EXTENSIONS OF AUTOCORRELATION INEQUALITIES WITH APPLICATIONS TO ADDITIVE COMBINATORICS

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ABSTRACT. In a 2019 paper, Barnard and Steinerberger show that for $f \in L^1(\mathbf{R})$, the following autocorrelation inequality holds:

$$\min_{0 \le t \le 1} \int_{\mathbb{R}} f(x) f(x+t)\, \mathrm{d}x \le 0.411 ||f||_{L^1}^2,$$

where the constant 0.411 cannot be replaced by 0.37. In addition to being interesting and important in their own right, inequalities such as these have applications in additive combinatorics. A set of integers S is called a *difference basis* for a positive integer n if for each $1 \le k \le n$, there are $a, b \in S$ so that $a - b = k$. The minimum size of these sets is a central question in additive combinatorics, and this problem can be encapsulated by a convolution inequality similar to the above integral. In their paper, Barnard and Steinerberger suggest that future research may focus on the existence of functions extremizing the above inequality. We prove necessary conditions for the existence of such extremal f. Specifically, we show that for f to be extremal under the above, we must have

$$\max_{x_1 \in \mathbb{R}} \min_{0 \le t \le 1} \left[f(x_1 - t) + f(x_1 + t) \right] \le \min_{x_2 \in \mathbb{R}} \max_{0 \le t \le 1} \left[f(x_2 - t) + f(x_2 + t) \right].$$

Our central technique for deriving this result is local perturbation of f to increase the value of the autocorrelation, while leaving $||f||_{L^1}$ unchanged. We examine the behavior of this condition in some classes of functions, focusing on f convex or concave, when they take simplified form.

These perturbation methods can be extended to examine a more general notion of autocorrelation. Let $d, n \in \mathbb{Z}^+$, $f \in L^1$, A be a $d \times n$ matrix with real entries and columns a_i for $1 \le i \le n$, and C be a constant. For a broad class of matrices A, we prove necessary conditions for f to extremize autocorrelation inequalities of the form

$$\min_{\mathbf{t} \in [0,1]^d} \int_{\mathbb{R}} \prod_{i=1}^{n} f(x + \mathbf{t} \cdot a_i)\, \mathrm{d}x \le C ||f||_{L^1}^n.$$

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1. INTRODUCTION

In discrete mathematics, Ramsey theory is the study of questions of the following form: "how large must a global structure be, in order to guarantee that a smaller substructure appears?" A classic problem in Ramsey theory is the study of Ramsey numbers. Given $r, b \in \mathbb{N}$, let $R(r, b) =: n$ be the smallest natural number such that a 2-coloring of the edges of K_n (with colors red and blue) must contain a red copy of K_r or a blue copy of K_b. Ramsey's Theorem asserts that $R(r, b)$ exists for all $r, b \in \mathbb{N}$. *Continuous Ramsey Theory* is the study of Ramsey-type questions in the continuous setting. An example such question is the problem of symmetric subsets, studied by Martin and O'Bryant [MO]. A subset of $[0, 1]$ is called *symmetric* if it is invariant under some reflection. Let λ denote the 1-dimensional Lebesgue measure and let

$$D(x) \ := \ \sup\{r \in \mathbb{R}^+ \text{so that } \forall A \subset [0,1] \text{ with } \lambda(A) = x, \ \exists\, S \subset A \text{ symmetric with } \lambda(S) \geq r\}. \quad (1.1)$$

By placing bounds on $D(x)$, Martin and O'Bryant analyzed the size of symmetric sets S found within larger sets A.

We study another continuous Ramsey theory problem. Given a function $f \in L^1$, how does the function

$$g(t) = \min_{t \in [0,1]} \int_{\mathbb{R}} f(x) f(x+t) \, \mathrm{d}x \quad (1.2)$$

behave? If we replace f by $c \cdot f$ for some $c \in \mathbb{R}$, then $g(t) = c^2 g(t)$, and so we must take into account the size of f. The L^1 norm is a natural choice for measuring the size of a function $f : \mathbb{R} \to \mathbb{R}$. Working in the well studied L^1 function space also allows any results to apply to problems in probability. For our measurement of substructure, we follow a recent paper of Bernard and Steinerberger [BS] and investigate

$$\frac{\min\limits_{t \in [0,1]} \int_{\mathbb{R}} f(x) f(x+t) \mathrm{d}x}{||f||_{L^1}^2}, \quad (1.3)$$

where we normalize by $||f||_{L^1}^2$ in order to provide invariance under the scaling discussed above.

This problem is fundamentally connected to a problem in additive combinatorics. Given some $n \in \mathbf{N}$, a set of integers $A \subset \mathbf{Z}$ is called a *difference basis* with respect to n if, for $A - A := \{a_1 - a_2 | a_1, a_2 \in A\}$, $\{1, \dots n\} \subset A - A$. The value

$$H(n) := \min\{|A|, \ A \subset \mathbf{Z} \text{ and } \{1, \dots, n\} \subset A - A\} \quad (1.4)$$

has been studied extensively. The connection to equation (1.3) is through probability; if $f(n)$ is a probability distribution on $n \in \mathbf{Z}$, $g(t)$ is the probability distribution given by taking the difference $f - f$. The function $H(n)$ was proposed and studied in [PKHJ, EG, HL, B]. Lower bounds on $H(n)$ as $n \to \infty$ were proved in [L] and later improved in [BT], while upper bounds were shown in [G]. Since $|A - A|$ is at most quadratic in $|A|$, it is immediate that $H(n) \geq \sqrt{2n}$. In fact, these are the correct asymptotics; the best known results are that

$$\sqrt{2.435n} \leq H(n) \leq \sqrt{2.645n}. \quad (1.5)$$

This connection to additive combinatorics motivates our investigation, and it is possible that the discrete and continuous problems, while distinct, could inform one another.

The central question in studying equation (1.3) is similar to that found in the Ramsey problem. In the classical discrete setting, we are given an edge coloring of some graph of known size, and asked to find which monochromatic subgraphs must appear. Just as there are some colorings which have huge amounts of structure, there are some functions with trivial behavior under $\min_{t \in [0,1]} g(t)$. For example, if $\operatorname{supp}(f) \subseteq [0, 1]$, then $\min_{t \in [0,1]} g(t) = 0$. However, the value

$$\sup_{f \in L^1} \min_{t \in [0,1]} g(t) \quad (1.6)$$

(corresponding to those graph edge colorings which are least structured) is not well understood. By Fubini's Theorem we have

$$\min_{t\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\mathrm{d}x \le \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} f(x)f(x+t)\mathrm{d}x\mathrm{d}t \tag{1.7}$$

$$= \frac{1}{2}||f||_{L^1}^2. \tag{1.8}$$

This is a crude approximation, since we replace a minimum with an averaging integral. This bound was improved in [BS], where the authors show that in fact

$$\min_{t\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\mathrm{d}x \le 0.411||f||_{L^1}^2. \tag{1.9}$$

This result was obtained using techniques from Fourier analysis, in particular the Wiener-Khintchine theorem. Barnard and Steinerberger also give explicit examples of functions for which the LHS of (1.9) is large, showing that 0.411 cannot be reduced to 0.37.

Our paper provides necessary conditions for the existence of a function f maximizing equation (1.3). This is a question which applies only to continuous Ramsey theory (as opposed to the discrete form). In a discrete problem, such as the study of Ramsey numbers, these extremal structures trivially exist. Given that $R(r, b) = n$, it is clear that there must exist some coloring of K_{n-1} which contains no red copy of K_r or blue copy of K_b. Furthermore, since there are but a finite number of such colorings, we know there is only a finite number of such extremal graphs, none 'more extreme' than any other. In the continuous case, it is not clear if there exist function(s) maximizing equation (1.3).

Our methods are based on perturbation theory. Given a candidate extremal function f, we attempt to increase its value under equation (1.3) by adding a function g which is small in L^1 norm. Because this is a quite general tactic, we may apply it to more general convolution-type problems. In fact, our perturbation techniques can be extended to prove results on generic convolution-type integrals. If d, n are positive integers, A a $d \times n$ matrix with columns a_i for $1 \le i \le n$, and $f \in L^1$, then we study

$$\frac{\min\limits_{\mathbf{t}\in[0,1]^d} \int_{\mathbb{R}} \prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\mathrm{d}x}{||f||_{L^1}^n}. \tag{1.10}$$

In Section 2, we state Theorem 2.1, our main result, as well as Corollary 2.3, Corollary 2.4, and Corollary 2.5. In Section 3, we prove Theorem 2.1. In Section 4, we discuss potential directions in which our work might be extended.

2. Main Results

In Section 2.1, we state our main result, Theorem 2.1. The theorem statement relies on a technical result, Lemma 2.2, which we state and prove in Section 2.2. In Section 2.3, we state Corollary 2.3, Corollary 2.4, and Corollary 2.5, which are special cases of Theorem 2.1. Finally, in Section 2.5, we state conditions under which the continuity hypothesis of Theorem 2.1 can be relaxed.

2.1. Statement of Theorem 2.1.

We now present our main results on the existence of functions f maximizing (1.10). First we present the theorem in its full generality.

Theorem 2.1. *Let $d, n \in \mathbb{N}$ and A a $d \times n$ matrix with columns a_i for $1 \le i \le n$ satisfying Lemma 2.2. Then a continuous function f maximizing equation* (1.10) *must satisfy both*

$$\max_{x_1\in\mathbb{R}} \min_{\mathbf{t}\in[0,1]^d} \sum_{i=1}^{n} \prod_{i=1, i\neq j}^{n} f(x_1+\mathbf{t}\cdot(a_i-a_j)) \;\le\; \frac{n}{||f||_{L^1}} \min_{\mathbf{t}\in[0,1]^d} \int\limits_{\mathbb{R}} \prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\mathrm{d}x \tag{2.1}$$

and

$$\max_{x_1\in\mathbb{R}} \min_{\mathbf{t}\in[0,1]^d} \sum_{j=1}^{n} \prod_{i=1,i\neq j}^{n} f(x_1+\mathbf{t}\cdot(a_i-a_j)) \;\le\; \min_{x_2\in\mathrm{supp}(f)} \max_{\mathbf{t}\in[0,1]^d} \sum_{j=1}^{n} \prod_{i=1,i\neq j}^{n} f(x_2+\mathbf{t}\cdot(a_i-a_j)). \quad (2.2)$$

2.2. **Technical lemma.** Our goal is to study the existence of functions f maximizing (1.10). However, there exist choices of A for which (1.10) is unbounded. For example, let $A=\mathbf{0}_{d\times n}$; then equation (1.10) is not necessarily even finite for individual f. In the specific case $n=d=2$, $\int_{\mathbb{R}}|f(x)|\mathrm{d}x<\infty$ does not imply that $\int_{\mathbb{R}} f(x)^2\mathrm{d}x<\infty$.

Analogous to the reasoning in (1.7), we can use Fubini's Theorem to give a sufficient condition on A for which (2.1) is bounded from above. These include the choice of A studied in [BS].

Lemma 2.2. *If the* $d+1$ *by* n *matrix*

$$B=\left[\frac{1\cdots 1}{A}\right]$$

has rank at least n*, then equation* (1.10) *is finite for all choices of* $f\in L^1$.

Proof. First we see that $d\ge n-1$ is implied by the rank criteria on A. Then we observe the right-multiplication

$$\begin{bmatrix} x & t_1 & \dots & t_d\end{bmatrix}\cdot B=\begin{bmatrix} x+\mathbf{t}\cdot a_1 & x+\mathbf{t}\cdot a_2 & \dots & x+\mathbf{t}\cdot a_n\end{bmatrix}. \quad (2.3)$$

Since B has rank at least n, there exists an invertible linear transformation $C:\mathbf{R}^d\to\mathbf{R}^d$ so that

$$\begin{bmatrix} x & \mathbf{t}\cdot C\end{bmatrix}\cdot B=\begin{bmatrix} x & x+t_1 & \dots & x+t_{n-1}\end{bmatrix}. \quad (2.4)$$

Then we return to our problem and use the sequence of upper bounds

$$\min_{\mathbf{t}\in[0,1]^d}\int_{\mathbb{R}}\prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\mathrm{d}x \le \int_{\mathbf{t}\in[0,1]^d}\int_{\mathbb{R}}\prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\mathrm{d}x\,\mathrm{d}\mathbf{t} \quad (2.5)$$

$$\le \int_{\mathbf{t}\in\mathbb{R}^d}\int_{\mathbb{R}}\prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\mathrm{d}x\,\mathrm{d}\mathbf{t}. \quad (2.6)$$

We exchange the $\mathbf{t}\cdot a_i$ for t_i by applying C,

$$\int_{\mathbf{t}\in\mathbf{R}^d}\int_{\mathbb{R}}\prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\mathrm{d}x\,\mathrm{d}\mathbf{t} \le \int_{\mathbf{t}\in C^{-1}\mathbb{R}^d}\int_{\mathbb{R}} f(x)\prod_{i=1}^{n-1} f(x+t_i)\mathrm{d}x\,\mathrm{d}\mathbf{t} \quad (2.7)$$

$$\le ||f||_{L^1}^n. \quad (2.8)$$

Thus (1.10) is necessarily finite. □

2.3. **Theorem 2.1 for specific *d*, *n*, *A*.** Corollary 2.3 addresses a question asked in [BS]. If we examine Theorem 2.1 where $n=2, d=1$ and $A=\begin{bmatrix}0 & 1\end{bmatrix}$, we find a corollary which relates to the autocorrelation ratio studied in [BS].

Corollary 2.3. *A continuous function* f *maximizing*

$$\frac{\min_{\mathbf{t}\in[0,1]}\int_{\mathbb{R}} f(x)f(x+t)\mathrm{d}x}{||f||_{L^1}^2} \quad (2.9)$$

must satisfy both

$$\max_{x_1\in\mathbb{R}}\min_{\mathbf{t}\in[0,1]}[f(x_1-t)+f(x_1+t)] \;\le\; \frac{2}{||f||_{L^1}}\min_{\mathbf{t}\in[0,1]}\int_{\mathbb{R}} f(x)f(x+t)\mathrm{d}x \quad (2.10)$$

and

$$\max_{x_1\in\mathbb{R}} \min_{\mathbf{t}\in[0,1]} [f(x_1-t)+f(x_1+t)] \;\le\; \min_{x_2\in\mathbb{R}} \max_{\mathbf{t}\in[0,1]} [f(x_2-t)+f(x_2+t)]. \tag{2.11}$$

Additionally, setting $d=n$ and $A=I$ in Theorem 2.1, we find the following.

Corollary 2.4. *Let n be a positive integer, then a continuous function f maximizing*

$$\frac{\displaystyle\min_{\mathbf{t}\in[0,1]^n} \int_{\mathbb{R}} \prod_{i=1}^{n} f(x+t_i)\mathrm{d}x}{||f||_{L^1}^n} \tag{2.12}$$

must satisfy both

$$\max_{x_1\in\mathbb{R}} \min_{\mathbf{t}\in[0,1]^n} \sum_{i=1}^{n} \prod_{i=1,i\neq j}^{n} f(x_1+t_i-t_j) \;\le\; \frac{n}{||f||_{L^1}} \min_{\mathbf{t}\in[0,1]^n} \int_{\mathbb{R}} \prod_{i=1}^{n} f(x+t_i)\mathrm{d}x \tag{2.13}$$

and

$$\max_{x_1\in\mathbb{R}} \min_{\mathbf{t}\in[0,1]^n} \sum_{j=1}^{n} \prod_{i=1,i\neq j}^{n} f(x_1+t_i-t_j) \;\le\; \min_{x_2\in\mathrm{supp}(f)} \max_{t\in[0,1]^n} \sum_{j=1}^{n} \prod_{i=1,i\neq j}^{n} f(x_2+t_i-t_j). \tag{2.14}$$

2.4. **Theorem 2.1 for convex or concave functions.** Corollary 2.5 follows from further simplifying the result of Theorem 2.1 under the additional assumptions that $d=1$ and f is concave or convex. The value

$$\max_{x_1\in\mathbb{R}} \min_{\mathbf{t}\in[0,1]^d} \sum_{j=1}^{n} \prod_{i=1,i\neq j}^{n} f(x_1+\mathbf{t}\cdot(a_i-a_j)) \tag{2.15}$$

found in Theorem 2.1 engenders some discussion on how it is connected to the structure of f. Consider the function $r:\mathbb{R}\to\mathbb{R}$ which takes

$$r(x_1) \;=\; \min_{\mathbf{t}\in[0,1]^d} \sum_{j=1}^{n} \prod_{i=1,i\neq j}^{n} f(x_1+\mathbf{t}\cdot(a_i-a_j)) \tag{2.16}$$

so that we are interested in $\max_x r(x)$. The value r takes on at a given point x_1 is not truly global; we may alter f outside of an interval around x_1 without altering the value taken there. Nor is it truly local; no amount of information on an ε-ball around x_1 can provide enough information to determine this value, because we allow t to extend to 1.

In the case $d=1$, equation (2.15) reduces to

$$\max_{x_1\in\mathbb{R}} \min_{t\in[0,1]} [f(x-t)+f(x+t)]. \tag{2.17}$$

In the special cases when f is convex, the minimum is always obtained for $t=0$, since increasing t increases the symmetric sum about the point x. Conversely, for f concave the minimum occurs when $t=1$. This provides the following corollary.

Corollary 2.5. *A continuous convex function f maximizing*

$$\frac{\displaystyle\min_{\mathbf{t}\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\mathrm{d}x}{||f||_{L^1}^2} \tag{2.18}$$

must satisfy both

$$\max_{x_1\in\mathbb{R}} f(x_1) \;\le\; \frac{1}{||f||_{L^1}} \min_{\mathbf{t}\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\mathrm{d}x \tag{2.19}$$

and

$$2\max_{x_1\in\mathbb{R}} f(x_1) \;\le\; \min_{x_2\in\mathbb{R}}[f(x_2-1)+f(x_2+1)]. \tag{2.20}$$

Similarly, a concave function maximizing (2.18) must satisfy both

$$\max_{x_1\in\mathbb{R}} f(x_1-1)+f(x_1+1) \;\le\; \frac{2}{||f||_{L^1}} \min_{\mathbf{t}\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\mathrm{d}x \tag{2.21}$$

and

$$\max_{x_1\in\mathbb{R}}[f(x_1-1)+f(x_1+1)] \;\le\; 2\min_{x_2\in\mathbb{R}} f(x_2). \tag{2.22}$$

2.5. **Theorem 2.1 for discontinuous functions.** We may relax the hypothesis on the continuity of f if, in the conditions given in the theorem, we replace $f(x_0)$ with

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{x_0-\varepsilon/2}^{x_0+\varepsilon/2} f(x)\mathrm{d}x$$

whenever we evaluate f at a point x_0. When f is continuous, this limit is $f(x_0)$. If f has a removable or jump discontinuity (such as those found in some constructions), the limit is no longer identical to function evaluation, but still may be calculated.

3. Proof of Theorem 2.1

Before proving Theorem 2.1, we recall the theorem statement.

Theorem 2.1. *Let $d, n \in \mathbb{N}$ and A a $d\times n$ matrix with columns a_i for $1\le i\le n$ satisfying Lemma 2.2. Then a continuous function f maximizing equation* (1.10) *must satisfy both*

$$\max_{x_1\in\mathbb{R}}\min_{\mathbf{t}\in[0,1]^d}\sum_{i=1}^{n}\prod_{i=1,i\neq j}^{n} f(x_1+\mathbf{t}\cdot(a_i-a_j)) \;\le\; \frac{n}{||f||_{L^1}}\min_{\mathbf{t}\in[0,1]^d}\int_{\mathbb{R}}\prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\mathrm{d}x \tag{2.1}$$

and

$$\max_{x_1\in\mathbb{R}}\min_{\mathbf{t}\in[0,1]^d}\sum_{j=1}^{n}\prod_{i=1,i\neq j}^{n} f(x_1+\mathbf{t}\cdot(a_i-a_j)) \;\le\; \min_{x_2\in\mathrm{supp}(f)}\max_{\mathbf{t}\in[0,1]^d}\sum_{j=1}^{n}\prod_{i=1,i\neq j}^{n} f(x_2+\mathbf{t}\cdot(a_i-a_j)). \tag{2.2}$$

Proof of Theorem 2.1. For $\varepsilon > 0$ and $x_1\in\mathbb{R}$, set $g(x) := \varepsilon\chi_{[x_1-\varepsilon/2,x_1+\varepsilon/2]}$. We show that if the given conditions fail, $f+g$ is an improvement over f. That is, we wish to show that

$$\frac{\min_{\mathbf{t}\in[0,1]^d}\int_{\mathbb{R}}\prod_{i=1}^{n}(f+g)(x+\mathbf{t}\cdot a_i)\mathrm{d}x}{||f+g||_{L^1}^n} > \frac{\min_{\mathbf{t}\in[0,1]^d}\int_{\mathbb{R}}\prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\mathrm{d}x}{||f||_{L^1}^n}. \tag{3.1}$$

By the triangle inequality it suffices to show that

$$\frac{\min_{\mathbf{t}\in[0,1]^d}\int_{\mathbb{R}}\prod_{i=1}^{n}(f+g)(x+\mathbf{t}\cdot a_i)\mathrm{d}x}{(||f||_{L^1}+||g||_{L^1})^n} > \frac{\min_{\mathbf{t}\in[0,1]^d}\int_{\mathbb{R}}\prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\mathrm{d}x}{||f||_{L^1}^n}. \tag{3.2}$$

Now the simple form of g allows us to compute $||g||_{L^1}=\varepsilon^2$. By letting $\varepsilon\to 0$, we find it is enough to see

$$\frac{\min_{\mathbf{t}\in[0,1]^d}\int_{\mathbb{R}}\prod_{i=1}^{n}(f+g)(x+\mathbf{t}\cdot a_i)\mathrm{d}x}{||f||_{L^1}^n+n\varepsilon^2||f||_{L^1}^{n-1}+O(\varepsilon^4)} > \frac{\min_{\mathbf{t}\in[0,1]^d}\int_{\mathbb{R}}\prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\mathrm{d}x}{||f||_{L^1}^n}. \tag{3.3}$$

Since g is $O(\varepsilon)$, the product on the left hand side will be dominated by those products containing only one g. Breaking up the minimum we find the sufficient condition

$$\frac{\min_{\mathbf{t}\in[0,1]^d} \int_{\mathbb{R}} \sum_{i=1}^{n} g(x+\mathbf{t}\cdot a_j) \prod_{i=1, i\neq j}^{n} f(x+\mathbf{t}\cdot a_i)\mathrm{d}x + O(\varepsilon^3)}{n\varepsilon^2 ||f||_{L^1}^{n-1} + O(\varepsilon^4)} > \frac{\min_{\mathbf{t}\in[0,1]^d} \int_{\mathbb{R}} \prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\mathrm{d}x}{||f||_{L^1}^n}. \tag{3.4}$$

With f continuous, $g(x+\mathbf{t}\cdot a_j)$, as we integrate over x, approximates $f(x_1 - \mathbf{t}\cdot a_j)$, so that again by letting $\varepsilon \to 0$ we need

$$\frac{\varepsilon^2 \min_{\mathbf{t}\in[0,1]^d} \sum_{i=1}^{n} \prod_{i=1, i\neq j}^{n} f(x_1+\mathbf{t}\cdot (a_i - a_j)) + O(\varepsilon^3)}{n\varepsilon^2 ||f||_{L^1}^{n-1} + O(\varepsilon^4)} > \frac{\min_{\mathbf{t}\in[0,1]^d} \int_{\mathbb{R}} \prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\mathrm{d}x}{||f||_{L^1}^n}. \tag{3.5}$$

Therefore as higher order terms are eliminated, we find

$$\min_{\mathbf{t}\in[0,1]^d} \sum_{i=1}^{n} \prod_{i=1, i\neq j}^{n} f(x_1+\mathbf{t}\cdot (a_i - a_j)) > \frac{n}{||f||_{L^1}} \min_{\mathbf{t}\in[0,1]^d} \int_{\mathbb{R}} \prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\mathrm{d}x, \tag{3.6}$$

and taking the maximum over x_1 yields the first half of Theorem 2.1.

For the second half, we take a second point $x_2 \in \mathrm{supp}(\mathbb{R})$ and in the spirit of g set $g_1 := \varepsilon\chi_{[x_1-\varepsilon/2, x_1+\varepsilon/2]}$ and $g_2 := \varepsilon\chi_{[x_2-\varepsilon/2, x_2+\varepsilon/2]}$. Then by taking ε small enough we know that $||f+g_1-g_2||_{L^1} = ||f||_{L^1}$, and, so long as $x_1 \neq x_2$, g_1 and g_2 have disjoint support. Then to prove that

$$\frac{\min_{\mathbf{t}\in[0,1]^d} \int_{\mathbb{R}} \prod_{i=1}^{n} (f+g_1-g_2)(x+\mathbf{t}\cdot a_i)\mathrm{d}x}{||f+g_1-g_2||_{L^1}^n} > \frac{\min_{\mathbf{t}\in[0,1]^d} \int_{\mathbb{R}} \prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\mathrm{d}x}{||f||_{L^1}^n} \tag{3.7}$$

we show that

$$\min_{\mathbf{t}\in[0,1]^d} \int_{\mathbb{R}} \prod_{i=1}^{n} (f+g_1-g_2)(x+\mathbf{t}\cdot a_i)\mathrm{d}x > \min_{\mathbf{t}\in[0,1]^d} \int_{\mathbb{R}} \prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\mathrm{d}x. \tag{3.8}$$

By breaking open the minimum and expanding the product on the left hand side, we find

$$\min_{\mathbf{t}\in[0,1]^d} \int_{\mathbb{R}} \sum_{j=1}^{n} (g_1-g_2)(x+\mathbf{t}\cdot a_j) \prod_{i=1, i\neq j}^{n} f(x+\mathbf{t}\cdot a_i)\mathrm{d}x > 0. \tag{3.9}$$

Transferring those negative g_2 terms to the right we find

$$\min_{\mathbf{t}\in[0,1]^d} \int_{\mathbb{R}} \sum_{j=1}^{n} g_1(x+\mathbf{t}\cdot a_j) \prod_{i=1, i\neq j}^{n} f(x+\mathbf{t}\cdot a_i)\mathrm{d}x > \max_{\mathbf{t}\in[0,1]^d} \int_{\mathbb{R}} \sum_{j=1}^{n} g_2(x+\mathbf{t}\cdot a_j) \prod_{i=1, i\neq j}^{n} f(x+\mathbf{t}\cdot a_i)\mathrm{d}x. \tag{3.10}$$

Once again, g_1 and g_2, when integrated against a product, return the value of that product evaluated at a specific value of x as $\varepsilon \to 0$, giving

$$\min_{\mathbf{t}\in[0,1]^d} \sum_{j=1}^{n} \prod_{i=1, i\neq j}^{n} f(x_1+\mathbf{t}\cdot (a_i - a_j)) > \max_{\mathbf{t}\in[0,1]^d} \sum_{j=1}^{n} \prod_{i=1, i\neq j}^{n} f(x_2+\mathbf{t}\cdot (a_i - a_j)). \tag{3.11}$$

Taking the best possible x_1, x_2 gives the second half of Theorem 2.1. □

4. FUTURE WORK

In this article we have shown certain conditions which must hold for f maximizing equation (1.10), but we were not able to show whether or not such functions exist. We have defined a much broader class of convolution-type inequalities than the one studied in [BS]. There, the authors' focus is on placing upper and lower bounds on equation (1.3). Can we find (the best possible) constants C_1, C_2, and function f_0, all depending on d, n, A, such that for any choice of $f \in L^1$, we have

$$\frac{\min\limits_{\mathbf{t}\in[0,1]^n} \int\limits_{\mathbb{R}} \prod_{i=1}^{n} f(x+t_i)\mathrm{d}x}{||f||_{L^1}^n} \geq C_1 \tag{4.1}$$

while

$$\frac{\min\limits_{\mathbf{t}\in[0,1]^n} \int\limits_{\mathbb{R}} \prod_{i=1}^{n} f_0(x+t_i)\mathrm{d}x}{||f_0||_{L^1}^n} \geq C_2? \tag{4.2}$$

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