMBER THEORY PROBLEM SESSIONS: '09-'16

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ABSTRACT. These notes are a summary of the problem session discussions at various CANT (Combinatorial and Additive Number Theory Conferences). Currently they include all years from 2009 through 2016 (inclusive); the goal is to supplement this file each year. These additions will include the problem session notes from that year, and occasionally discussions on progress on previous problems. If you are interested in pursuing any of these problems and want additional information as to progress, please email the author.

For more information, visit the conference homepage at

<http://www.theoryofnumbers.com/>

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Warning: Many of these notes were LaTeX-ed in real-time by Steven J. Miller; all errors should be attributed solely to him.

The typist was supported by NSF grants DMS0600848, DMS0600848 and DMS1265673. These notes would not have been possible without the help of the participants, especially Zack McGuirk, Kevin O'Bryant, Steven Senger, my students, especially Olivia Beckwith, Alan Chang, Ginny Hogan, Jasmine Powell, Ryan Ronan, Maddie Weinstein and of course the organizer Mel Nathanson.

CONTENTS

2009. CANT Problem Sessions: 2009	3
2009.1. Problem Session I: Tuesday, May 26th (Chair Kevin O'Bryant)	3
2009.2. Problem Session II: Wednesday, May 27th	5
2009.3. Problem Session III: Thursday, May 28th	7
2009.4. Problem Session IV: Saturday, May 30	8
2009.5. Speakers and Participants Lists	9
2010. CANT Problem Sessions: 2010	11
2010.1. Problem Session III: Friday, May 28th (Chair)	11
2010.2. Problem Session IV: Saturday, May 29th	13
2010.3. Speaker List	14
2011. CANT Problem Sessions: 2011	15
2011.1. Problem Sessions	15
2011.2. Speaker List	17
2012. CANT Problem Sessions: 2012	19
2012.1. Problem Session I: Tuesday, May 22nd (Chair Renling Jin)	19
2012.2. Problem Session II: Wednesday, May 23rd (Chair Steven J Miller)	20
2012.3. Problem Session III: Thursday, May 24th (Chair Alex Iosevich)	21
2012.4. Problem Session IV: Friday, May 25th (Chair Kevin O'Bryant)	23
2012.5. Speaker List	24
2013. CANT Problem Sessions: 2013	25
2013.1. Problem Session I: Tuesday, May 21st (Chair Steven Miller)	25
2013.2. Problem Session II: Wednesday, May 23rd (Chair Seva Lev)	26
2013.3. Problem Session III: Thursday, May 24th (Chair Kevin O'Bryant)	27
2013.4. Problem Session IV: Friday, May 24th (Chair Renling Jin)	27

2013.5. Additional Problems	28
2013.6. Speaker List	28
2014. CANT Problem Sessions: 2014	30
2014.1. Problem Session I: Wednesday, May 28th (Chair Steven J Miller)	30
2014.2. Problem Session II: Thursday, May 29th (Chair Kevin O'Bryant)	30
2014.3. Problem Session III: Friday, May 24th (Chair Mel Nathanson)	31
2014.4. Speaker List	32
2015. CANT Problem Sessions: 2015	33
2015.1. Phase Transitions in MSTD sets: Steven J Miller	33
2015.2. An accidental sequence: Satyanand Singh	33
2015.3. Kevin O'Bryant	33
2015.4. Steven Senger (repeat from previous years):	33
2015.5. Nathan Kaplan:	33
2015.6. Nathan Kaplan (repeat from previous years):	34
2015.7. Kevin O'Bryant:	34
2015.8. Speaker List	35
2016. CANT Problem Sessions: 2016	36
2016.1. Problem Session I: Tuesday, May 24th (Nathanson Chair)	36
2016.2. Problem Session II: Wednesday, May 25th (Iosevich Chair)	37
2016.3. Problem Session III: Thursday, May 26th (Miller Chair)	38
2016.4. Problem Session IV: Friday, May 27th (O'Bryant Chair)	39
2016.5. Speaker List	41
Countries represented	42

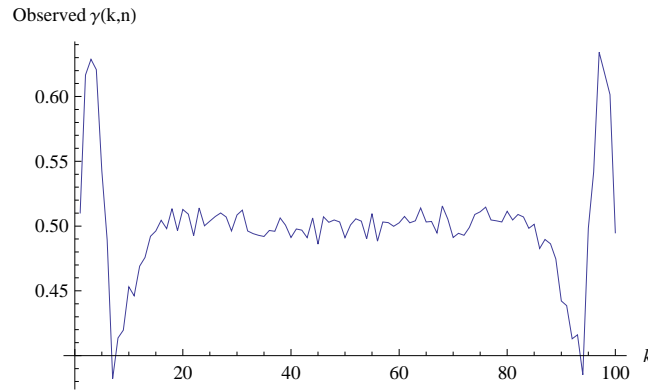


FIGURE 1. Observed $\gamma(k, 100)$, random sample 4458 MSTD sets.

2009. CANT PROBLEM SESSIONS: 2009

2009.1. **Problem Session I: Tuesday, May 26th (Chair Kevin O'Bryant).**

2009.1.1. *Steven J Miller: I (sjm1@williams.edu).* **Probability an element is in an MSTD**

Let $\gamma(k, n)$ be the probability that k is in an MSTD set A with $A \subset [0, n]$; see for instance the figure below

Conjecture: Fix a constant $0 < \alpha < 1$. Then $\lim_{n \rightarrow \infty} \gamma(k, n) = 1/2$ for $\lfloor \alpha n \rfloor \leq k \leq n - \lfloor \alpha n \rfloor$.

Questions:

- How big are the spikes? Do the sizes of the spikes tend to zero as $n \rightarrow \infty$?
- Is the spike up equal to the spike down?
- Study more generally $g(n) \leq k \leq n - g(n)$; is it sufficient for $g(n) \rightarrow \infty$ monotonically at any rate to have all k in the region above having probability $1/2$ of being in an MSTD set? Can we take $g(n) = \log \log \log n$, or do we need $g(n) = \alpha n$?
- A generic MSTD set has about $n/2$ elements; what other properties of a generic set are inherited by an MSTD set?
- How big are the fluctuations in the middle?
- Do we want to look at all MSTD sets containing 1 and n , or do we want to just look at all subsets of $[1, n]$ that are MSTD sets

Note since the end of the conference: Kevin O'Bryant has observations relevant to this and other problems proposed by Miller.

Note added in 2014: Many of these claims were later proved by Zhao.

2009.1.2. *Steven J Miller: II (sjm1@williams.edu).* With Dan S. and Brooke O. we constructed very dense families of MSTD sets in $[0, r]$ such that these families were C/r^4 of all subsets of $[0, r]$. This isn't a positive percentage of sets, but it is significantly larger than any previous family. Can one do better? Can one get a positive percentage?

2009.1.3. *Peter Hegarty: I (hegarty@chalmers.se).* Smallest size of an MSTD is 8 elements: $A = \{0, 2, 3, 4, 7, 10, 12, 14\}$; remove 4 and symmetric about 7. If look in \mathbb{Z}^2 , can construct an MSTD set of size 16 from this: take $A \times \{0, 1\}$ (any set of size two would work). Can you construct MSTD sets in \mathbb{Z}^2 without going through an MSTD set in \mathbb{Z} . Need a computer to show this set A was minimal (about 15 hours to find all MSTD sets of size 8, and thus see that this set A is minimal). To find all MSTD sets up to isomorphism of a certain size is a finite computation, but practically impossible for 9.

2009.1.4. *Peter Hegarty: II (hegarty@chalmers.se)*. **Question:** What are the possible orders of a basis for \mathbb{Z}_n ?

Let $A \subset \mathbb{Z}_n$. We say A is a basis of order h if $hA = \mathbb{Z}_n$ but $(h-1)A \neq \mathbb{Z}_n$. A is a basis of some order if and only if $(\gcd(A), n) = 1$.

$A = \{a_1, \dots, a_k\}$, $|hA| = O(k^h)$, order should be about $\log n / \log k$, so $k^h = n$. Order for a random set to be a basis, should be of logarithmic order. Can write down a very inefficient basis where need long summands to cover all of \mathbb{Z}_n . To do this, take $A = \{0, 1, \dots, k-1\}$. Order of this will be essentially $\frac{n}{k-1}$.

Conjecture: If the order of a basis is $\Theta(n)$ then the order must be very close to n/k for some k . So no number between $n/2$ and n can be the order. Gives gaps. See paper by Dukes and Herke.

Question from participants: this can't go on forever?

Answer from Peter: Can go on forever. Fix a k , let $n \rightarrow \infty$, the way you would phrase to make it precise: Fix a k . For $n \gg 0$ if the order of a basis is greater than $\frac{n}{k+1} + O(k)$ then the order must be within $O(k)$ of n/ℓ for some $\ell \leq k$.

Note : Since the end of the conference, Peter Hegarty has solved this problem. His result is available at <http://arxiv.org/abs/0906.5484>

2009.1.5. *Kevin O'Bryant (obryant@gmail.com)*. Take $g_0 = 0$, g_i to be the least positive integer such that $\{g_0, g_1, \dots, g_i\}$ has no solutions to $5w + 2x = 5y + 2z$. This is building a set greedily.

Let $a_0 = 0$ and

$$a_i = \lfloor \frac{5 + 7 \sum_{j=0}^{i-1} a_j}{2} \rfloor. \quad (1)$$

Let A equal the sum of distinct a_i 's.

Conjecture: $G = \{5x + y : x \in A, 0 \leq y \leq 4\}$.

Appears computationally that there is some description of this sort when one number is at least twice as large as another; can replace $(5, 2)$ with $(11, 4)$ without trouble, but not with $(4, 3)$.

Question from audience: Why 5 and 2?

Answer: 5, 2 smallest haven't solved and have done the most computation.

Question from audience: How many other cases investigated?

Answer: Calculated all terms up to about 100,000 if both numbers at most 12 (and can exclude cases, such as cases with common prime factors). Nice structure if one is twice the other, else irregular and nothing to say (though all irregular in the same way).

2009.1.6. *Ruzsa (through Simon Griffiths through Kevin O'Bryant)*. Let $A \subset \mathbb{Z}$, $|A| = n$, and

$$S_k = \left\{ \sum_{a \in B} a : B \subset A, |B| = k \right\}.$$

Note $|S_k| = |S_{n-k}|$. For example, if $A = \{1, 2, 4, 8, \dots, 2^{n-1}\}$ then $|S_{k+1}| = \binom{n}{k+1} = \frac{n-k}{k+1} |S_k|$.

Question: $|S_{k+1}| \leq \frac{n-k}{k+1} |S_k|$ whenever $k < n/2$?

Theorem (Ruzsa): Yes, when $n > \frac{k^2 + 7k}{2}$.

Exercise: $|S_{k+1}| \leq \frac{n}{k+1} |S_k|$.

2009.2. Problem Session II: Wednesday, May 27th.

2009.2.1. *Comments after Nathanson's Talk (Mel Nathanson:*

melvyn.nathanson@lehman.cuny.edu). Paper is online at <http://arxiv.org/pdf/0811.3990>.

Take G_i with A_i of generators. Only one direct product, but many sets of generators that can construct from generators of individual groups. Could take direct product of generators. That's complicated. Given groups and generating sets, many ways to put together new sets of generators. Never thought about finite groups because thinking about geometric group theory. For finite groups know at some point all spheres empty.

Question: A result like this might not be true for semi-groups: bunch of things with finite spheres then empty at some point. Additive sub-model of integers, all positive integers exceeding 1000. Can you have infinite sphere, finite sphere, infinite sphere, finite sphere.... Answer: don't know. Wanted to create oscillating sets of spheres, turned out couldn't.

2009.2.2. *Constructing MSTD Sets (Kevin O'Bryant, communicated to Steven Miller).*

Theorem: $d_i \in \{3, 4, 5\}$ independent uniformly distributed, $x_1 = 4$, $x_2 = 5$, $x_i = x_{i-1} + d_i$, $A = \{1\} \cup \{0, \pm x_1, \dots, \pm x_n\}$. Then $|A + A| > |A - A|$ with probability 1.

Note

$$\begin{aligned} A + A &= (X + X) \cup (X + 1) \cup \{2\} \\ A - A &= (X - X) \cup (X - 1) \cup (1 - X) \cup \{0\} \\ &= (X + X) \cup (X - 1) \cup (1 + X) \cup \{0\}, \end{aligned}$$

where X is the set of the x_i 's.

2009.2.3. *David Newman (davidsnewman@gmail.com)*. Suppose we have a basis for the non-negative integers, that is a set so that for any non-negative number we can find two elements of the set whose sum is this given number. If we arrange the numbers in the set in ascending order then we can cut it off at a certain point and look at the first N terms of this basis.

Question: Can this beginning of a basis be extended into a minimal basis? By minimal basis I mean a basis where if you remove any element it is no longer a basis.

Has to have 0 and 1 as a start. I think the answer is yes is because I haven't seen the beginning of a basis I couldn't extend to a minimal basis. I have an algorithm implemented in Mathematica and in a few seconds gives a set which is a minimal basis. That's about all the info I have, other than one family of bases that I can always extend to a minimal basis.

Another problem (from the theory of partitions): Consider

$$(1 + x)(1 + x^2) \cdots = \sum a_n x^n,$$

which is the generating function for partitioning into distinct parts. Now put in minus signs:

$$(1 - x)(1 - x^2) \cdots = 1 - x - x^2 + x^5 + x^7 + \cdots,$$

where all coefficients are in $\{0, \pm 1\}$. Now do partitions into unrestricted parts:

$$(1 + x + x^2 + x^3 + \cdots)(1 + x^2 + x^4 + x^6 + \cdots) \cdots = \sum b_n x^n.$$

Question: can we change some of the signs above into minus signs so that the b_n 's are also in $\{0, \pm 1\}$.

Note since the conclusion of the conference: Peter Hegarty and David Newman have made progress on this. They are currently working on a paper: Let $h > 1$ be an integer, for any basis A for \mathbb{N}_0 of order h and any $n \in \mathbb{N}_0$ the initial segment $A \cap [0, n]$ can be extend to a basis of A' of order h which is also a minimal asymptotic basis of this order.

2009.2.4. *Infinitude of Primes (Steven Miller)*. Two types of proofs of the infinitude of primes, those that give lower bounds and those that don't (such as Furstenberg's topological proof). What category does $\zeta(2) = \pi^2/6 \neq \mathbb{Q}$ fall under? It implies there must be infinitely many primes, as this is $\zeta(2) = \prod_p (1 - p^{-2})^{-1}$; if we knew how well π^2 can be approximated by rationals, we could convert this to knowledge about spacings between primes. Unfortunately while we know the irrationality exponent for π^2 is at most 5.441243 (Rhin and Viola, 1996), their proof uses the prime number theorem to estimate $\text{lcm}(1, \dots, n)$. This leads to $\pi(x) \gg \log \log x / \log \log \log x$ infinitely often; actually, I can show: let $g(x) = o(x/\log x)$ then $\pi(x) \geq g(x)$. A preprint of my paper (with M. Schiffrman and B. Wieland) is online at

http://arxiv.org/PS_cache/arxiv/pdf/0709/0709.2184v3.pdf

and I hope to have a final, cleaned up version in a few months. I'm looking for a proof of the finiteness of the irrationality measure of $\zeta(2)$ that doesn't assume the prime number theorem. **Note added in 2014: Miller is currently working on this with some of his students.**

2009.2.5. *Kent Boklan (boklan@boole.cs.qc.edu)*. There are infinitely many primes, don't know much about twin primes. Know sum of reciprocals of twins converges by Brun's theorem. This is a hard theorem – I want to do elementary things. How do you show there are infinitely many primes which are not twin primes. Trivial proof: There are infinitely many primes of the form $15k + 7$ by Dirichlet, and not prime if add or subtract 2. But Dirichlet isn't elementary!

2009.2.6. *Mel Nathanson II*. $A = \{a_1, \dots, a_k\}$ finite set of integers, $n = \sum_{i=1}^k a_i x_i$ is solvable for all n if and only if $\gcd(A) = 1$. In geometric group theory, can deduce algebraic properties of the group by seeing how it acts on geometric objects. Fundamental lemma of geometric group theory says the following: G is a group and acts on a set S (metric space), want action to be an isometry for any fixed g in the group. Acts isometrically on the metric space S . Suppose the space is nice (Heine-Borel, want that: any closed and bounded set is compact, call this a proper space). G acts properly discontinuously on X if intersection non-empty for only finitely many g . Example: \mathbb{Z}^n acts on \mathbb{R}^n by $(g, x) = g + x$. Have $G \backslash X$, send x to its orbit $\langle x \rangle$. Put a quotient topology on $G \backslash X$ that makes projection map continuous. Example: $\mathbb{Z}^n \backslash \mathbb{R}^n$ is the n -torus. Let $K \subset X$ compact, for every $x \in X$ there is a $y \in K$ such that $gy = x$. For example, $n = 1$: \mathbb{Z} acts on \mathbb{R} by translation, take unit interval $[0, 1]$ (compact), and every number is congruent modulo 1 to something in unit interval. The fundamental lemma of geometric group theory: Group acts as isometry and properly discontinuously on proper metric space then G must be finitely generated. Know nothing about if it is finitely or infinitely generated, but if acts geometrically in this nice way, that can only happen if the group is finitely generated. Proof goes by finding a compact set K with exactly the property above. What we know about K since group action properly discontinuous, group action under K only finitely many, that is a finite set of generators.

Suppose we specialize to elementary number theory: integers acting on reals by translations, compact set K such that every real number is congruent modulo 1 to an element of K . Then we get a finite set of generators for the group, but the group is the integers and a finite set of integers is a finite set of relatively prime integers. Get certain sets of relatively prime integers. What finite sets of integers can we get geometrically in this way? Every finite set of relatively prime set of integers can be obtained this way. Curious thing is that there is this geometric way to describe these sets. Look at lattice points in two dimensions, seems quite complicated.

Article might be: <http://arxiv.org/pdf/0901.1458>.

2009.3. Problem Session III: Thursday, May 28th.

2009.3.1. Gang Yu (yu@math.kent.edu). $C \subset \mathbb{N}$ is an infinite sequence, $C(N) = C \cap [N]$ (where $[N] = \{0, \dots, N\}$ or perhaps it starts at 1), $h \geq 2$ fixed, call $A \subset [N]$ an h -basis of $C(N)$ if $hA \supset C(N)$.

Trivial estimate: $|A| \geq h!|C(N)|^{1/h}$.

Interesting cases: C is sparse but arithmetically nice:

$$\begin{aligned} C &= \{n^2\} \\ C &= \{n^\alpha\} \\ C &= \{f(n)\} \end{aligned}$$

where f is a degree 2 polynomial. Let

$$\Gamma_h/C = \overline{\lim_{N \rightarrow \infty}} \frac{|C(N)|}{D_h(C, N)^h}, \quad (2)$$

where

$$D_h(C, N) = \min_{\substack{A \subset [N] \\ A \text{ is an } h\text{-basis of } C(N)}} |A|. \quad (3)$$

Question: Is $\Gamma_h(C) = 0$ for polynomial C (ie, degree at least 2)?

Audience: Is it true for any sequence?

Gang: Don't know. For $AP = C$, bounded away from 0. Specifically, $A \subset [N]$, $A + A \supset \{n^2 : D \leq n \leq \sqrt{N}\}$, $N^{1/4} = o(|A|)$?

After the conference it was noted: Some information available at <http://arxiv.org/pdf/0711.1604>.

2009.3.2. Simon Griffiths (sg332@cam.ac.uk). An n -sum of a sequence x_1, \dots, x_r is a sum of the form $x_{i_1} + \dots + x_{i_n}$ where $i_1 < \dots < i_n$, i.e. an element that can be obtained as the sum of an n -term subsequence.

EGZ: Every sequence $x_1, \dots, x_{2n-1} \in \mathbb{Z}_n$ has 0 as an n -sum.

Bollobás-Leader: Let $x_1, \dots, x_{n+r} \in G$ and suppose D is not an n -sum, then you have at least $r + 1$ n -sums.

Examples: EGZ is tight as demonstrated by the sequence of $n - 1$ 0s and $n - 1$ 1s; Bollobás-Leader is tight as demonstrated by the seq of $n - 1$ 0s and $r + 1$ 1s.

What about finite abelian groups more generally?

$D(G)$ is Davenport constant, the minimum r where every r -term sequence has a non-trivial subsequence with sum 0. For example: not difficult to show $D(\mathbb{Z}_n) = n$.

Example: Let $x_1, \dots, x_{D(G)-1}$ be a sequence in G with no non-trivial subsequence summing to 0, and adjoin $n - 1$ 0s by setting $x_{D(G)}, \dots, x_{n+D(G)-2} = 0$. Then, by an easy check, we see that this sequence, of length $n + D(G) - 2$ does not have 0 as an n -sum.

Gao: Every sequence $x_1, \dots, x_{n+D(G)-1}$ has 0 as an n -sum.

Question: EGZ is to Bollobás-Leader as Gao is to ..?..?..

One Answer: A theorem of Gao and Leader.

Why do we need another answer: Both of the results, Bollobás-Leader and Gao-Leader allow us to see the set of n -sums grow as the length of the underlying sequence increases. However perhaps in the case of general abelian groups there may be a different way to see this growth - to see this growth as a growth of dimension in the sense described below.

Our approach to defining the dimension of a subset $S \subset G$ is similar to describing the dimension of a subspace via the maximum dimension of an independent subspace. Call a sequence zero-sum-free if no non-trivial subsequence has sum 0. Let $\dim(S)$ equal $D(G)$ minus the minimum r such that for every zero-sum-free sequence

y_1, \dots, y_r there exists an $s \in S$ and a subsequence I such that
 $s + \sum_I y_i = 0$.

If $0 \in S$ then we take the minimum to be zero. Thus,

Examples: $\dim(G) = D(G)$. $S = G - \{0\}$ implies $\dim(S) = D(G) - 1$. $S = \emptyset$ implies $\dim(S) = 0$.

Conjecture: x_1, \dots, x_{n+r} either 0 as an n -sum or $\dim(\{n - \text{SUMS}\}) \geq r + 1$.

2009.4. Problem Session IV: Saturday, May 30.

2009.4.1. *Urban Larsson*. How small can a *maximal* AP-free set be? Specifically, how large is the smallest maximal (with respect to not having 3 terms in arithmetic progression) subset of $[n]$? Set

$$\mu(n) := \min_{\substack{A \\ A \text{ is 3-free}}} |A \cap [n]|.$$

Examples: The greedy subset of $\{0, 1, \dots\}$ with 3-term APs is $\{n \in \mathbb{N} : \text{base-3 expansion of } n \text{ has no '2's}\}$. This shows that $\mu(3^t) \leq 2^t$. A better example is the set of natural numbers whose base-4 expansions have neither '2's nor '3's. This gives $\mu(4^t) \leq 2^t$.

Each pair of elements of A , which is 3-free, forbids at most three other numbers from A , so $3 \binom{|A|}{2} \geq n - |A|$, so that $|A| \geq c\sqrt{n}$. Attention to detail gives $|A| \geq \sqrt{2n/3}$. If A is uniformly distributed mod 4, and u.d. in $[n]$, then many pairs will not forbid three other numbers, and this gives $|A| \geq \sqrt{420n/401}$.

I conjecture that $\mu(4^t) = 2^t$, and $\mu(n) \geq \sqrt{n}$ for all n .

2009.4.2. *Renling Jin*. Let $\underline{d}(A) = \liminf_{n \rightarrow \infty} A(n)/n$ be the lower asymptotic density of A , and let \mathcal{P} be the set of primes. Clearly

$$\forall A \subseteq \mathbb{N} \quad (\underline{d}(A + \mathcal{P}) \geq f(\underline{d}(A)))$$

for $f(x) = x$. What is the right f ?

Using Plünnecke and $\underline{d}(3\mathcal{P}) = 1$ (due to Easterman, van der Corput, and possibly others independently) we get $f(x) = x^{2/3}$.

Audience: Can replace \mathcal{P} with any h -basis and still have $f(x) = x^{1-1/h}$.

Note that Erdős proved the existence of A with $A(n) \sim \log n$ and $A + \mathcal{P} \sim \mathbb{N}$, so the primes are not a typical basis.

Audience: Can $x^{2/3}$ be improved assuming the Goldbach conjecture? Answer: Goldbach gives only $\sigma(4\mathcal{P}) = 1$, so no.

2009.4.3. *Mel Nathanson*. Clarifying earlier problem. We say that two points in \mathbb{R}^n are congruent if their difference is in \mathbb{Z}^n . Suppose that $K \subseteq \mathbb{R}^n$ is compact and for each $x \in \mathbb{R}$ there is a $y \in K$ such that $x \equiv y$.

Theorem: $A := (K - K) \cap \mathbb{Z}^n$ is a finite set and generates the additive group \mathbb{Z}^n .

For $n = 1$, there is a K that will give any set of generators that contains 0 and is symmetric about 0. For $n = 2$, which sets of generators arise in this fashion? Specifically, is there a K (compact and hitting

every residue class modulo 1) such that $A \subseteq \{(x, y) : xy = 0\}$? Even more specifically, is there a K with $A = \{(0, 0), (\pm 1, 0), (0, \pm 1)\}$?

Note since the end of the conference: Renling Jin has solved this problem. Independently, and by different methods, Mario Szegedy has obtained a partial solution.

2009.5. Speakers and Participants Lists.

2009.5.1. *Speakers.*

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2009.5.2. *Participants.*

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 Simon Griffiths, University of Montreal
 Li Guo, Rutgers University-Newark
 Mariah Hamel, University of Georgia
 Piper Harris, Princeton University

Derrick Hart, Rutgers University - New Brunswick
Tanya Haxhoviq, CUNY Graduate Center
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Shanta Laishram, University of Waterloo
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David Newman, New York
Hoi H. Nguyen, Rutgers University-New Brunswick
Lan Nguyen, University of Michigan
Kevin O'Bryant, College of Staten Island (CUNY)
Brooke Orosz, Essex County College
Gina-Louise Santamaria, Montclair State University
Steven Senger, University of Missouri - Columbia
Satyanand Singh, CUNY Graduate Center
Jonathan Sondow, New York
Craig Spencer, Institute for Advanced Study
Jacob Steinhardt
Mario Szegedy, Rutgers University-New Brunswick
Jonathan Wang
Benjamin Weiss, University of Michigan
Julia Wolf, Rutgers University-New Brunswick
Thomas Wright, Johns Hopkins University
Gang Yu, Kent State University

2010. CANT PROBLEM SESSIONS: 2010

2010.1. Problem Session III: Friday, May 28th (Chair).

2010.1.1. *Nathanson: Classical Problems in Additive Number Theory.* N. G. de Bruijn had two papers:

- (1) *On bases for the set of integers*, 1949.
- (2) *On number systems*, 1956.

Very few references to these papers. The second paper: he stated and solved a problem; in the first he stated a problem but neither he nor others could solve. Lately, however, these have become of interest to people in harmonic analysis.

These are related to the idea of complementing sets. Given a finite set A , can you find an infinite set B such that $A \oplus B = \mathbb{Z}$? De Bruijn considered a slightly different problem, but in the same spirit. Given a family of sets $\{A_i\}_{i \in I}$ with $I = \mathbb{N}$ or $\{1, \dots, n\}$, we are interested in sets with the property that $\mathbb{N}_0 = \bigoplus_{i \in I} A_i$; in other words, every non-negative integer is of the form $\sum_{i \in I} a_i$ and $a_i \neq 0$ only finitely often. De Bruijn calls this a *British number system*. Years ago 12 pence in a shilling, The British number system (pence, shillings, pounds) is the motivation for notation. If you have 835 pence that is 3 pounds, 9 shillings and 7 pence. The British number system is based on 12 and 20.

Using 12 pence is 1 shilling and 20 shillings is 1 pound. Take $\{g_i\}_{i \in I}$, $g_i \geq 2$, $G_0 = 1$, $G_1 = g_1$, $G_2 = g_1 g_2$, $G_i = g_1 g_2 \cdots g_n$,

$$A_n = G_{n-1} * [0, 1, 2, \dots, g_{n-1}) = G_{i-1} * [0, g_i),$$

and

$$\begin{aligned} A_1 &= \{0, 1, 2, \dots, g_1-1\} \\ A_2 &= G_1 * \{0, 1, \dots, g_2-1\} \\ A_3 &= G_2 * \{0, 1, 2, \dots, g_3-1\} \\ A_{n+1} &= G_n * \mathbb{N}_0, \end{aligned}$$

where

$$d * A = \{da : a \in A\}.$$

Are there other sets? Yes. Let $\{A_i\}_{i \in I}$ and $I = \bigcup_{j \in J} I_j$ with $I_j \cap I_{j'} = \emptyset$ for $j \neq j'$, $B_j = \sum_{i \in I_j} A_i$, $\{B_j\}_{j \in J}$. Comes down to choosing sequence of g 's to be prime numbers to get indecomposable sequence.

Consider a set B of integers such that every $n \in \mathbb{Z}$ has a unique representation in the form

$$n = \sum_{b \in B} \epsilon_b b$$

where $\epsilon_b \in \{0, 1\}$ and $\epsilon_b = 1$ finitely often. Let $A_i = \{0, b_i\}$, $\bigoplus_{i=1}^{\infty} A_i = \mathbb{Z}$.

Take set of powers of 2: $\{2^i\}_{i=0}^{\infty}$: get all non-negative integers. Suppose we look at $\{\epsilon_i 2^i\}_{i=0}^{\infty}$ where $\epsilon_i \in \{\pm 1\}$. Need infinitely many +1s and -1s to be a basis. No other condition necessary. Works if there are infinitely many +1s and infinitely many -1s. What infinite sets B have this property? Want subset sums to give each integer once and only once. First thing can say is that we better not have everything even, so must have at least one odd integer in the set. Then de Bruijn proves something clever: not only at least one odd integer, but at most one odd integer. Was a conjecture of someone else, de Bruijn proves this conjecture.

Think about this for a minute. Exactly one odd number. If you are going to represent an even number it cannot have that odd number, and thus if divide all even numbers by 2 get another system of this form, so one of these and only one of these is divisible by 2 and not 4. By induction, see for every power of 2 there is one and only one number x in this set such that $2^7 || x$. We can thus write $b_i = d_n 2^i$ with d_i odd. So B comes from a sequence of odd numbers. Let's call this sequence of odd numbers $\{d_i\}_{i=1}^{\infty}$ okay; sequence of odd numbers. In other words, it is an additive basis. Just restated the problem – what sequences of odd integers are okay?

No one knows what sequences of odd numbers are okay. De Bruijn proved the following: suppose $\{d_1, d_2, d_3, \dots\}$ is an okay sequence; this is an okay sequence if and only if $\{d_2, d_3, d_4, \dots\}$ is okay. Can throw off any bunch – do again. Can really screw around with an okay sequence – can chop at any point, any finite sequence of garbage in the beginning. This is an interesting problem. I went this morning to MathSciNet to see what papers have referenced this paper of de Bruijn. There was a gap of about 50 years, but now relevant for something in harmonic analysis (they can't solve this problem, but it is in the same spirit as something they are interested in).

2010.1.2. *Schnirelman*. When did additive number theory start? In 1930s Schnirelman proved that every even number is the sum of a large number of primes; he did this by proving a theorem about sumsets. Before this the results were beautiful (Fermat, Lagrange, Gauss, Hardy, Ramanujan, Littlewood, Vinogradov); Schnirelman had a general theorem about integers. Let

$$A + B = \{a + b : a \in A, b \in B\}.$$

Counting function

$$A(n) = \sum_{\substack{a \in A \\ 1 \leq a \leq n}} 1.$$

Let's say $0 \in A \cap B$ to be safe. Defining

$$\delta(A) = \inf \frac{A(n)}{n}.$$

Schnirelman proved

$$\delta(A + B) \geq \delta(A) + \delta(B) - \delta(A)\delta(B).$$

Norwegians are funny – go off to the mountains and come down with a great theorem. In WWI de Bruijn goes up to mountains and invents a sieve method that allows him to prove things about Goldbach and Twin Primes. No one could understand the paper. Landau couldn't understand it, didn't try. Schnirelman did understand and used it to get his results, which made the result / method fashionable. Now people studied de Bruijn's paper. Landau's exposition in one of the seminal journals became the standard exposition for the de Bruijn sieve. Same thing happened with Selberg. WWII started, he was captured by Germans, released if promised not to stay in Oslo, went to family home and proved results on zeros of $\zeta(s)$.

Could also look at

$$\delta_L(A) = \lim_{n \rightarrow \infty} \frac{A(n)}{n}.$$

Say $A \sim B$ if there is an N such that for all $n \geq N$ we have $n \in A$ if and only if $n \in B$. Embarrassment: if every element is even then all sums even, must be careful. If $\delta_L(hA) > 0$ for some h then if $d = \gcd(A)$ and $0 \in A$ there is an h_0 such that $h_0 A \sim d * N_0$. First time appears is in a paper with John C. M. Nash (the son).

Let $0 \in A$ and $\delta_L(hA) = 0$ for all $h \geq 1$. Assume

$$A \subseteq 2A \subseteq 3A \subseteq 4A \subseteq \dots \subseteq hA \subseteq \dots.$$

If any set has positive density then get all integers from some point onward. Maybe in this case some infinite case appears. The question is: take a set of non-negative integers containing 0 such that all of these sets have asymptotic density zero. Get an increasing sequence. As add to itself more and more times, does any structure appear? Can you say something that interests your friends mathematically about this? Is there anything that must happen?

2010.1.3. *Alex Kontorovich*. Not convinced problem is difficult, but we haven't made progress. The question is the additive energy in $SL(2, \mathbb{Z})$. This means that we take elements $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ in $SL(2, \mathbb{Z})$ in a ball B_N and want to know how many there are such that $\gamma_1 + \gamma_2 = \gamma_3 + \gamma_4$. The number of points in a ball (4 variables, 1 quadratic equation) gives cN^2 .

We have 16 variables (unknowns), 4 quadratic equations (the determinants equaling 1) and 4 linear equations. Want an upper bound of the form $\ll N^{4+\epsilon}$. Have a trivial lower bound of N^4 . Easy thing to prove is N^5 for the following reason. Let

$$\eta(\omega) = \#\{\gamma_1 \gamma_2 \in B_N(SL_2) : \gamma_1 - \gamma_2 = \omega\} = \begin{cases} N^2 & \text{if } \omega = 0 \\ N^{1+\epsilon} & \text{if } \omega \neq 0. \end{cases}$$

What about non-trivial bounds? See

<http://arxiv.org/pdf/1310.7190v1.pdf>

for more on this problem.

2010.1.4. *Peter Hegarty*. This is a problem on Phase Transitions inspired by Hannah Alpert's talk. Let G be an Abelian group, A a set of generators, everything infinite. Have $S(1), S(2), S(3), \dots, S(r), \dots$, where the sequence is $\infty, \infty, \dots, \infty, (n), 0, 0, \dots$ (where we may or may not have the n term). We should be able to compare the sizes of infinite, i.e., their measure. Suppose G is a compact Abelian group, such as the circle, and let A be a measurable set. Want to look at Lebesgue (or Haar) measure of the sets: $\mu(S(1)), \mu(S(2)),$ et cetera. The sequence should be unimodal (regular).

David Neumann looked at something similar. For finite groups looking at the sizes, did a lot of computations with different groups and generating sets. Did find an example where it wasn't the case, but typically do have unimodality. Hegarty conjectured that for any finite group (not necessarily Abelian) can always find a set of generators such that the sequence is unimodal.

2010.2. Problem Session IV: Saturday, May 29th.

2010.2.1. *Peter Hegarty*. Let $A \subset \mathbb{N}$,

$$r(A, n) = \#\{(a_1, a_2) : a_1 + a_2 = n\}.$$

What sequences of non-negative integers can be asymptotic representation functions? Of course there are restrictions if start from 0. Obviously only one way to represent 0 (0+0). Given a sequence of numbers, want the sequence to equal $r(A, n)$ starting at some point. Assuming Erdős-Turan, cannot be bounded and simultaneously not have infinitely many zeros.

Comment from Nathanson: Matter of choice whether take $r(A, n)$ or the function

$$\mathfrak{r}(A, n) = \#\{(a_1, a_2) : a_1 + a_2 = n, a_1 \leq a_2\}.$$

More generally, say $|S| = \infty$ and $S \subseteq G$, A is an asymptotic basis (of order 2) for S if $S \subseteq A + A$ (up to finite sets). Let

$$r_A(s) = \#\{(a_1, a_2) : a_1 + a_2 = s\}.$$

We have $r_A : S \rightarrow \mathbb{N} \cup \{\infty\}$. Hardest problem is what we had earlier.

2010.2.2. *Mel Nathanson*. Erdős-Renyi Method: Let

$$\Omega = \{\text{all sequences of non-negative integers}\}.$$

Let $0 \leq p(n) \leq 1$ for $n = 0, 1, 2, \dots$. Then there exists a probability measure P_r on Ω such that

$$P_n(E_n) := P_n(\{A \in \Omega : n \in A\}) = p(n)$$

then the events E_n are independent.

If choose $p(n)$ to be something like a logarithm over a power of n , say $\alpha \frac{\log^\beta n}{n^\gamma}$ with $1/3 < \gamma \leq 1/2$ – want a result that doesn't use any probability. If put this probability measure on the sequence of integers, then if $A \subset \mathbb{N}_0$ with $A + A \sim \mathbb{N}_0$ and $S(n) = \{a \in A : n - a \in A\}$, then for $m \neq n$ we have

$$|S(m) \cap S(n)| \leq \frac{2}{3\gamma - 1}$$

for all but finitely many pairs of integers.

Below is an example of where this result was used. An asymptotic basis means every number from some point onward can be written as $a + a'$ with $a, a' \in A$. An asymptotic basis A is minimal if no proper subset of A is an asymptotic basis. This means we have the set of integers with the property that if throw away any number then all of a sudden infinitely many numbers cannot be represented. Came up in an attempt to construct a counter-example to the Erdős-Turan conjecture. Not every asymptotic basis contains a minimal basis. There is a theorem that says that if $r_A(n) \rightarrow \infty$ and $|S_A(m) \cap S_A(n)| = O(1)$ then A contains a minimal asymptotic basis.

Theorem: If have a sequence with $r_A(n) > c \log n$ for some $c > 1/\log(4/3) \approx 3.47606$ and $n \geq n_0$ then A contains a minimal asymptotic basis.

2010.3. **Speaker List.**

- Hannah Alpert, University of Chicago
- Paul Baginski, Universite Claude Bernard Lyon, France
- Gautami Bhowmik, Universite Lille, France
- Kent Boklan, Queens College (CUNY)
- Mei-Chu Chang, University of California-Riverside
- Scott Chapman, Sam Houston State University
- Brian Cook, University of British Columbia
- David Covert, University of Missouri
- Aviezri Fraenkel, Weizmann Institute of Science, Israel
- John Friedlander, University of Toronto
- John Griesmer, University of British Columbia
- Sinan Gunturk, Courant Institute, NYU
- Peter Hegarty, Chalmers University of Technology and University of Gothenburg
- Charles Helou, Penn State Brandywine
- Alex Iosevich, University of Missouri
- Renling Jin, College of Charleston
- William J. Keith, Drexel University
- Alex Kontorovich, Institute for Advanced Study
- Brandt Kronholm, SUNY at Albany
- Urban Larsson, Chalmers University of Technology and University of Gothenburg
- Jaewoo Lee, Borough of Manhattan Community College (CUNY)
- Zeljka Lujic, CUNY Graduate Center
- Neil Lyall, University of Georgia
- Steven J. Miller, Williams College
- Rishi Nath, York College (CUNY)
- Mel Nathanson, Lehman College (CUNY)
- Lan Nguyen, University of Michigan
- Kevin O'Bryant, College of Staten Island (CUNY)
- Alex Rice, University of Georgia
- Steven Senger, University of Missouri
- Jonathan Sondow, New York
- John Steinberger, Institute for Theoretical Computer Science, Tsinghua University

2011. CANT PROBLEM SESSIONS: 2011

2011.1. **Problem Sessions.** There was an issue with my computer and the original file was lost for 2011; the items below are restored from earlier copies, though I have lost who spoke on what day and thus have run this as one entry.

2011.1.1. *Seva Lev.* **Problem:** Let $A \subset \mathbb{F}_2^n$, $p \in \mathbb{F}_2[x_j]_{j=1}^n$, and for all $a, b \in A$ if $a \neq b$ then $p(a+b) = 0$. Does this imply that $p(0) = 0$?

For example, if $A = \{a, b\}$ then $p(a+b) = 0$ does not imply $p(0) = 0$.

If A is large and the degree of p is small, what is true? For a given p , how large must $|A|$ be for this to be true? We have the following:

$\deg P$	need
0	$ A \geq 2$
1	$ A \geq 3$
2	$ A \geq n+3$
3	$ A \geq 2n$
$\leq (\frac{1}{2}o(1))n$???

2011.1.2. *Giorgis Petridis.* P-R: $D_2 \geq 1$ implies that there exists v_0 vertex disjoint paths of length 2 in G .

Problem: What can be said when $D_2 \geq k \in \mathbb{Z}$?

Guess: there exist v_0 vertex disjoint trees in G each having at least k_i vertices in V_i . Note: there is an example which shows that one cannot hope to prove this guess using max flow - min out. Guess confirmed in $k = |V_0| = 2$ by Petridis.

2011.1.3. *Mel Nathanson.* Believe the following is an unsolved problem by Hamidoune (he proposed it and no one has solved it):

Problem: Let G be a torsion free group, $G \neq \{e\}$. Let S be a finite subset of G , $e \in S$,

$$\kappa_k(S) = \min \{|XS| - |X| : \text{finitesets } X \subset G, |X| \geq k\}.$$

Hamidoune conjectured that there is an $A \subset G$ with $|AS| - |A| = \kappa_k(S)$ and $|A| = k$.

True for $k = 1$, unknown for $k \geq 2$. It is true for ordered groups. As every free abelian group of finite rank can be ordered, true here. In general for $k = 2$ still unknown.

2011.1.4. *Matthew DeVos.* **Problem:** Let G be a multiplicative group, $S \subset G$ a finite set, and set

$$\Pi(S) = \{s_1 \cdots s_k : s_i \in S, s_i = s_j \iff i = j\} \cup \{1\}.$$

Not allowed to use an element multiple times. Conjecture: there is a $c > 0$ such that for every group G and set $S \subset G$ there exists $H \subset G$ with $|\Pi(S)| \geq |H| + c|H| \cdot |S \setminus H|^2$.

True with $c = 1/64$ when G is abelian.

2011.1.5. *David Newman.* **Problem:** How many partitions are there where no frequency is used more than once?

For example, the partitions of 4 are $\{4\}$, $\{3, 1\}$, $\{2, 2\}$, $\{2, 1, 1\}$ and $\{1, 1, 1, 1\}$. The ones that are okay are all but $\{3, 1\}$. The problem here is that the two decompositions each occur just once: we have one 3 and one 1.

2011.1.6. *Steven J. Miller, Sean Pegado, Luc Robinson.* **Problem:** For each positive integer k , consider all A such that $|kA + kA| > |kA - kA|$ and $1 \in A$ (for normalization purposes). Let C_k be the smallest of the largest elements of such A 's. What can you say about the growth of C_k ?

$$C_1 = 15, C_2 = 31, \dots$$

2011.1.7. *Speaker unremembered.* **Problem:** Assume that you have A, B in a general group and $|AB| < \alpha|A|$ and $|AbB| \leq \alpha|A|$ for all $b \in B$. Does there exist an absolute c such that $X \subset A$ then $|XB^h| \leq \alpha^{ch}|X|$?

Rusza showed that if you have $|A+B_j| \leq \alpha_j|A|$ for $j = 1, 2$, then there is an X such that $|X+B_1+B_2| \leq \alpha_1\alpha_2|X|$.

Problem: Is there a prescription for X given that Rusza's theorem shows the existence of X .

2011.1.8. *Speaker unremembered.* **Problem:** Let \mathcal{B} be a partition of n . Consider the partition where $c_1 + \dots + c_k = n$, and $1^{d_1} 2^{d_2} \dots n^{d_n}$. The d_i 's are the number of the c_j 's and $d_1 + \dots + d_n = m$. Consider

$$\sum_{\mathcal{B} \in \mathcal{P}(n)} \binom{n}{c_1, \dots, c_k} \binom{n+m+1}{n+1, d_1, \dots, d_n} \left[\frac{1}{m+n+1} \right].$$

What can be said?

Try putting in an r^n and summing over n . Maybe this is a holomorphic part of a non-holomorphic Maass form.

2011.1.9. *Peter Hegarty.* **Problem:** Consider the least residue of n modulo q , denoted $[n]_q$, which is in $\{-q/2, \dots, q/2\}$. Want a function from $\pi : \{1, \dots, 27\}$ to itself (a permutation, so 1-1) with the property that given any a, b, c not all equal with $|[a+c-2b]_{27}| \leq 1$ then $|[\pi(a) + \pi(c) - 2\pi(b)]_{27}| \geq 2$.

Motivation: replace 27 with n , ..., have a permutation avoiding a progression. Conjecture that a permutation of \mathbb{Z}_n exists for every n sufficiently large.

2011.1.10. *Speaker unremembered.* **Problem:** Define $h : \{1, \dots, N\} \rightarrow \mathbb{Z}/N\mathbb{Z}$; call it a partial homomorphism if it is a bijection such that whenever $a, b, ab \in \{1, \dots, N\}$ then $h(ab) = h(a) + h(b) \pmod{N}$. Does such a function exist for all N ?

Have built by hand for all N up to 64?

2011.1.11. *Steven Senger.* The basic idea is that an additive shift will destroy multiplicative structure. Given a large, finite set, $A \subset \mathbb{N}$, suppose that $|AA| = n$. We know that there exists no generalized geometric progression, G , of length $c_1 n$, such that $|(AA+1) \cap G| \geq c_2 n$, where c_1 and c_2 do not depend on n . The question is, given the same conditions on A , do there exist sets $E, F \subset \mathbb{N}$, such that the following hold for c_3, c_4 independent of n , and $\delta > 0$:

- $|E|, |F| \geq n^\delta$
- $|EF| = c_3 n$
- $|(AA+1) \cap EF| \geq c_4 n$

Even partial results would be interesting to me. Also, considering the problem over \mathbb{R} would be interesting to me.

2011.1.12. *Urban Larsson*. 2 pile Nim can be described as the set of moves on a chessboard made by a rook, moving only down and left. Players take turns moving the rook, and the person to move it to the lower-left corner is the winner. The set of legal moves is defined to be

$$\{(0, x), (x, 0)\}.$$

In this case, the positions which guarantee victory following perfect play, or *p-positions* are along the diagonal. That is, the player who consistently moves the rook to the diagonal will eventually win.

In Wythoff Nim, the piece is replaced by a queen, and the diagonal move is added. The set of legal moves for Wythoff Nim is

$$\{(0, x), (x, 0), (x, x)\}.$$

This game has p-positions close to the lines of slope ϕ and ϕ^{-1} , where, ϕ denotes the golden ratio. For example, the points $(\lfloor \phi x \rfloor, \lfloor \phi^2 x \rfloor)$ are p-positions in Wythoff Nim.

Now, adjoin the multiples of the last possible p-positions from Wythoff Nim which are not in Wythoff Nim, namely the multiples of the knight's move. The legal moves of the new game are

$$\{0, x), (x, 0), (x, x), (x, 2x), (2x, x)\}.$$

The p-positions for this game appear to split along lines of slopes nearly 2.25 and 1.43. Why?

2011.1.13. *Thomas Chartier*. Let $n, k \in \mathbb{N}$, and $p = nk + 1$ be prime. Exclude 1 and 2. Fixing n does there exist a k such that

$$1^k, 2^k, 3^k, \dots, n^k$$

are distinct mod p ? The conjecture is that such a k exists for every non-trivial n .

2011.1.14. *Mel Nathanson*. Recall the classical sum-product problem of Erdős. Given a large set of positive integers, $A \subset \mathbb{N}$, either the set of sums or the set of products should be large. The conjecture is that, for such an A , with c independent of n , for any $\epsilon > 0$,

$$\max\{|A + A|, |AA|\} \geq cn^{2-\epsilon}.$$

2011.2. Speaker List.

- (1) Paul Baginski, Universite Claude Bernard Lyon, France
- (2) Mei-Chu Chang, University of California-Riverside
- (3) Scott Chapman, Sam Houston State University
- (4) Jonathan Cutler, Montclair State University
- (5) Matthew DeVos, Simon Fraser University
- (6) Aviezri Fraenkel, Weizmann Institute of Science, Israel
- (7) Peter Hegarty, University of Gothenburg, Sweden
- (8) Charles Helou, Penn State Brandywine
- (9) Jerry Hu, University of Houston - Victoria
- (10) Alex Iosevich, University of Missouri
- (11) Renling Jin, College of Charleston
- (12) Nathan Kaplan, Harvard University
- (13) Mizan R. Khan, Eastern Connecticut State University
- (14) Omar Kihel, Brock University, Canada
- (15) Alex Kontorovich, SUNY at Stony Brook
- (16) Urban Larsson, University of Gothenburg, Sweden
- (17) Thai Hoang Le, Institute for Advanced Study
- (18) Vsevolod Lev, University of Haifa, Israel
- (19) Zeljka Lujic, CUNY Graduate Center
- (20) Neil Lyall, University of Georgia
- (21) Steven J. Miller, Williams College
- (22) Rishi Nath, York College (CUNY)

- (23) Mel Nathanson, Lehman College (CUNY)
- (24) Hoi H. Nguyen, University of Pennsylvania
- (25) Lan Nguyen
- (26) Sean Pegado, Williams College
- (27) Giorgis Petridis, University of Cambridge
- (28) Luc Robinson, Williams College
- (29) Steve Senger, University of Missouri
- (30) Jonathan Sondow, New York

2012. CANT PROBLEM SESSIONS: 2012

2012.1. Problem Session I: Tuesday, May 22nd (Chair Renling Jin).

- From Renling Jin, *jinr@cofc.edu*: Define a subset of the natural numbers B to be an essential component if for all $A \subset \mathbb{N}$, $\sigma(A+B) > \sigma(A)$ if $0 < \sigma(A) < 1$. B is an extraordinary component if

$$\liminf_{\sigma(A) \rightarrow 0} \frac{\sigma(A+B)}{\sigma(A)} = \infty.$$

Here

$$\sigma(A) = \inf_{x \geq 1} \frac{A(x)}{x}.$$

Ruzsa conjectured that every essential component is an extraordinary component.

What are the essential components we know? If

$$B = \{k^2 : k \in \mathbb{N}\}$$

then

$$\sigma(A+B) \geq \sigma(A)^{1-1/4}$$

since B is a basis of order four. We get

$$\frac{\sigma(A+B)}{\sigma(A)} \geq \frac{1}{\sqrt[4]{\sigma(A)}}.$$

Similar for cubes or k -powers.

- From Steven J. Miller, *sjm1@williams.edu*: We say a set A is a More Sums Than Differences Set, or an MSTD set, if $|A+A| > |A-A|$, where

$$\begin{aligned} A+A &= \{a_i + a_j : a_i, a_j \in A\} \\ A-A &= \{a_i - a_j : a_i, a_j \in A\}. \end{aligned}$$

As addition is commutative and subtraction is not, it's expected that 'most' sets are difference dominated; however, Martin and O'Bryant proved that a positive percentage of sets are sum-dominated. There are explicit constructions of infinite families of sum-dominant sets. Initially the best result was a density of $n^{c2^{n/2}/2^n}$, then $1/n^4$ (or $1/n^2$), and now the record is $1/n$ (where our sets A are chosen uniformly from subsets of $\{0, 1, \dots, n-1\}$). Can you find an 'explicit' family that is a positive percentage.

- From Urban Larsson, *urban.larsson@yahoo.se*: Let $A = \{0, 1, 3, 4, \dots\}$ for a set that avoids arithmetic progression, thought to be best set to avoid arithmetic progression but not (comes from a greedy construction). Equivalence with a base 3 construction: $A = \{0, 1, 10, 11, 100, 101, \dots\}$ gives $A((3^n + 1)/2) = 2^n$, where $A(n) = \#\{i \in A \mid i < n\}$. Hence, for all n , $A(n) < Cn^{\log 2 / \log 3} \approx n^{2/3}$. Study impartial heap games. Is it possible to find a game such that the P and N-positions correspond to the numbers in this construction? (A position is in N if and only if the first player wins.) In some sense such that:

$$\begin{array}{ccccccccc} & P & P & N & P & P & N & N & \\ \hline & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \end{array}$$

We rather use three heaps of sizes in three-term arithmetic progression. A legal move is to erase the largest pile and then to announce one of the smaller piles as the new largest pile. Notation (x, y) , where x is the number of tokens in the smallest heap and y in the second smallest. In the table below, the first entry is the outcome, the second is the position, the third is the Grundy value, and

the fourth are the options.

P	$(0, 1)$	0
N	$(0, 2)$	1 $(0, 1)$
P	$(0, 3)$	0 $(1, 2)$
N	$(0, 4)$	0 $(0, 2), (2, 3)$
N	$(1, 2)$	1 $(0, 1)$
N	$(1, 3)$	0 $(1, 2),$
P	$(1, 4)$	0 $(0, 2), (2, 3)$
N	$(2, 3)$	2 $(0, 1), (1, 2)$
N	$(2, 4)$	3 $(1, 2), (0, 2), (2, 3)$
N	$(3, 4)$	0 $(1, 2), (0, 2), (2, 3)$

The P positions (Grundy value 0) have both lower heap sizes in the set A . The N positions have Grundy values > 0 , defined as the *minimal exclusive* of the Grundy values of the options. What are they? Is it possible to extend the game by *adjoining moves* to obtain $\limsup A(n)/n^{\log 2 / \log 3} = \infty$? The game generalizes to k -term arithmetic progressions and the Sidon-condition for example.

How do we extend such games? We need a general definition for the family of games. A *ruleset* is a set of finite sets of positive integers. From a *position* consisting of a set S of non-negative integers, choose one of the numbers $s \in S$ and a set M of numbers from the given ruleset. The next position, which is a set of nonnegative numbers, is $\{s - m \mid m \in M\}$, provided $\max M \leq s$. We get a recursive definition of the set A which determines the P-positions for a given M . A position S is in P if and only if $S \subset A$. That is S is in N iff $S \cap A \neq \emptyset$. In this sense we can abuse notation and regard A as the set of “P-positions”. A game extension of M is $M \cup M'$, for M' a set of finite sets of nonnegative numbers. For our game the set M is $M = \{\{d, 2d\} \mid d > 0\}$. One first example of a game extension is $M = \{\{d, 2d\} \mid d > 0\} \cup \{\{1\}\}$. Question: does the set A become less dense for this game than for our original AP-avoiding game?

2012.2. Problem Session II: Wednesday, May 23rd (Chair Steven J Miller).

- From Steven J Miller, sjm1@williams.edu: We investigated in <http://arxiv.org/pdf/1109.4700v2.pdf> properties of $|A + A|$ and $A + A$ as A varies uniformly over all subsets of $\{0, 1, \dots, n - 1\}$. How does the behavior change if we change the probability of choosing various A 's (see for example my work with Peter Hegarty: <http://arxiv.org/pdf/0707.3417v5>).

Another related problem is to ‘clean-up’ the formula we have for the variance. This involves sums of products of Fibonacci numbers – can the answer be simplified?

What about the expected values of $2kA$ versus $kA - kA$.

- From Ryan Ronan, ryan.p.ronan@gmail.com: Earlier today I discussed joint work on generalized Ramanujan primes, <http://arxiv.org/pdf/1108.0475>. One natural question is whether or not for each prime p there is some constant c_p such that p is a c_p -Ramanujan prime.

Another question is the distribution of c -Ramanujan primes among the primes, in particular the length of runs of these and non-these. It can take awhile for the limiting behavior of primes to set in; it's dangerous to make conjectures based on small sized data sets. Are the calculations here sufficiently far enough down the number line to have hit the limiting behavior? For a related question, perhaps the Cramer model is not the right model to use to build predictions, and instead we should use a modified sieve to construct ‘random primes’. It would be worthwhile to do so and see what happens / what the predictions are.

- *From Steven Senger, senger@math.udel.edu:* Have a subset A of a finite field \mathbb{F}_q satisfying for all ϵ and δ positive (1) $|A| |AA| \geq q^{3/2+\epsilon}$, (2) $|AA| \leq q^{1-\delta}$. For all generalized geometric progressions G with $|G| \approx |AA|$ we have $|(AA + 1) \setminus G| \geq q^\delta$. Can reduce the size constraint (1)? Can we increase the size of $|(AA + 1) \setminus G| \geq q^\delta$?
- *From Kevin O'Bryant, obryant@gmail.com:* How far out can you go $\{x_1, x_2, x_3, x_4, \dots\}$ such that the first four are in the first four quadrant, the first nine in the first nine subdivisions (3×3), the first 16 in the first 4×4 and so on.... We know this can't go on forever, violates Schmidt.

The discrepancy of the sequence $\{x_i\}$ is

$$\text{Disc}(\{x_i\}_{i=1}^d) = \sup_R \left| \frac{\#\{x_i \in R\}}{d} - A(R) \right|.$$

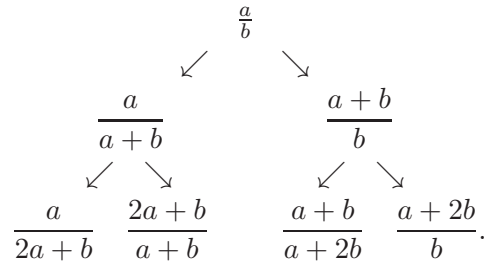
We have $\text{Disc}(\{x_i\}_{i=1}^d) \geq C \frac{\log d}{d}$. If we spread the points too well, the discrepancy gets very low.

Let me rephrase – I strongly believe that this logarithmic factor will kill this arrangement.

2012.3. Problem Session III: Thursday, May 24th (Chair Alex Iosevich).

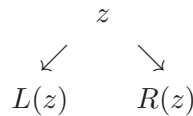
- *From Jerry Hu, HuJ@uhv.edu:*

This problem is related to Nathanson's talk "The Calkin-Wilf tree and a forest of linear fractional transformations" from Tuesday. Recall the form of the Calkin-Wilf tree, where we have:



When a and b are both initialized as 1, each positive rational number appears on the tree exactly once.

The question is: how can we generalize this? More specifically, do there exist other trees of the form



in which every positive rational number appears exactly once? Can we find all, or any, nontrivial functional pairs L, R such that this condition holds?

- *From Nathan Kaplan, nathanckaplan@gmail.com:* Here is a problem about counting lines among points in \mathbb{F}_3^n . I will give two different kinds of motivation for why someone might be interested in this.

The card game SET is played with 81 distinct cards, each of which has four attributes (number, color, shading, and shape), where each attribute has three possibilities. We can identify a card with a 4-tuple (x_1, x_2, x_3, x_4) , where each $x_i \in \mathbb{F}_3$. The game is played by collecting sets. A set is a collection of three cards (x, y, z) such that for each of the four attributes each card is the same or all three cards are different. It is equivalent that the vectors in \mathbb{F}_3^4 represented by our three cards take the form $(x, y, -(x+y))$, or equivalently, $(x, x+d, x+2d)$. Therefore, we see that what we are looking for is a three term arithmetic progression in \mathbb{F}_3^n . In \mathbb{F}_3^n a 3-term AP is equivalent to a line. A set of vectors with no 3-term AP is called a cap set. The cap set problem asks, "What is the maximum size of a cap set in \mathbb{F}_3^n ?" This problem is very hard and has been well-studied. Exact answers are known only for $n \leq 6$. We note that for $n = 3$ the cap set problem is equivalent to asking for the maximum number of SET cards one can have so that there is no set among them. The

answer to this is 20 and an argument is given in the paper *The Card Game Set* by Benjamin Davis and Diane MacLagan.

There is a related problem motivated by SET which does not seem to have appeared in the literature. The game is usually played by dealing out 12 cards. We know that it is possible to have no sets at all, but we could ask for the largest number of sets which could occur among 12 cards. I can show that this is 14, but the argument is sort of ad hoc and not so satisfying. I have not found anything written before about the following question. What is the maximum number of lines that m points in \mathbb{F}_3^n can contain? Note that any two points determine a unique line, so if a set contains many lines, then it determines few lines. Equivalently we could ask for the minimum number of lines determined by m points in \mathbb{F}_3^n . This question is very general and includes the cap set problem as a subcase. This is because the number of lines contained in a subset of \mathbb{F}_3^n determines the number of lines contained in its complement, so if we know the maximum number of lines among any collection of m points for all m , then we also know the minimum number of lines among m points.

Here is the actual problem I am asking. In the argument for the maximum number of lines among 12 points in \mathbb{F}_3^4 is 14, it is clear that the maximum number of lines among 12 points in \mathbb{F}_3^n is 14 for any $n \geq 3$. That is, if we want lots of lines, the best thing that we can do is to put our points into the smallest possible dimensional subspace that can contain them.

Conjecture 2012.1. Fix $m \geq 0$ and let $d = \lceil \log_3(m) \rceil$. For any $n \geq d$, the maximum number of lines contained among m points in \mathbb{F}_3^n is equal to the maximum number of lines contained among m points in \mathbb{F}_3^d .

I think that this is probably true and that the proof for it is probably easy. One could also ask similar questions for \mathbb{F}_q^n for other q .

Here is some extra motivation that the cap set problem is interesting. Tic-Tac-Toe on a $3 \times 3 \times 3$ board can never end in a draw no matter how many moves are made by each player. This is the first case of a more general phenomenon, the Hales-Jewett Theorem. Given k , there exists a d such that Tic-Tac-Toe on a $k \times \cdots \times k = [k]^d$ board (where it takes k in a row to win) cannot end in a draw no matter how many times each player moves. A more precise statement is that for large enough n , either a set or its complement must contain a combinatorial line. I won't define exactly what a combinatorial line is, but it is a slightly more restrictive condition than a Tic-Tac-Toe line, which is slightly more restrictive than the type of line described above in the discussion of SET.

A few years ago, the initial Polymath project organized by Tim Gowers was focused on giving a combinatorial proof of the Density Hales-Jewett Theorem. The only previous proof of this theorem involved arguments from ergodic theory. Let $c_{n,k}$ be the largest number of points of $[k]^n$ which does not contain a combinatorial line. Let $c'_{n,k}$ be the largest number of points of $[k]^n$ which does not contain a geometric line (you can think of this as a Tic-Tac-Toe line. These are called Moser numbers. Finally, let $c''_{n,k}$ be the largest number of points of $[k]^n$ without a line of the type described above. Clearly $c''_{n,k} \leq c'_{n,k} \leq c_{n,k}$.

Theorem 2012.2 (Density Hales-Jewett). Fix $k \geq 1$. Then

$$\lim_{n \rightarrow \infty} \frac{c_{n,k}}{n^k} = 0.$$

This result is important in understanding the growth of cap sets. The Polymath project also proved the best known lower bound for $c_{n,k}$. It is quite difficult to compute these numbers in general, even for small k . We mentioned above that $c'_{4,3} = 20$ and it is also known that $c'_{5,3} = 45$ and that $c'_{6,3} = 112$. This last statement determines the maximum number of lines among $3^6 - 112$ points in $[3]^6$, for example. The Polymath project also determined more values of $c_{n,3}$ and $c'_{n,3}$ than previously known.

Since so much work has gone into understanding large subsets of $[k]^n$ with no lines, it seems reasonable to study collections of points which contain the largest possible number of lines.

2012.4. **Problem Session IV: Friday, May 25th (Chair Kevin O'Bryant).** The following papers are relevant for the problems proposed by Steven Miller.

- <http://arxiv.org/abs/1107.2718>
- <http://arxiv.org/abs/1008.3204>
- <http://arxiv.org/abs/1008.3202> (the gap paper referenced below is in preprint stage, but available upon request).
- <http://www.emis.de/journals/INTEGERS/papers/j57/j57.pdf> (Hannah Alpert).

Proposed problems.

- *From Steven J. Miller, sjm1@williams.edu:* The following problems are related to Zeckendorf decompositions. **Many of these are currently being studied by my summer REU students in the Williams 2012 SMALL program. If you are interested in working on these, please email me at sjm1@williams.edu.**

◊ We know every number has a unique Zeckendorf decomposition, and appropriately localized the number of summands converges to being a Gaussian. What happens if we have a decomposition where some integers have multiple representations? What if there are some integers that have no representations? Instead of counting the total number of summands, what if you just count how many of each summand one has (so in decimal 4031 wouldn't count as $4+0+3+1$ but $1+0+1+1$).

◊ We have formulas for the limiting distribution of gaps between summands of Fibonacci and some generalized Fibonacci sequences. Try to find formulas for general recurrence relations as a function of the coefficients of the relations. Do this for the signed Fibonacci decomposition (see Hannah Alpert's paper; can we generalize signed distributions to other recurrence relations). What about the distribution of the largest gap (that should grow with n for numbers between H_n and H_{n+1}). If we appropriately normalize it, does it have a nice limiting distribution?

- *From Mizan Khan, khanm@easternct.edu:* Let

$$\mathcal{H}_n := \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : xy \equiv 1 \pmod{n}, 1 \leq x, y \leq n-1\}.$$

Consider the convex closure of \mathcal{H}_n — what can we say about the number of vertices in this convex closure? Let $v(n)$ be the number of vertices. Easily, $v(n) \geq 2(\tau(n-1)-1)$, where τ is the number of positive divisors.

It is easy to see that $\limsup v(n) = \infty$. Can we show that $\lim_{n \rightarrow \infty} v(n) = \infty$?

Also, consider $D(n) = v(n) - 2(\tau(n-1)-1)$. We know that $D(n) > 0$ for a set of density 1 in the naturals and furthermore $D(n) = 0$ on a set which is $\gg \frac{x}{\log x}$. Can we improve the second estimate?

- *From Steven Senger, senger@math.udel.edu:* We will call a family of sets, $P_n \subset [0, 1]^2$, s -adaptable if they satisfy the following bound:

$$\frac{1}{\binom{n}{2}} \sum_{x \neq y; x, y \in P_n} |x - y|^{-s} \leq 1.$$

The Szemerédi-Trotter incidence theorem says that for a set of n points and m “reasonable” curves in the plane, the number of incidences of points and curves is bounded above by

$$I \leq (nm)^{\frac{2}{3}} + n + m.$$

Can we get better incidence bounds for s -adaptable sets? Specifically, can we get tighter bounds in the case of n points and n circles centered at those points?

- *From Nathan Pflueger, pflueger@math.harvard.edu:* Suppose S is a numerical semigroup, $S \subset \mathbb{N}_+$, closed under addition, i.e., $S + S \subset S$. Let $G := \mathbb{N}_+ \setminus S$. Define the weight of S to be $w(s) = |\{(x, y) \in S \times G : 0 < x < y\}|$. Define the irreducible elements of S to be the minimal generators. Define the effective weight of S to be $w_{eff}(s) = |\{(x, y) \in S_{irred} \times G : 0 < x < y\}|$. Let the genus of S be $g = |G|$.

For example, $S = \langle 3, 5 \rangle$. Then $w(s) = 4$, and $w_{eff}(s) = 3$.

Can we characterize the genus g subgroups of largest effective weight? We believe the largest is $\approx \frac{g^2}{4}$, and in the form $\langle a, a+1, \dots, b-1, b \rangle$, where $b < 2a$.

This comes from algebraic geometry. Pick a point p on an algebraic curve or surface. $S = \{ord_p(f) : f \text{ is a rational function}\}$, where $ord_p(f)$ is the order of the single pole at p of f .

2012.5. **Speaker List.**

- John Bryk, John Jay College (CUNY)
- Mei-Chu Chang, University of California-Riverside
- Emel Demirel, Bergen County College
- Frederic Gilbert, Ecole Polytechnique, Paris
- Christopher Hanusa, Queens College (CUNY)
- Charles Helou, Penn State Brandywine
- Jerry Hu, University of Houston - Victoria
- Alex Iosevich, University of Rochester
- Geoff Iyer, University of Michigan
- Renling Jin, College of Charleston
- Nathan Kaplan, Harvard University
- Mizan R. Khan, Eastern Connecticut State University
- Sandra Kingan, Brooklyn College (CUNY)
- Alex Kontorovich, Yale University
- Urban Larsson, Chalmers University of Technology and University of Gothenburg
- Oleg Lazarev, Princeton University
- Xian-Jin Li, Brigham Young University
- Neil Lyall, University of Georgia
- Steven J. Miller, Williams College
- Rishi Nath, York College (CUNY)
- Mel Nathanson, Lehman College (CUNY)
- Kevin O'Bryant, College of Staten Island (CUNY)
- Kerry Ojakian, St. Joseph's College, New York
- Ryan Ronan, Cooper Union
- Steven Senger, University of Delaware
- Jonathan Sondow, New York
- Liyang Zhang, Williams College
- Wei Zhang, Columbia University

2013. CANT PROBLEM SESSIONS: 2013

2013.1. Problem Session I: Tuesday, May 21st (Chair Steven Miller).

2013.1.1. *MSTD sets and their Generalizations*. Proposed by Steven J. Miller and expanded on by the audience: There are many problems one can ask about More Sums Than Differences sets. Here are just a few.

- We know that, in the uniform model, a positive percentage of the 2^n subsets of $\{0, 1, \dots, n-1\}$ are sum-dominant. Unfortunately these proofs are non-constructive, in that one shows with high probability almost anything thrown between two specially chosen fringes work. Early constructions of explicit families often involved tweaking arithmetic progressions (which are balanced). While these early families were often sub-exponential in terms of their relative size, work of Miller, Scheinerman and Orosz proved that one can find 'explicit' families with density $1/n^2$; Zhao obtained a density of $1/n$ through the use of bidirectional ballot sequences. Can one find an explicit formula with a better density (or, dare to dream, one that is a positive percentage?).
- Continue to investigate phase transitions, and the natural of the relative size function, for more summands with different combinations of size. **This is currently being studied by students in Miller's 2013 REU at Williams.**
- Instead of looking at $A + A$ and $A - A$, choose A and B randomly and study $A + B$ and $A - B$ (of course, $A - B$ allow both $a - b$ and $b - a$ for $a \in A$ and $b \in B$).
- In determining if A is sum-dominant or difference-dominant, it doesn't matter how much larger one is than the other. Try and find a natural weighting on the sets, try to take into account by how much one beats the other.
- Is there a set A such that $|A + A| > |A - A|$ and $|A \cdot A| > |A/A|$? If yes, can you find an explicit, infinite family? What is the density of such sets? **Note: Miller finds this problem interesting and wants to bring this to his REU students. Anyone interested in collaborating please email sjm1@williams.edu.**
- Instead of looking at subsets of the integers or finite groups, look at subsets of \mathbb{Z}^d , intersected with different regions (say spheres, boxes). These sets have different fringe structures. How does the shape of the fringe affect the answer? We can play with the relative sizes of the length and width of a box in two dimensions, for example. **This is currently being studied by students in Miller's 2013 REU at Williams.**
- Can we say anything about MSTD sets in the continuous case? Is this related to some results on measures? What about subsets of fractals or other special objects (similar to the modular hyperbolas Amanda mentioned).
- (Mizan Khan): Speaking of Amanda's talk, the 84% lower bound mentioned is almost surely not the true answer. What do numerical investigations suggest? What is the correct limiting behavior?

2013.1.2. *Weakened Convex Functions*. Problem proposed by Seva Lev.

Consider functions $f : [0, 1] \rightarrow \mathbb{R}$ that satisfy (1) $\max\{f(0), f(1)\} \leq 0$ and (2) for any $0 \leq x_1 \leq \dots \leq x_m \leq 1$ we have

$$f\left(\frac{x_1 + \dots + x_m}{m}\right) \leq \frac{f(x_1) + \dots + f(x_m)}{m} + (x_m - x_1).$$

Note that convex functions satisfy this.

Set

$$F_m(x) = \sup\{f(x) : f \in \mathcal{F}_m\},$$

where $F_m \in C([0, 1])$ and $F(0) = F(1) = 0 < F(x)$ for $0 < x < 1$. What is F_m explicitly?

Results are known for $m \in \{2, 3, 4\}$. For such m we have

$$F_m(x) = \sum_{k=1}^{\infty} m^{-k} \min\{\|m^{k-1}x\|, 1/m\}.$$

When $m = 2$ we have $2\omega(x)$, where

$$\omega(x) = \sum_{k=0}^{\infty} 2^{-k} \|2^k x\|.$$

What about $m = 4$?

2013.2. **Problem Session II: Wednesday, May 23rd (Chair Seva Lev).**

2013.2.1. *Matrices and Curves*. Problem proposed by Seva Lev.

Consider an $m \times n$ matrix A whose entries are 0 or 1. Consider n points in the plane $\{p_1, \dots, p_n\}$, with each point corresponding to a column of A . If there exists m curves (continuous, no self-intersection) $\{c_1, \dots, c_m\}$ with each curve corresponding to a column of A , such that

- curve c_i passes through point p_j if $A_{i,j} = 1$ and does not pass through point p_j if $A_{i,j} = 0$, and
- any two curves intersect at most once,

we will call A realizable by curves.

The following are questions we can ask:

- What conditions can we put on A to guarantee A is realizable? Note: requiring the dot product of any two rows of A to be at most one does not guarantee A is realizable.
- Can you find a small matrix that is not realizable?
- Lastly, if A is realizable, does this mean A^T is realizable? The speaker does not see a reason this should be true, but hasn't found a counterexample yet.

This problem might be related to planar graphs.

2013.2.2. *Enumerating Points in the Plane with Polynomials*. Problem proposed by Mel Nathanson.

Consider the set $P = \{(x, y) : x \geq 0, 0 \leq y \leq \alpha x\}$. Does there exist a bijective polynomial $f : P \rightarrow \mathbb{N} \cup \{0\}$?

For instance, if $\alpha = 1$, then $f(x, y) = \frac{x(x+1)}{2} + y$. Notice when $y = 0$, $f(x, 0)$ is a triangular number. However, even when $\alpha = 2/3$, it is not clear what f should be or if f even exists.

2013.2.3. *Sumsets*. Problem posed by Dmitry Zhelezov.

Let B be a set such that $|B| = n$. Let

$$B + B \supseteq A = \{a_0 < \cdots < a_n\},$$

where A is concave ($a_1 - a_0 > a_2 - a_1 > \cdots > a_n - a_{n-1}$) or convex ($a_1 - a_0 < a_2 - a_1 < \cdots < a_n - a_{n-1}$). Is it true that $|A| = O(n^2)$?

Problem posed by Steven Senger.

Let A and B be “large” subsets of \mathbb{N} (or \mathbb{R} , or \mathbb{Z} , ...). Do there exist “large” subsets of \mathbb{N} , C and D , such that $|((A \cdot B) + 1) \cap (C \cdot D)| = |A \cdot B|^{1-\epsilon}$?

2013.3. **Problem Session III: Thursday, May 24th (Chair Kevin O’Bryant).**

2013.3.1. *3-term Geometric Progressions in Sets of Positive Density*. Problem proposed by Kevin O’Bryant.

As motivation for this problem, recall Van der Waerden’s Theorem: given any partition of \mathbb{N} , at least one part has an arithmetic progression of arbitrarily large length. Similarly, we have Szemerédi’s Theorem: given any set of positive density in \mathbb{N} , there exists an arithmetic progression of arbitrarily large length. Here we are defining the density of $A \subseteq \mathbb{N}$ as $d(A) = \lim_{n \rightarrow \infty} \frac{|A \cap [1, n]|}{n}$.

It is known that Van der Waerden’s Theorem holds for geometric progressions as well. We would like to consider Szemerédi’s Theorem for geometric progressions, but unfortunately it is not true: the square-free integers provide a simple counter-example. Currently, there is work being done on which densities we can obtain with no geometric progressions.

The original problem proposed in the session was: if $A \subseteq \mathbb{N}$ has density 1, does A have a three-term geometric progression? After some Googling by Nathan Kaplan, a 1996 paper by Brienne Brown and Daniel M. Gordon, “On Sequences Without Geometric Progressions”, was found which stated that if $A \subseteq \mathbb{N}$ has a density and has no 3-term geometric progressions, then the density of A is bounded by .869.

The revised problem proposed is: Given a subset $A \subseteq \mathbb{N}$, which densities of A guarantee 3-term geometric progressions?

2013.3.2. *Convex Subsets of Sumsets*. Problem proposed by Dmitry Zhelezov and requested by Giorgis Petridis.

We consider a variant of the Erdős-Newman conjecture, but replace the idea of squaring a set with sumsets.

The problem proposed is: does there exist any set B with $|B| = n$ such that $B + B \supseteq A$ for some convex set A with $|A| = \Omega(n^2)$?

2013.4. **Problem Session IV: Friday, May 24th (Chair Renling Jin).**

2013.4.1. *Sumsets*. Problem proposed by Renling Jin.

Let $A, B \subseteq \mathbb{N}$ such that $\max A \geq \max B$, $0 = \min A = \min B$, and $\gcd(A, B) = 1$. Let $\delta = 1$ if $\max A = \max B$ and 0 otherwise. If $|A + B| = |A| + |B| - 2\delta$, what structure can $A + B$ have? We can also ask the same question if δ is replaced by $\delta(A, B)$, where $\delta(A, B) = 1$ if $A \subseteq B$ and 0 otherwise.

2013.5. **Additional Problems.** Proposed by Vsevolod F. Lev.

For integer $m \geq 2$, let \mathcal{F}_m denote the class of all real-valued functions f , defined on the interval $[0, 1]$ and satisfying the boundary condition $\max\{f(0), f(1)\} \leq 0$ and the “relaxed convexity” condition

$$f\left(\frac{x_1 + \dots + x_m}{m}\right) \leq \frac{f(x_1) + \dots + f(x_m)}{m} + (x_m - x_1),$$

$0 \leq x_1 \leq \dots \leq x_m \leq 1$. Now, let $F_m := \sup\{f : f \in \mathcal{F}_m\}$. It is easy to prove that $F_m \in C[0, 1]$, $0 = F_m(0) = F_m(1) < F_m(x)$ for all $x \in (0, 1)$, $F_m(1 - x) = F_m(x)$ for all $x \in [0, 1]$, and, somewhat surprisingly, $F_m \in \mathcal{F}_m$ (meaning that F_m is the maximal function of the class \mathcal{F}_m). What is F_m , explicitly? We have

$$F_m(x) = \sum_{k=0}^{\infty} m^{1-k} \min\{\|m^k x\|, 1/m\}, \quad m \in \{2, 3, 4\}$$

(where $\|x\|$ denotes the distance from x to the nearest integer), but for $m \geq 5$ this fails to hold.

It is easy to see that for any 0-1 matrix, say M , one always can find a system of simple curves and a system of points in the plane so that their incidence matrix is exactly the matrix M . Suppose now that any pair of curves is allowed to intersect in at most one point (belonging or not to our system of points), and let's say that M is *realizable* if such curves and points can be found. Clearly, a necessary condition for this is that the scalar product of any two rows of M be at most 1, but this condition is insufficient: say, for q large enough, by the Trotter-Szemerédi theorem, the point-line incidence matrix of $PG(q, 2)$ has two many incidences to be realizable. What are other reasonable necessary / sufficient conditions for M to be realizable? What are “small” examples of non-realizable 0-1 matrices?

2013.6. **Speaker List.**

- Paul Baginski, Smith College
- Arnab Bhattacharyya, DIMACS, Rutgers University
- Gautami Bhowmik, Universite de Lille, France
- Thomas Bloom, University of Bristol, UK
- Tomas Boothby, Simon Fraser University, Canada
- Amanda Bower, University of Michigan-Dearborn
- Jeff Breeding II, Fordham University
- Javier Cilleruelo, University of Madrid, Spain
- David Covert, University of Missouri - St. Louis
- Matthew Devos, Simon Fraser University, Canada
- Mauro Di Nasso, University of Pisa, Italy
- Mohamed El Bachraoui, United Arab Emirates University, UAE
- Leopold Flatto, City College (CUNY) and Bell Labs
- George Grossman, Central Michigan University
- Christopher R. H. Hanusa, Queens College (CUNY)
- Derrick Hart, Kansas State University
- Kevin Henriot, Universite de Montreal
- Ginny Hogan, Stanford University
- Jerry Hu, University of Houston - Victoria
- Renling Jin, College of Charleston
- Delaram Kahrobaei, New York City Tech (CUNY)
- Nathan Kaplan, Harvard University
- Omar Kihel, Brock University, Canada

- Sandra Kingan, Brooklyn College (CUNY)
- Thai Hoang Le, University of Texas
- Seva Lev, University of Haifa, Israel
- Neil Lyall, University of Georgia
- Richard Magner, Eastern Connecticut State University
- Steven J. Miller, Williams College
- Rishi Nath, York College (CUNY)
- Mel Nathanson, Lehman College (CUNY)
- Kevin O'Bryant, College of Staten Island (CUNY)
- Brooke Orosz, Essex County College
- Giorgis Petridis, University of Rochester
- Alex Rice, Bucknell University
- Tom Sanders, Oxford University, UK
- Steven Senger, University of Delaware
- Satyanand Singh, New York Tech (CUNY)
- Jonathan Sondow, New York
- Dmitry Zhelezov, Chalmers Institute of Technology, Sweden

2014. CANT PROBLEM SESSIONS: 2014

2014.1. Problem Session I: Wednesday, May 28th (Chair Steven J Miller).

2014.1.1. *Steve Senger*. From last year from a talk of Dmitry Zhelezov.

Let $A \subset \mathbb{R}$, $|A| < \infty$, let P be the longest arithmetic progression in $AA = \{ab : a, b \in A\}$. We have $|P| \leq cn^{1+\epsilon} \ll n^2$.

Dmintry (possibly from Hegarty): What if instead of \mathbb{R} we have \mathbb{F}_q , the finite field with q elements? Due to Grosu we get the longest progression is at most $cn^{1+\epsilon}$ if $n \leq c \log \log \log p$ where $q = p$ a prime. Question: What bounds can I get on the size of P if we replace \mathbb{R} with \mathbb{F}_q ? Here q can be anything.

2014.1.2. *Steven J Miller*. The following builds on my talk from earlier.

- How does the structure of number affect the answer or the rate of convergence?
- How does the answer depend on c ?
- What is the best way to compute all the k -symmetric means for a given n ? What if we want just a certain one (such as $k = n/2$)?
- Find other sequences and compute these means – is there an interesting phase transition?

2014.1.3. *Steven J Miller*. Consider the 196 game (or problem). Take an n -digit number; if it is not a palindrome reverse the digits and add. If the sum is not a palindrome continue, else stop. Lather, rinse and repeat. It's called the 196 problem as 196 is the first number where we don't know if the process terminates (in a palindrome) or goes off to infinity. We know numbers that do not terminate in base 2 (as well as powers of 2, base 11, base 17 and base 26).

What can you say about this problem? What about other bases than 10? What about other decomposition schemes? See <http://www.math.niu.edu/~rusin/known-math/96/palindrome>.

2014.1.4. *Nathan Kaplan*. Let C be a cubic curve in $\mathbb{P}^2(\mathbb{F}_q)$. Want a large subset so that there are no three points on a line.

The set of \mathbb{F}_q -points form an abelian group G . Three points sum to zero if and only if they lie on a line.

Given group G what's the largest subset H s.t. $x + y + z = 0$ with x, y, z distinct has no solutions in H ?

For example take $G = \mathbb{Z}/2\mathbb{Z} \times G'$ and take $(1, g)$.

What if $G = \mathbb{Z}/p\mathbb{Z}$, consider $\{0, 1, 2, \dots, \lfloor p/3 \rfloor\}$.

What if $\mathbb{Z}/5\mathbb{Z} \times G'$, take things of the form $(1, g)$ and $(4, g)$. For each group ask such a question.

One thing you can do is look at a greedy construction. What is the best percentage you can get?

2014.2. Problem Session II: Thursday, May 29th (Chair Kevin O'Bryant).

2014.2.1. *Kevin O'Bryant* ? A k -GP cover of $[N]$ is a family of \mathcal{F} of k -term geometric progressions with

$$[N] \subseteq \bigcup_{F \in \mathcal{F}} F.$$

Set

$$\gamma_k = \lim_{N \rightarrow \infty} \inf_{\mathcal{F}} \frac{|\mathcal{F}|}{N},$$

the infimum being over all k -GP covers of $[N]$.

It is easy to see that $\gamma_3 \geq \gamma_4 \geq \dots$, and by basic counting $\gamma_k \geq 1/k$. The cover

$$\mathcal{F} = \left\{ b \cdot 2^{ki} \cdot \{1, 2, \dots, 2^{k-1}\} : 1 \leq b \cdot 2^{ki} \leq N, i \geq 0, b \text{ odd} \right\}$$

shows that

$$\gamma_k \leq \frac{2^k}{2(2^k - 1)},$$

so that $\lim_{k \rightarrow \infty} \gamma_k$ exists and is in $[0, 1/2]$. I conjecture that the limit is positive.

Comment from Bloom (in audience): The GP $\{1, 2, \dots, 2^{N-1}\}$ is covered by the APs $\{1, 2, 3, 4\}, \{8, 16, 24, 32\}, \dots$, so the analog of γ_k does go to 0.

Comment from Xiaoyu (in audience): Each GP has at most 2 squarefree numbers, so $\gamma_k \geq 3/\pi^2$. This proves the conjecture, but leaves open the precise limit.

2014.2.2. *David Newman*. About partitions. Finding two sets of partitions which are equal. Finding a type of partition which can divide into two different sets. $\prod(1+x^n)$ counts partition into distinct parts. Change plus sign into minus signs:

$$\prod(1-x^n) = 1 - x - x^2 + x^5 + x^7 + \dots$$

Most of the time, the partitions (?) are equal, but for 1, 2, 5, 7, ... , they differs by one

$$\prod \frac{1}{1-x^n} = (1+x+x^2+\dots)(1+x^2+x^4+\dots)\dots$$

Question: if you change some of the plus signs on the RHS to minus signs, is it possible to get all the coefficients (when you expand) to be $-1, 0, 1$.

"I have an example where it works up to x^{101} ."

Kevin O'Bryant: throw in lots of ϵ (taking value in ± 1), becomes a SAT problem.

2014.3. Problem Session III: Friday, May 24th (Chair Mel Nathanson).

2014.3.1. *Mel Nathanson*. Think of $\frac{a}{b}$ as being a parent of 2 children. Left child is $\frac{a}{a+b}$. Right child is $\frac{a+b}{b}$. Starting with 1 as root, this gives tree with rows

$$1 \tag{4}$$

$$1/2, 2 \tag{5}$$

$$1/3, 3/2, 2/3, 3/1 \tag{6}$$

$$1/4, 4/3, 3/5, 5/2, 2/5, 5/3, 3/4, 4 \tag{7}$$

$$\vdots \tag{8}$$

"Calkin-Wilf tree"

Every fraction occurs exactly once in this tree.

Start with z (variable) at root instead of 1. Apply Calkin-Wilf: $z \mapsto (\frac{z}{z+1}, z+1)$. Get linear fractional transformations. $f(z) = \frac{az+b}{cz+d}$.

Rule from parent to children: apply matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

Depth formula (involving continued fraction) holds in the Calkin-Wilf tree for z .

$$f(z) = [q_0, q_1, \dots, q_{k-1}, q_k + z] \text{ if } k \text{ is even}$$

$$f(z) = [q_0, q_1, \dots, q_{k-1}, q_k, z] \text{ if } k \text{ is odd}$$

The form is different for k even/odd. However, if you use that formula above for k even when k is odd, you get a fractional linear transformation with $\det -1$. (i.e., like starting a tree with $1/z$, gives tree with $\det -1$.) For a given determinant, only finitely many orphans (i.e., no parent) of that determinant.

Question from Nathanson: For a given determinant, how many orphans?

Question from Harald Helfgott: What if the parent to children rules use the matrices $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ instead?

Question from Thao Do: what happens if you start with i instead of 1 as the root? (i.e., let $z = i$) you get tree with elements of $\mathbb{Q}[i]$.

Nathanson: if $z = -1$, then $z \mapsto \frac{z}{z+1}$ gives you -1 again... if you're looking at complex numbers.

Thomas Bloom: use fields of characteristic p ? (Nathanson: “I don’t know anything about char p .”)

2014.3.2. *Thomas Bloom*. You have the sum set, difference set, product set, ratio set.

Because of commutativity, you expect difference set to be larger than sum set.

Question: Is there some subset $A \subset \mathbb{N}$ that is both MSTD and MPTR?

MSTD “more sums than differences”: $|A + A| > |A - A|$

MPTR “more products than ratios”: as well as $|A \cdot A| > |A/A|$.

Nathanson: what is the probability measure? We’ve seen MSTD sets before but not the multiplicative.

Note: from an MSTD set, can exponentiate to get an MPTR set.

Comments from Thao Do: in order to have MSTD, must have “almost symmetric form” i.e., smallest + largest = 2nd smallest + 2nd largest = 3rd smallest + 3rd largest, etc. (then change around a little and be clever) $a_1 + b_1 = a_2 + b_2$ implies $a_1 - b_2 = a_2 - b_1$.

$I = \{(a_1, b_1), (a_2, b_2) : a_1 + b_1 = a_2 + b_2\}$.

$J = \{(a_1, b_1), (a_2, b_2) : a_1 - b_1 = a_2 - b_2\}$.

Nathanson: tell Miller to have students working on MSTD/MPTR over summer. **Note from Miller: done!**

Bloom: I don’t think these sets exist.

2014.4. **Speaker List**. Talks here: <http://www.theoryofnumbers.com/CANT2014-program.pdf>.

- Sukumar Das Adhikari, Harish-Chandra Research Institute, India
- Paul Baginski, Fairfield University
- Thomas Bloom, University of Bristol
- Bren Cavallo, CUNY Graduate Center
- Alan Chang, Princeton University
- Jean-Marc Deshouillers, IPB-IMB Bordeaux, France
- Charles Helou, Penn State Brandywine
- Nathan Kaplan, Yale University
- Sandra Kingan, Brooklyn College (CUNY)
- Angel Kumchev, Towson State University
- Thai Hoang Le, University of Texas
- Eshita Mazumdar, Harish-Chandra Research Institute, Allahabad, India
- Nathan McNew, Dartmouth College
- Steven J. Miller, Williams College
- Mel Nathanson, Lehman College, CUNY
- Lan Nguyen, University of Wisconsin-Parkside
- Kevin O’Bryant, College of Staten Island, CUNY
- Alberto Perelli, University of Genova, Italy
- Giorgis Petridis, University of Rochester
- Luciane Quoos, Instituto de Matemática, UFRJ, Rio de Janeiro, Brasil
- Steven Senger, University of Delaware
- Satyanand Singh, New York City Tech (CUNY)
- Jonathan Sondow, New York
- Yonutz V. Stancescu, Afeka College, Tel Aviv, Israel
- Tim Susse, CUNY Graduate Center
- Johann Thiel, New York City Tech (CUNY)

The abstracts are here: <http://www.theoryofnumbers.com/CANT2014-abstracts.pdf>.

2015. CANT PROBLEM SESSIONS: 2015

2015.1. Phase Transitions in MSTD sets: Steven J Miller. In previous years I talked about phase transitions in the behavior of $|A + A|$ and $|A - A|$ when each element in $\{0, \dots, N\}$ is chosen independently with probability $p(N) = N^{-\delta}$ as δ hits $1/2$. What happens with three summands? Four?

What happens if we restrict A to special types of sets? How does the additional structure affect the answer?

Also, is there an 'explicit' construction of an infinite family of MSTD sets? The word 'explicit' is deliberately not being defined; I would like some nice, concrete procedure that does not involve randomness.

Finally, is this the 14th or the 13th CANT?

2015.2. An accidental sequence: Satyanand Singh. The two outer graphs which form an envelope around $\gamma_3((6j+2)^5)$ illustrate that:

$$\left(\frac{\ln(6j+2)^5}{3 \ln 3} \right) < \gamma_3((6j+2)^5) < \left(\frac{\ln(6j+2)^5}{\ln 3} \right).$$

The upper bound is easily seen by finding the power of 3 that is closest to $(6j+2)^5$ but does not exceed it. We can also say for certain that $\gamma_3((6j+2)^5) \geq 3$, since Bennett dispensed of the two term case in [?] and equality occurs when $2^5 = 3^3 + 3^1 + 2$. We were not able to prove the lower bound suggested by the experimental results, i.e., $\gamma_3((6j+2)^5) > \left(\frac{\ln(6j+2)^5}{3 \ln 3} \right)$ for $j \geq 1$. This would completely resolve the case for $q = 5$.

Problem 1. For both a and b odd, where $a > b > 0$, find all solutions to the diophantine equation $3^a + 3^b + 2 = (6j+2)^5$ or show that the only solution is $(a, b, j) = (3, 1, 0)$.

Problem 2. For any positive integer n , with $(n, 3) = 1$, find all solutions to $\gamma_3(n^q) \leq 3$ for q a prime number where $q > 1000$?

Problem 3. For any positive integer n , with $(n, 3) = 1$, we conjecture that $\gamma_3((6j+2)^5) > c \ln(6j+2)^5$ where c is a constant such that $0 < c < 1/(3 \ln(3))$.

2015.3. Kevin O'Bryant. Let b_1, b_2, \dots be an infinite binary sequence, and let A be the set of real numbers of the form $a_i := \sum_{n=1}^{\infty} b_{n+i} \cdot 2^{-n}$. If A has no infinite decreasing subsequence, that is, if A is an ordinal, what are the possibilities for the order type of A ? In particular, can the order type be ϵ_0 ?

Blair, Hamkins, and O'Bryant [forthcoming] have shown that the order type, if infinite, must be at least ω^2 , and can be as large as $\omega \uparrow\uparrow n$ for any n .

2015.4. Steven Senger (repeat from previous years): Given a large finite subset, A , of real numbers, and any non-degenerate, generalized geometric progression, G , with $|G| \approx |AA|$, can we get a nontrivial bound on $|(AA+1) \cap G|$?

2015.5. Nathan Kaplan: We say a set of points, $P \subset \mathbb{R}^2$, is in general position if no curve has more points than it "ought to". That is there are no three points on a line, no six points on a conic, etc.... The original problem posed by Jordan Ellenberg is "How does the minimum height of a set of completely generic points grow with the number of points?" It is available at:

<https://quomodocumque.wordpress.com/2014/04/05/puzzle-low-height-points-in-general-position/>

Can anything be said, even if we choose a greedy construction for our points.

2015.6. **Nathan Kaplan (repeat from previous years):** In \mathbb{F}_3^n , define the function $f(n, m)$ to be the maximum number of lines completely contained in any set of m points. Is it true that the simplest greedy construction (filling in lower dimensional subspaces), is the best possible? That is, is $f(n, m) = f(\lceil \log_3(m) \rceil, m)$?

2015.7. **Kevin O'Bryant:** Given a large finite subset, A , of real numbers, is it true that $|AA+A| \geq |A+A|$?

Oliver Roche-Newton has asked on Math Overflow

(<http://mathoverflow.net/questions/204020/is-the-set-aaa-always-at-least-as-large-as-aa/>)

if it is possible for

$$|A \cdot A + A| < |A + A|$$

with A being a set of real numbers. Some observations.

- For a random set A of k real numbers, $A \cdot A + A$ has $\sim k^3/2$ elements while $A + A$ has only $\sim k^2/2$, so any example needs to have some special structure.
- Modulo 13, the set $A = \{2, 5, 6, 7, 8, 11\}$ (the set of positive quadratic non-residues, with a modulus $p \equiv 1 \pmod{4}$), is an example. ✗
- if $|\cdot|$ means Lebesgue measure, then $A = [0, 1/2]$ is an example, as $A \cdot A + A = [0, 3/4]$ but $A + A = [0, 1]$.
- if A is a set of 3 or more positive integers, then it cannot be an example, as

$$a_1 + a_n A, a_2 + a_n A, \dots, a_n + a_n A$$

, where $a_n = \max A$, are necessarily disjoint (reduce modulo a_n) and contain at least $|A|^2$ elements altogether, while $|A + A| \leq \binom{|A|+1}{2}$.

The audience asked what was known for Hausdorff measure, and suggested considering the problem over the integers, positive rationals, and complexes.

2015.8. Speaker List.

- Paul Baginski, Fairfield University
- Bela Bajnok, Gettysburg College
- Dakota Blair, CUNY Graduate Center
- Lisa Bromberg, CUNY Graduate Center
- Mei-Chu Chang, University of California
- Scott Chapman, Sam Houston State University
- David John Covert, University of Missouri St. Louis
- Robert Donley, Queensborough Community College (CUNY)
- Leonid Gurvits, City College (CUNY)
- Sandie Han, New York City Tech (CUNY)
- Charles Helou, Penn State Brandywine
- Alex Iosevich, University of Rochester
- Renling Jin, College of Charleston
- Nathan Kaplan, Yale University
- Mizan Khan, Eastern Connecticut State University
- Sandra Kingan, Brooklyn College (CUNY)
- Diego Marques, University of Brasilia
- Ariane Masuda, New York City Tech (CUNY)
- Nathan McNew, Dartmouth College
- Steven J. Miller, Williams College
- Mel Nathanson, Lehman College (CUNY)
- Kevin O'Bryant, College of Staten Island (CUNY)
- Cormac O'Sullivan, Bronx Community College (CUNY)
- Jasmine Powell, Northwestern University
- Alex Rice, University of Rochester
- Steven Senger, Missouri State University
- Satyanand Singh, New York City Tech (CUNY)
- Jonathan Sondow, New York
- Johann Thiel, New York City Tech (CUNY)
- Yuri Tschinkel, NYU
- Bart Van Steirteghem, Medgar Evers College (CUNY)
- Madeleine Weinstein, Harvey Mudd College

A list of talks and abstracts is available online here:

<http://www.theoryofnumbers.com/CANT2015-abstracts.pdf>.

2016. CANT PROBLEM SESSIONS: 2016

2016.1. Problem Session I: Tuesday, May 24th (Nathanson Chair).

Mel Nathanson: Alexander Borisov in his 2005 arXiv paper, “Quotient singularities, integer ratios of factorials and the Riemann Hypothesis,” discussed integer valued ratios of factorials and their relation to problems in number theory and algebraic geometry. Historically the application of such ratios to number theory goes back at least to Pafnuty Chebyshev, who used them to obtain the order of magnitude of $\pi(x)$. Since $\binom{2n}{n} = \frac{(2n)!}{n!n!}$ is an integer, it is not hard to see that its prime decomposition must include all primes p such that $n < p < 2n$, and so

$$4^n > \frac{(2n)!}{n!n!} \geq \prod_{n < p < 2n} p$$

where p is a prime number. Chebyshev used this fact to show that

$$\frac{x}{\log(x)} \ll \pi(x) \ll \frac{x}{\log(x)}$$

It is a theorem of Eugène Charles Catalan that for any $k \in \mathbb{N}$ one has

$$\frac{(2n)!(2k)!}{n!k!(n+k)!} \in \mathbb{Z}$$

There are combinatorial proofs of this identity for $k = 0, 1, 2$.

Question: For $3 \leq k \leq n$, prove

$$\frac{(2n)!(2k)!}{n!k!(n+k)!} \in \mathbb{Z}$$

by a counting argument.

One can show that both

$$\frac{(9n)!n!}{(5n)!(3n)!(2n)!}$$

and

$$\frac{(14n)!(3n)!}{(9n)!(7n)!(n)!}$$

are integers for all positive integers n .

Question: Find all quintuples $(a, b, c, d, e) \in \mathbb{N}^5$ such that

$$a + b = c + d + e$$

$$\gcd(a, b, c, d, e) = 1$$

and

$$\frac{(an)!(bn)!}{(cn)!(dn)!(en)!} \in \mathbb{Z}?$$

Show that there are only 29 such quintuples.

Question: Can one deduce something interesting about the distribution of the primes from an integer identity of the form $\frac{(an)!(bn)!}{(cn)!(dn)!(en)!}$?

Thomas Blume: Does there exist $A \subset \mathbb{Z}$ such that

$$(1) |A + A| > |A - A|,$$

$$(2) |A \times A| > |A/A|.$$

Mel Nathanson: J. A. Haight showed in a paper from 1973 that for all h and l , there exists a modulus m and A a subset of $\mathbb{Z}/m\mathbb{Z}$ such that $A - A = \mathbb{Z}/m\mathbb{Z}$ but hA omits l consecutive residues in $\mathbb{Z}/m\mathbb{Z}$. He then used this algebraic result to show the following.

Theorem 2016.1. *There exists $E \subset \mathbb{R}$ such that $E - E = \mathbb{R}$ but $\mu(hE) = 0$ for all $h \geq 1$, where μ is Lebesgue measure.*

This is a kind of reverse MSTD result for the reals.

Question: Let $\epsilon > 0$. Does there exist $A \subset \mathbb{R}$ such that $A - A = \mathbb{R}$ and $\mu(2 * A - A) < \epsilon$.

Using Haight's results, Ruzsa was able to show that for any fixed h , there exists an $A \subset \mathbb{N}$ such that $|A - A|$, but $|hA|$ is small.

Question: Does there exist $A \subset \mathbb{R}$ such that $A - A = \mathbb{R}$ and $\mu(2 * A - A) < \epsilon, \forall \epsilon > 0$.

Specifically, Ruzsa showed the following.

Theorem 2016.2. *For any $h > 1$, and any $\epsilon > 0$, there exists a modulus m and an $A \subset \mathbb{Z}/m\mathbb{Z}$ such that $A - A = \mathbb{Z}/m\mathbb{Z}$, and $|hA| < \epsilon \cdot m$.*

If one defines $\Phi(t_1, t_2, \dots, t_h) = \sum_{i=1}^h t_i$ and $\Upsilon(t_1, t_2) = t_1 - t_2$, then a slight reinterpretation of the above theorem says that $|\Phi(A)| < \epsilon \cdot m$ and $\Upsilon(A) = \mathbb{Z}/m\mathbb{Z}$.

Question: What, if anything, is special about these linear forms? If one takes $\Phi(t_1, t_2, \dots, t_h) = \sum_{i=1}^h \phi_i t_i$, for some function ϕ on the index set of Φ and similarly a ψ for $\Upsilon(t_1, t_2, \dots, t_g) = \sum_{i=1}^g \psi_i t_i$, for what functions ψ and ϕ does one get a Haight like result?

Consider $(\phi_1, \phi_2, \dots, \phi_h)$ and $I = \{1, 2, \dots, h\}$. Then let $S_I^{(\Phi)} = \sum_{i \in I} \phi_i$ and $S_I^{(\Upsilon)}$, then

Exercise: Show that when $\Upsilon(t_1, t_2) = t_1 - t_2$ and $\Phi(t_1, t_2) = 2t_1 - t_2$ one gets a Haight like result, (i.e., $\exists A \subset \mathbb{R}$ such that $A - A = \mathbb{R}$ and $\mu(2 * A - A) < \epsilon$).

Question: What if the measure μ in the above statements is replaced with Hausdorff dimension? Are there Haight like results that one can describe where the difference set is of dimension 1, and the h fold sum set is of fractional dimension?

William Keith: Let $P = \prod_{i=1}^{\infty} (1 + q^{2^i})$ and $Q = \prod_{i=1}^{\infty} (1 + q^{2^{4^i}})$ for q a prime. Note that $(1 - q)^2 \equiv (1 + q^2)$ and $P \cdot Q \equiv 1 + q + q^2 + \dots$

Question: When is it that $F = \sum f(m)q^m$, where $f(m)$ is odd with positive density, that $(F)^k$ has zero density for the odd coefficients?

Larsen Urban: Let $A \subset \mathbb{N}^d$, if $A + A = A^c \setminus T_A$, where $T_A = \{z \in \mathbb{N}^d \mid z \text{ is unrelated to anything in } A\}$, then what can one say about A ? Let $A, B \subset \mathbb{N}^d$, when is it that $\emptyset = A \cap B$, and $A + B = (A \cup B)^c$.

2016.2. **Problem Session II: Wednesday, May 25th (Iosevich Chair).**

Urban Larsson: Suppose X is a sum-free set on $\mathbb{Z}_{\geq 0}$; such sets are known, but now require $\min X = \infty$

$\mathbb{Z}_{\geq 0}$. What is the maximum density of such a set X if it is sum-free and has smallest element k ? What if we vary k ? Has this been studied? Is it

$$X = \{ip_k + k, \dots, ip_k + 2k - 1 : i \in \mathbb{Z}_{\geq 0}, p_k = 3k - 1.\}$$

Brendan Murphy: Inspired by Alex and Tom's talks, say you color \mathbb{F}_q^* , $E \subset \mathbb{F}_q^d$, $d \geq 2$, x and y connected by an edge if $\|x - y\| = t \in \mathbb{F}_q^*$, with $\|x\| = x_1^2 + \dots + x_d^2$. How large must E be so that we have monochromatic $(\|x - y\|, x \cdot y)$. Comment from audience: hope. Alex: for large sets might be ok (maybe $|E| > Cq^{(d+1)/2}$), small sets....

Sarfraz Ahmad: Goal is to prove that for all $i > 0$ we have

$$\sum_{j=0}^i \frac{(-1)^j (i - j + 1/2)^i}{j!(i - j)!} = 1.$$

Comment from audience: is this a difference order problem?

$$\Delta_h^i f(x) = \sum_{j=0}^i \binom{i}{j} (-1)^j f(x + j),$$

where

$$\Delta_h^{i_1+i_2} = \Delta_h^{i_1} (\Delta_h^{i_2}).$$

2016.3. Problem Session III: Thursday, May 26th (Miller Chair).

Steven Miller: Prove unconditionally that there are infinitely many subsets of the primes that are MSTD sets. What about other special sets? Answered at lunch and in talk: Use Green-Tao, follows immediately.

Mel Nathanson: For all $A, B \subseteq \mathbb{R}^n$, we have $(A \cap \mathbb{Z}^n) + (B \cap \mathbb{Z}^n) \subseteq (A + B) \cap \mathbb{Z}^n$. The special case $A = B$ is interesting. For example, in \mathbb{R}^3 , the Reeve polytope A is the convex hull of the set $(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 2)$. The lattice points in $A \cap \mathbb{Z}^3$ are $\{(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 2)\}$. The sumset $2A$ contains the lattice point $(1, 1, 1) = (1/2, 1/2, 0) + (1/2, 1/2, 1)$, but it is not the sum of two lattice points in A . Thus, $(A \cap \mathbb{Z}^3) + (A \cap \mathbb{Z}^3)$ is a proper subset of $2A \cap \mathbb{Z}^3$. Describe the lattice polytopes A such that $2(A \cap \mathbb{Z}^n) = (2A) \cap \mathbb{Z}^n$.

In the plane there exist lattice triangles where the sum of the triangles contains a lattice point that is not the sum of lattice points in the triangle. For example, the triangles with vertices at $(0, 0), (1, 0), (0, 1)$ and at $(0, 1), (1, 3)$ and $(2, 4)$ have this property.

Kevin O'Bryant: A couple of months ago on Google+, Harald Bögeholz found an arrangement of the integers from 1 to 32 such that any adjacent pair adds up to a square. See his graphic in Figure 2, or <https://plus.google.com/u/0/106537406819187054768/posts/Y9qaWEwiLuv?cfem=1>.

For $N \leq 31$ it is impossible to arrange the numbers up to N on a circle so that each adjacent pair sums to a perfect square. For example, if $N = 19$ what goes next to 16? Could have 9 but then in trouble as need two neighbors. Considering the graph with vertices $\{1, 2, \dots, N\}$ and edges connecting numbers that sum to a square, we are asking for a Hamiltonian cycle. Unique ways for $N = 32, 33$; number of ways of doing is not monotonic.

Implied question: is it possible for every $N \geq 32$? For infinitely many $N \geq 32$?

Could add all the sums, that gets each number twice, so that would be $N(N + 1)$, has to be a linear combination of the squares, gives a Diophantine condition which maybe could be easily checked. For some N , this condition can be satisfied even though there is no Hamiltonian cycle.

Generalizations: What about three in a row added? What about a cube? A sphere?

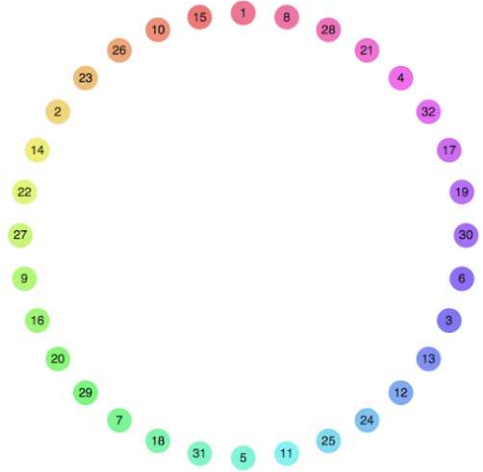


FIGURE 2. Bögeholz's arrangement of numbers such that sums of adjacent elements are squares.

Sergei Konyagin: What is the cardinality of the maximal subset of $\{1, \dots, N\}$ such that A does not contain an MSTD subset? Comment: if have an arithmetic progression of length 15 fail.

Colin Defant: Define $\sigma_c(n) = \sum_{d|n} d^c$, look at $\sigma_c(\mathbb{N})$, gives a set of complex numbers, look at closure, for which complex numbers c is $\overline{\sigma_c(\mathbb{N})}$ connected? If $\operatorname{Re}(c) < -3.02$ (approximately) not connected, this bound is probably not optimal.

2016.4. Problem Session IV: Friday, May 27th (O'Bryant Chair).

Sergei Konyagin: Given a natural number, $r \geq 2$, consider the equation over natural numbers, $1 \leq x_i, y_i \leq N$.

$$\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_r} = \frac{1}{y_1} + \frac{1}{y_2} + \dots + \frac{1}{y_r}. \quad (9)$$

There are roughly $r!N^r$ trivial solutions of Equation 9, where the x_i are permutations of the y_i . Define $\mathcal{F}_{r,n}$ to be the number of nontrivial solutions to Equation 9. Konyagin and Korolev have shown that as $N \rightarrow \infty$,

$$\mathcal{F}_{r,n} \leq O\left(N^{r-\frac{1}{4}}\right).$$

They conjecture that the upper bound should really be $O\left(N^{r-1+o(1)}\right)$. Consider also the following example. For any choice of z with $1 \leq z \leq N$.

$$\frac{1}{2z} + \frac{1}{4z} + \frac{1}{x_3} + \dots + \frac{1}{x_r} = \frac{1}{3z} + \frac{1}{3z} + \frac{1}{x_3} + \dots + \frac{1}{y_r}.$$

To bar such examples, we could consider the assumption that $x_i \neq y_i$ for all i in Equation 9. For this variant, we could have $x_1 = x_2 = 2z$, $y_1 = z$, $x_3 = z'$, $y_2 = y_3 = 2z'$, etc... The conjecture for the variant is that the number of nontrivial solutions to Equation 9 with this additional restriction should be no more than

$$O\left(N^{\lfloor \frac{2r}{3} \rfloor + o(1)}\right).$$

We know that, for some positive constants c and C , $\mathcal{F}_{r,n} \approx r!N^r$ when $r \leq c\left(\frac{\log N}{\log \log N}\right)^{\frac{1}{3}}$, but that $\mathcal{F}_{r,n} > r!N^r$ when $r > C\left(\frac{\log N}{\log \log N}\right)^{\frac{1}{3}}$. This is known when one considers the choices of y_j to be the smooth numbers.

Question 1 (Shparlinski): Consider the solutions to Equation 9 with $M+1 \leq x_i, y_i \leq M+N$.

Question 2 (Konyagin): Consider the solutions to the following equation

$$\frac{a_1}{x_1} + \frac{a_2}{x_2} + \cdots + \frac{a_r}{x_r} = \frac{1}{y_1} + \frac{1}{y_2} + \cdots + \frac{1}{y_r},$$

where a_j are nonzero rational numbers, and the x_j and y_j are as before.

Question 3 (Senger): Consider the solutions to Equation 9, except with different numbers of terms on each side.

Kamil Bulinski: Let the group $G = \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/M\mathbb{Z}$, and

$$G = \bigsqcup_{i=1}^m (a_i + H_i),$$

where the H_i are subgroups of G . Must $H_i = H_j$ for some $i \neq j$? Note that this is false for the case that $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, as

$$G = \{(0, 0, 0), (1, 0, 0)\} \cup \{(0, 0, 1), (0, 1, 1)\} \cup \{(1, 1, 0), (1, 1, 1)\} \cup \{(0, 1, 0), (1, 0, 1)\},$$

a union of four disjoint lines.

Kevin O'Bryant: Sun's Conjecture: If $a_1 + H_1, a_2 + H_2, \dots, a_m + H_m$ are disjoint, then there exist $i < j$ such that $\gcd([G : H_i], [G : H_j]) \geq m$.

Brian Hopkins: Let $p(n, 3)$ denote the number of partitions of a natural number, n , into exactly three parts. It is known that $p(n, 3)$ is the nearest integer to $\frac{n^2}{12}$. This tells us that for Pythagorean triples, a, b , and c , where $a^2 + b^2 = c^2$, we have that

$$p(a, 3) + p(b, 3) = p(c, 3). \quad (10)$$

Question 1: Is there a direct bijective proof of Equation 10?

Question 2: If a triple, (a, b, c) satisfies Equation 10, it may not be a Pythagorean triple. Characterize the triples for which Equation 10 holds.

Steven Senger: Tom Sanders spoke on colorings of the natural numbers where there exists a quadruple, $(x, y, x + y, xy)$, whose entries are all the same color.

Question 1: Can one show that any four-coloring (with equally dense sets of colors) of the natural numbers will guarantee the existence of a quadruple $(x, y, x + y, xy)$ are all different colors? Note that if each residue class modulo 4 is given a different color, then restricting possible colors to fixed places may render the answer as no, so we must allow any color to be in any entry.

Answer (Ryan Alweis): NOPE! Actually, even this restriction is irrelevant, as giving each residue class modulo 4 will render the answer negative. To see this, note that if x and y are both even, $x + y$ will also be even, so they cannot come from three distinct equivalence classes modulo 4. To see this, note that if x and y are both odd, then xy will also be odd, so they cannot come from three distinct equivalence classes modulo 4. Therefore, x and y must have different parity. This will also not work, as can be seen by checking each case.

Question 2: Perhaps more colors could work?

Tom Bloom: Let $A \subset \{1, 2, \dots, 2N\}$ is N -circular (as discussed by Kevin O'Bryant earlier, there exists a permutation of $\{1, 2, \dots, N\}$ such that pairwise, $a + b \in A$).

Question 1: For which $p(n)$ is it true that a random subset of $[2N]$ is N -circular with high probability.

Question 2: If $\{1, 2, \dots, N\} = X \sqcup Y$ such that $(X + Y) \cap A = \emptyset$, then A is not N -circular. Are there any other natural obstructions?

2016.5. **Speaker List.****CANT 2016 Speakers****Fourteenth Annual Workshop on
Combinatorial and Additive Number Theory**CUNY Graduate Center
May 24–27, 2016

- (1) **Sarfraz Ahmad**, Comsats Institute of Information Technology, Lahore, Pakistan
- (2) **Paul Baginski**, Fairfield University
- (3) **Gautami Bhowmik**, Universit of Lille, France
- (4) **Pierre Bienvenu**, University of Bristol, UK
- (5) **Arnab Bose**, University of Lethbridge, Canada
- (6) **Kamil Bulinski**, University of Sydney, Australia
- (7) **Sam Cole**, University of Illinois at Chicago
- (8) **Colin Defant**, University of Florida
- (9) **Mohamed El Bachraoui**, United Arab Emirates University, United Arab Emirates
- (10) **George Grossman**, Central Michigan University
- (11) **Sandie Han**, New York City Tech (CUNY)
- (12) **Brian Hopkins**, St. Peter's University
- (13) **Alex Iosevich**, University of Rochester
- (14) **William J. Keith**, Michigan Technological University
- (15) **Mizan Khan**, Eastern Connecticut State University
- (16) **S. V. Konyagin**, Steklov Mathematical Institute of the Russian Academy of Sciences, Russia
- (17) **Ben Krause**, University of British Columbia, Canada
- (18) **Urban Larsson**, Dalhousie University, Halifax, Canada
- (19) **Jiange Li**, University of Delaware
- (20) **Ray Li**, Carnegie-Mellon University
- (21) **Neil Lyall**, University of Georgia
- (22) **Akos Magyar**, University of Georgia
- (23) **Ariane Masuda**, New York City Tech (CUNY)
- (24) **Nathan McNew**, Towson State University
- (25) **Steven J. Miller**, Williams College
- (26) **Amanda Montejano**, UMDI-Facultad de Ciencias, Universidad Nacional Autónoma de México, Querétaro, México
- (27) **Brendan Murphy**, University of Rochester
- (28) **Rishi Nath**, York College (CUNY)
- (29) **Mel Nathanson**, Lehman College (CUNY)
- (30) **Péter Pál Pách**, Budapest University of Technology and Economics, Hungary
- (31) **Zhao Pan**, Carnegie-Mellon University
- (32) **Giorgis Petridis**, University of Rochester
- (33) **Bradley Rodgers**, University of Michigan

- (34) **Ryan Ronan**, CUNY Graduate Center
- (35) **Tom Sanders**, Oxford University, UK
- (36) **James Sellers**, Pennsylvania State University
- (37) **Steven Senger**, Missouri State University
- (38) **Satyanand Singh**, New York City Tech (CUNY)
- (39) **Jonathan Sondow**, New York
- (40) **Yoni Stancescu**, Afeka College, Israel
- (41) **Johann Thiel**, New York City Tech (CUNY)
- (42) **Andrew Treglown**, University of Birmingham, UK
- (43) **Yuri Tschinkel**, New York University
- (44) **Van Vu**, Yale University
- (45) **Huanzhong Xu**, Carnegie-Mellon University
- (46) **Victor Xu**, Carnegie-Mellon University
- (47) **Xiaorong Zhang**, Carnegie-Mellon University

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- (1) Australia
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- (9) Russia
- (10) Sweden
- (11) United Arab Emirates
- (12) United Kingdom
- (13) United States

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