

A PAIR OF DIOPHANTINE EQUATIONS INVOLVING THE FIBONACCI NUMBERS

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ABSTRACT. Let $a, b \in \mathbb{N}$ be relatively prime. Previous work showed that exactly one of the two equations $ax + by = (a - 1)(b - 1)/2$ and $ax + by + 1 = (a - 1)(b - 1)/2$ has a nonnegative, integral solution; furthermore, the solution is unique. Let F_n be the n^{th} Fibonacci number. When $(a, b) = (F_n, F_{n+1})$, it is known that there is an explicit formula for the unique solution (x, y) . We establish formulas to compute the solution when $(a, b) = (F_n^2, F_{n+1}^2)$ and (F_n^3, F_{n+1}^3) , giving rise to some intriguing identities involving Fibonacci numbers. Additionally, we construct a different pair of equations that admits a unique positive (instead of nonnegative), integral solution.

1. INTRODUCTION

The study of Diophantine equations, particularly those associated with specific integer sequences, has been a topic of great interest in number theory (see [4, 8, 9, 10, 11, 13] for some recent work on Diophantine equations that involve Fibonacci numbers). We study a pair of Diophantine equations, first encountered by Beiter [2] in the study of cyclotomic polynomials $\Phi_{pq}(x)$ for primes $p < q$. The pair was later extended by Chu [5] to relatively prime numbers.

Theorem 1.1. [5, Theorem 1.1] *For relatively prime $a, b \in \mathbb{N}$, exactly one of the following equations has a nonnegative, integral solution*

$$ax + by = \frac{(a - 1)(b - 1)}{2}, \tag{1.1}$$

$$ax + by + 1 = \frac{(a - 1)(b - 1)}{2}. \tag{1.2}$$

Furthermore, the solution is unique.

Thanks to Theorem 1.1, we can define the function $\Gamma : \{(m, n) : \gcd(m, n) = 1\} \rightarrow \{1, 2\}$ as

$$\Gamma(m, n) = \begin{cases} 1, & \text{if (1.1) has a solution when } (a, b) = (m, n); \\ 2, & \text{if (1.2) has a solution when } (a, b) = (m, n). \end{cases}$$

It is well-known that two consecutive Fibonacci numbers are relatively prime. Chu [5] then studied $\Gamma(F_n, F_{n+1})$, where $(F_n)_{n=0}^{\infty}$ are the Fibonacci numbers defined as $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. It turned out that for $n \geq 3$,

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$\Gamma(F_n, F_{n+1})$ alternates between 1 and 2 in groups of three ([5, Theorem 1.6]). Continuing the work, Davala [6] examined $\Gamma(B_n, B_{n+1})$, $\Gamma(B_{2n-1}, B_{2n+1})$, $\Gamma(C_n, C_{n+1})$, and $\Gamma(B_n, C_n)$, with $(B_n)_{n=1}^\infty$ and $(C_n)_{n=1}^\infty$ being the so-called n^{th} balancing number and the n^{th} Lucas-balancing number.¹ These sequences satisfy the recurrence relations $B_1 = 1, B_2 = 6, B_{n+1} = 6B_n - B_{n-1}$ and $C_1 = 3, C_2 = 17, C_{n+1} = 6C_n - C_{n-1}$ for $n \geq 2$.

Recently, Arachchi et al. [1] provided a useful criterion on (a, b) to determine the value of $\Gamma(a, b)$. The authors then used the criterion to study $\Gamma(a_n, a_{n+1})$ for various sequences $(a_n)_{n=1}^\infty$, including the natural numbers raised to the k^{th} power, arithmetic progressions, shifted geometric sequences, and so on. One notable result is that for a fixed k , $\Gamma(n^k, (n+1)^k)$ eventually alternates between 1 and 2.

Our main results are inspired by the identities in [5, Theorem 1.6]:

$$\begin{aligned} \frac{F_{6k-1} - 1}{2} F_{6k} + \frac{F_{6k-1} - 1}{2} F_{6k+1} &= \frac{(F_{6k} - 1)(F_{6k+1} - 1)}{2}, \\ \frac{F_{6k+1} - 1}{2} F_{6k+1} + \frac{F_{6k-1} - 1}{2} F_{6k+2} &= \frac{(F_{6k+1} - 1)(F_{6k+2} - 1)}{2}, \\ \frac{F_{6k+1} - 1}{2} F_{6k+2} + \frac{F_{6k+1} - 1}{2} F_{6k+3} &= \frac{(F_{6k+2} - 1)(F_{6k+3} - 1)}{2}, \\ 1 + \frac{F_{6k+2} - 1}{2} F_{6k+3} + \frac{F_{6k+2} - 1}{2} F_{6k+4} &= \frac{(F_{6k+3} - 1)(F_{6k+4} - 1)}{2}, \\ 1 + \frac{F_{6k+4} - 1}{2} F_{6k+4} + \frac{F_{6k+2} - 1}{2} F_{6k+5} &= \frac{(F_{6k+4} - 1)(F_{6k+5} - 1)}{2}, \\ 1 + \frac{F_{6k+4} - 1}{2} F_{6k+5} + \frac{F_{6k+4} - 1}{2} F_{6k+6} &= \frac{(F_{6k+5} - 1)(F_{6k+6} - 1)}{2}. \end{aligned}$$

Therefore, not only is $\Gamma(F_n, F_{n+1})$ periodic, but there is also a formula for the solution (x, y) . We shall examine $\Gamma(F_n^2, F_{n+1}^2)$ and establish a formula for the solution (x, y) . Let us look at the data.

n	F_n^2	F_{n+1}^2	x_n	y_n	$\Gamma(F_n^2, F_{n+1}^2)$
2	1	4	0	0	1
3	4	9	3	0	1
4	9	25	5	2	2
5	25	64	20	4	1
6	64	169	51	12	1
7	169	441	83	52	2
8	441	1156	356	84	1
9	1156	3025	935	220	1
10	3025	7921	1513	934	2
11	7921	20736	6408	1512	1
12	20736	54289	16775	3960	1
13	54289	142129	27143	16776	2

Table 1. Data for $\Gamma(F_n^2, F_{n+1}^2)$.

¹Balancing numbers were introduced by Behera and Panda [3] to be solutions of the Diophantine equation $1 + 2 + \dots + (n-1) = (n+1) + (n+2) + \dots + (n+r)$ for some natural number r . The n^{th} balancing number is denoted by B_n , and $C_n = \sqrt{8B_n^2 + 1}$ is called the n^{th} Lucas number [12].

The data suggest that as n increases, $\Gamma(F_n^2, F_{n+1}^2)$ repeats the pattern 1, 1, 2. Furthermore, the unique nonnegative, integral solution (x_n, y_n) of the Diophantine pair

$$F_n^2 x_n + F_{n+1}^2 y_n = \frac{(F_n^2 - 1)(F_{n+1}^2 - 1)}{2}, \quad (1.3)$$

$$1 + F_n^2 x_n + F_{n+1}^2 y_n = \frac{(F_n^2 - 1)(F_{n+1}^2 - 1)}{2} \quad (1.4)$$

seems to have the property that for $n \not\equiv 1 \pmod{3}$,

$$F_n^2 - x - y = 1,$$

while

$$\begin{cases} 2x_n - F_n^2 = 1, & \text{for } n \equiv 4 \pmod{6}; \\ 2x_n - F_n^2 = -3, & \text{for } n \equiv 1 \pmod{6}. \end{cases}$$

The next theorem and proposition confirm that our observations hold for all $n \geq 2$:

Theorem 1.2. *For $n \geq 2$, we have the following identities:*

$$1 + \frac{F_n^2 - 3}{2} \cdot F_n^2 + \frac{F_n^2 - F_{n-1}^2 - 1}{2} \cdot F_{n+1}^2 = \frac{(F_n^2 - 1)(F_{n+1}^2 - 1)}{2}, \text{ if } n \text{ is odd;} \quad (1.5)$$

$$1 + \frac{F_n^2 + 1}{2} \cdot F_n^2 + \frac{F_n^2 - F_{n-1}^2 - 1}{2} \cdot F_{n+1}^2 = \frac{(F_n^2 - 1)(F_{n+1}^2 - 1)}{2}, \text{ if } n \text{ is even;} \quad (1.6)$$

$$\left(F_n^2 - \frac{F_{n-1}^2 + 1}{2}\right) \cdot F_n^2 + \frac{F_{n-1}^2 - 1}{2} \cdot F_{n+1}^2 = \frac{(F_n^2 - 1)(F_{n+1}^2 - 1)}{2}. \quad (1.7)$$

Proposition 1.3. *For $n \geq 0$ with $n \equiv 0 \pmod{3}$, F_n is even, while F_{n+1} and F_{n+2} are odd.*

Corollary 1.4. *For $n \geq 2$ and $n \equiv 0, 2, 3, 5 \pmod{6}$,*

$$(x, y) = \left(F_n^2 - \frac{F_{n-1}^2 + 1}{2}, \frac{F_{n-1}^2 - 1}{2}\right)$$

is the unique nonnegative, integral solution of (1.3).

For $n \geq 2$ and $n \equiv 1 \pmod{6}$,

$$(x, y) = \left(\frac{F_n^2 - 3}{2}, \frac{F_n^2 - F_{n-1}^2 - 1}{2}\right)$$

is the unique nonnegative, integral solution of (1.4).

For $n \geq 2$ and $n \equiv 4 \pmod{6}$,

$$(x, y) = \left(\frac{F_n^2 + 1}{2}, \frac{F_n^2 - F_{n-1}^2 - 1}{2}\right)$$

is the unique nonnegative, integral solution of (1.4).

Next, we investigate $\Gamma(F_n^3, F_{n+1}^3)$ and find formulas for the unique integral solution (x, y) of the pair

$$F_n^3 x_n + F_{n+1}^3 y_n = \frac{(F_n^3 - 1)(F_{n+1}^3 - 1)}{2}, \quad (1.8)$$

$$1 + F_n^3 x_n + F_{n+1}^3 y_n = \frac{(F_n^3 - 1)(F_{n+1}^3 - 1)}{2}. \quad (1.9)$$

Interestingly, formulas for the solution can be expressed as a (alternate) sum of Fibonacci numbers cubed. As before, we examine the data.

n	F_n^3	F_{n+1}^3	x_n	y_n	$\Gamma(F_n^3, F_{n+1}^3)$
2	1	8	0	0	1
3	8	27	8	1	1
4	27	125	18	9	2
5	125	512	106	36	1
6	512	2197	405	161	2
7	2197	9261	1791	673	1
8	9261	39304	7469	2870	2

Table 2. Data for $\Gamma(F_n^3, F_{n+1}^3)$.

From the data, $\Gamma(F_n^3, F_{n+1}^3)$ seems to alternate between 1 and 2, and

$$x_4 = 3^3 - x_3 - 1, \quad y_4 = y_3 + 2^3,$$

$$x_5 = 5^3 - x_4 - 1, \quad y_5 = y_4 + 3^3,$$

$$x_6 = 8^3 - x_5 - 1, \quad y_6 = y_5 + 5^3,$$

which suggests that for $n \geq 4$,

$$\begin{cases} x_n &= F_n^3 - x_{n-1} - 1, \\ y_n &= y_{n-1} + F_{n-1}^3. \end{cases}$$

These observations are verified for all $n \geq 3$ by our next theorem.

Theorem 1.5. *For $m \in \mathbb{N}_{\geq 2}$, we have the following identities:*

$$\left(\sum_{k=1}^{2m-1} (-1)^{k-1} F_k^3 \right) F_{2m-1}^3 + \left(\sum_{k=2}^{2m-2} F_k^3 \right) F_{2m}^3 = \frac{(F_{2m-1}^3 - 1)(F_{2m}^3 - 1)}{2}, \quad (1.10)$$

$$1 + \left(\sum_{k=1}^{2m} (-1)^k F_k^3 - 1 \right) F_{2m}^3 + \left(\sum_{k=2}^{2m-1} F_k^3 \right) F_{2m+1}^3 = \frac{(F_{2m}^3 - 1)(F_{2m+1}^3 - 1)}{2}. \quad (1.11)$$

We shall prove Theorems 1.2 and 1.5 in Sections 2 and 3, respectively. In Section 4, we construct another pair of Diophantine equations that resembles the original pair, (1.1) and (1.2), but admits a unique positive (instead of nonnegative), integral solution and is asymmetric in the sense that $\Gamma(a, b)$ may not be equal to $\Gamma(b, a)$. Finally, Section 5 is devoted to discussing several problems for future investigation.

2. THE CASE OF FIBONACCI NUMBERS SQUARED

The main goal of this section is to prove an analog of Theorem 1.1 for Fibonacci numbers squared. Our proof uses Cassini's identity and the Fibonacci recurrence relation to make similar terms appear and cancel themselves out.

Proof of Proposition 1.3. Let $k \geq 0$ such that $k \equiv 0 \pmod{3}$. Assume that F_k is even, and F_{k+1} and F_{k+2} are odd. By the linear recurrence,

$$\begin{aligned} F_{k+3} &= F_{k+1} + F_{k+2} \equiv 1 + 1 \equiv 0 \pmod{2}, \\ F_{k+4} &= F_{k+2} + F_{k+3} \equiv 1 + 0 \equiv 1 \pmod{2}, \\ F_{k+5} &= F_{k+3} + F_{k+4} \equiv 0 + 1 \equiv 1 \pmod{2}. \end{aligned}$$

Since $F_0 = 0$ and $F_1 = F_2 = 1$, Proposition 1.3 holds inductively. \square

Proof of Theorem 1.2. We start by proving (1.5). By Cassini's identity for odd n ,

$$F_n^2 - 1 = F_{n-1}F_{n+1}.$$

Hence,

$$F_n^4 - 2F_n^2 + 1 = F_{n-1}^2 F_{n+1}^2,$$

which gives

$$F_n^4 - 2F_n^2 + 1 - F_{n+1}^2 \cdot F_{n-1}^2 + (1 + F_{n+1}^2 \cdot F_n^2 - F_{n+1}^2 - F_n^2) = (1 + F_{n+1}^2 \cdot F_n^2 - F_{n+1}^2 - F_n^2).$$

Therefore,

$$2 + (F_n^2 - 3) \cdot F_n^2 + (F_n^2 - F_{n-1}^2 - 1) \cdot F_{n+1}^2 = (F_{n+1}^2 - 1)(F_n^2 - 1).$$

We obtain

$$1 + \frac{F_n^2 - 3}{2} \cdot F_n^2 + \frac{F_n^2 - F_{n-1}^2 - 1}{2} \cdot F_{n+1}^2 = \frac{(F_n^2 - 1)(F_{n+1}^2 - 1)}{2}.$$

Next, we prove (1.6). We again start with Cassini's identity for even n :

$$F_n^2 + 1 = F_{n-1}F_{n+1}.$$

Hence,

$$F_n^4 + 2F_n^2 + 1 = F_{n-1}^2 F_{n+1}^2,$$

which gives

$$F_n^4 + F_n^2 + 1 - F_{n-1}^2 F_{n+1}^2 + (1 + F_{n+1}^2 F_n^2 - F_{n+1}^2) = -F_n^2 + (1 + F_{n+1}^2 F_n^2 - F_{n+1}^2).$$

The above implies that

$$2 + (F_n^2 + 1) \cdot F_n^2 + (F_n^2 - F_{n-1}^2 - 1) \cdot F_{n+1}^2 = (F_{n+1}^2 - 1)(F_n^2 - 1).$$

Dividing both sides by 2, we obtain

$$1 + \frac{F_n^2 + 1}{2} \cdot F_n^2 + \frac{F_n^2 - F_{n-1}^2 - 1}{2} \cdot F_{n+1}^2 = \frac{(F_n^2 - 1)(F_{n+1}^2 - 1)}{2}.$$

It remains to prove (1.7). On the one hand, from $F_n = F_{n+1} - F_{n-1}$, we have

$$F_n^2 = F_{n+1}^2 + F_{n-1}^2 - 2F_{n+1}F_{n-1}.$$

Hence,

$$F_n^2 - F_{n-1}^2 - F_{n+1}^2 = -2F_{n-1}F_{n+1},$$

and so,

$$F_n^2(F_n^2 - F_{n-1}^2 - F_{n+1}^2) = -2F_n^2 F_{n-1} F_{n+1}. \quad (2.1)$$

On the other hand, by Cassini's identity,

$$(F_{n-1} F_{n+1} - F_n^2)^2 = 1,$$

which gives

$$F_n^4 - 2F_n^2 F_{n+1} F_{n-1} = 1 - F_{n-1}^2 F_{n+1}^2. \quad (2.2)$$

It follows from (2.1) and (2.2) that

$$\begin{aligned} 2F_n^4 - F_n^2 F_{n-1}^2 - F_n^2 F_{n+1}^2 &= F_n^4 - 2F_n^2 F_{n-1} F_{n+1} \\ &= 1 - F_{n-1}^2 F_{n+1}^2. \end{aligned}$$

Equivalently,

$$2F_n^4 - F_n^2 F_{n-1}^2 + F_{n-1}^2 F_{n+1}^2 = 1 + F_n^2 F_{n+1}^2,$$

from which we know that

$$2F_n^4 - F_n^2 F_{n-1}^2 + F_{n-1}^2 F_{n+1}^2 + (-F_n^2 - F_{n+1}^2) = 1 + F_n^2 F_{n+1}^2 + (-F_n^2 - F_{n+1}^2),$$

Therefore,

$$(2F_n^2 - F_{n-1}^2 - 1) \cdot F_n^2 + (F_{n-1}^2 - 1) \cdot F_{n+1}^2 = (F_n^2 - 1)(F_{n+1}^2 - 1).$$

Dividing both sides by 2, we obtain

$$\left(F_n^2 - \frac{F_{n-1}^2 + 1}{2}\right) \cdot F_n^2 + \frac{F_{n-1}^2 - 1}{2} \cdot F_{n+1}^2 = \frac{(F_n^2 - 1)(F_{n+1}^2 - 1)}{2},$$

as desired. \square

3. THE CASE OF FIBONACCI NUMBERS CUBED

This section proves an analog of Theorem 1.1 for Fibonacci numbers cubed. As for the case of Fibonacci numbers squared, our proof uses Cassini's identity and the Fibonacci recurrence relation. Our proofs also employ the identities presented by Frontczak [7] to replace the sum of cubes with a simpler expression and use the well-known identity $F_{3n} = 5F_n^3 + 3(-1)^n F_n$ (see [7, (2.6)]).

Theorem 3.1. [7, cf. Theorem 1] *For each $n \geq 1$,*

$$\sum_{k=1}^n F_k^3 = \frac{1}{4} (F_{3n+3} + F_{3n}) - F_{n+1}^3 - F_n^3 + \frac{1}{2}, \quad (3.1)$$

and

$$\sum_{k=1}^n (-1)^k F_k^3 = \frac{1}{4} ((-1)^n F_{3n+3} + (-1)^{n+1} F_{3n}) - (-1)^n F_{n+1}^3 - (-1)^{n+1} F_n^3 + \frac{1}{2}. \quad (3.2)$$

Proof of (1.10). It follows from (3.1) and (3.2) that

$$\begin{aligned} \sum_{k=1}^{2m-1} (-1)^{k-1} F_k^3 &= - \sum_{k=1}^{2m-1} (-1)^k F_k^3 = \frac{F_{6m}}{4} - \frac{F_{6m-3}}{4} - F_{2m}^3 + F_{2m-1}^3 - \frac{1}{2}, \text{ and} \\ \sum_{k=2}^{2m-2} F_k^3 &= -1 + \sum_{k=1}^{2m-2} F_k^3 = \frac{F_{6m-3}}{4} + \frac{F_{6m-6}}{4} - F_{2m-1}^3 - F_{2m-2}^3 - \frac{1}{2}. \end{aligned}$$

Hence, the left-hand side of (1.10) becomes

$$\begin{aligned} T_m &:= \left(\frac{F_{6m}}{4} - \frac{F_{6m-3}}{4} - F_{2m}^3 + F_{2m-1}^3 - \frac{1}{2} \right) F_{2m-1}^3 + \\ &\quad \left(\frac{F_{6m-3}}{4} + \frac{F_{6m-6}}{4} - F_{2m-1}^3 - F_{2m-2}^3 - \frac{1}{2} \right) F_{2m}^3 \\ &= \frac{F_{6m} - F_{6m-3}}{4} F_{2m-1}^3 + F_{2m-1}^6 + \frac{F_{6m-3} + F_{6m-6}}{4} F_{2m}^3 - \\ &\quad F_{2m-2}^3 F_{2m}^3 - 2F_{2m-1}^3 F_{2m}^3 - \frac{F_{2m-1}^3 + F_{2m}^3}{2}. \end{aligned}$$

Then the identity $T_m = (F_{2m-1}^3 - 1)(F_{2m}^3 - 1)/2$ is equivalent to

$$\begin{aligned} (F_{6m} - F_{6m-3})F_{2m-1}^3 + 4F_{2m-1}^6 + (F_{6m-3} + F_{6m-6})F_{2m}^3 \\ - 4F_{2m-2}^3 F_{2m}^3 - 10F_{2m-1}^3 F_{2m}^3 = 2. \end{aligned} \quad (3.3)$$

Since $F_{3m} = 5F_m^3 + 3(-1)^m F_m$, we write

$$F_{6m} = 5F_{2m}^3 + 3F_{2m}, \quad F_{6m-3} = 5F_{2m-1}^3 - 3F_{2m-1}, \quad \text{and} \quad F_{6m-6} = 5F_{2m-2}^3 + 3F_{2m-2}.$$

Hence,

$$(F_{6m} - F_{6m-3})F_{2m-1}^3 = 5F_{2m}^3 F_{2m-1}^3 + 3F_{2m} F_{2m-1}^3 - 5F_{2m-1}^6 + 3F_{2m-1}^4, \quad (3.4)$$

and

$$(F_{6m-3} + F_{6m-6})F_{2m}^3 = 5F_{2m-1}^3 F_{2m}^3 - 3F_{2m-1} F_{2m}^3 + 5F_{2m-2}^3 F_{2m}^3 + 3F_{2m-2} F_{2m}^3. \quad (3.5)$$

Using (3.4) and (3.5), we can rewrite the left-hand side of (3.3) as

$$S_m := 3F_{2m} F_{2m-1}^3 - F_{2m-1}^6 + 3F_{2m-1}^4 - 3F_{2m-1} F_{2m}^3 + F_{2m-2}^3 F_{2m}^3 + 3F_{2m-2} F_{2m}^3.$$

By Cassini's identity,

$$F_{2m-2} F_{2m} = (-1)^{2m-1} + F_{2m-1}^2 = -1 + F_{2m-1}^2,$$

which gives

$$F_{2m-2}^3 F_{2m}^3 = (-1 + F_{2m-1}^2)^3 = -1 + 3F_{2m-1}^2 - 3F_{2m-1}^4 + F_{2m-1}^6,$$

and

$$3F_{2m-2} F_{2m}^3 = 3F_{2m}^2 (-1 + F_{2m-1}^2) = -3F_{2m}^2 + 3F_{2m}^2 F_{2m-1}^2.$$

Therefore,

$$\begin{aligned}
S_m &= 3F_{2m}F_{2m-1}^3 - 3F_{2m-1}F_{2m}^3 - 1 + 3F_{2m-1}^2 - 3F_{2m}^2 + 3F_{2m}^2F_{2m-1}^2 \\
&= 3F_{2m}F_{2m-1}(F_{2m-1}^2 - F_{2m}^2 + F_{2m}F_{2m-1}) + 3F_{2m-1}^2 - 3F_{2m}^2 - 1 \\
&= 3F_{2m}F_{2m-1}(F_{2m-1}^2 - F_{2m}^2 + F_{2m}(F_{2m} - F_{2m-2})) + 3F_{2m-1}^2 - 3F_{2m}^2 - 1 \\
&= 3F_{2m}F_{2m-1}(F_{2m-1}^2 - F_{2m}F_{2m-2}) + 3F_{2m-1}^2 - 3F_{2m}^2 - 1 \\
&= 3F_{2m}F_{2m-1} + 3F_{2m-1}^2 - 3F_{2m}^2 - 1 \\
&= 3((F_{2m+1} - F_{2m-1})F_{2m-1} + F_{2m-1}^2 - F_{2m}^2) - 1 \\
&= 3(F_{2m+1}F_{2m-1} - F_{2m}^2) - 1 = 2,
\end{aligned}$$

as desired. \square

Proof of (1.11). Using Theorem 3.1, we can rewrite the left-hand side of (1.11) as

$$\begin{aligned}
T_m &:= 1 + \left(\frac{1}{4}(F_{6m+3} - F_{6m}) - F_{2m+1}^3 + F_{2m}^3 - \frac{1}{2}\right) F_{2m}^3 \\
&\quad + \left(\frac{1}{4}(F_{6m} + F_{6m-3}) - F_{2m}^3 - F_{2m-1}^3 - \frac{1}{2}\right) F_{2m+1}^3.
\end{aligned}$$

Hence, $T_m = (F_{2m}^3 - 1)(F_{2m+1}^3 - 1)/2$ is equivalent to

$$4F_{2m}^6 - 10F_{2m}^3F_{2m+1}^3 - 4F_{2m-1}^3F_{2m+1}^3 + F_{2m}^3(F_{6m+3} - F_{6m}) + F_{2m+1}^3(F_{6m} + F_{6m-3}) = -2. \quad (3.6)$$

We now use the identity $F_{3m} = 5F_m^3 + 3(-1)^m F_m$ to write

$$F_{6m-3} = 5F_{2m-1}^3 - 3F_{2m-1}, \quad F_{6m} = 5F_{2m}^3 + 3F_{2m}, \quad \text{and} \quad F_{6m+3} = 5F_{2m+1}^3 - 3F_{2m+1}.$$

Hence, (3.6) becomes

$$-F_{2m}^6 - 3F_{2m}^4 - 3F_{2m+1}F_{2m}^3 + 3F_{2m+1}^3F_{2m} + F_{2m-1}^3F_{2m+1}^3 - 3F_{2m+1}^3F_{2m-1} = -2. \quad (3.7)$$

By Cassini's identity,

$$F_{2m-1}F_{2m+1} = F_{2m}^2 + 1, \quad (3.8)$$

so

$$F_{2m+1}^3F_{2m-1}^3 = F_{2m}^6 + 3F_{2m}^4 + 3F_{2m}^2 + 1,$$

and

$$3F_{2m+1}^3F_{2m-1} = 3F_{2m+1}^2(F_{2m}^2 + 1) = 3F_{2m}^2F_{2m+1}^2 + 3F_{2m+1}^2. \quad (3.9)$$

Let S_m be the left side of (3.7). By (3.8) and (3.9), we have

$$\begin{aligned}
S_m &= -3F_{2m+1}F_{2m}^3 + 3F_{2m+1}^3F_{2m} + 3F_{2m}^2 + 1 - 3F_{2m}^2F_{2m+1}^2 - 3F_{2m+1}^2 \\
&= 3F_{2m}F_{2m+1}(F_{2m+1}^2 - F_{2m}^2 - F_{2m}F_{2m+1}) + 3F_{2m}^2 - 3F_{2m+1}^2 + 1 \\
&= 3F_{2m}F_{2m+1}(F_{2m+1}^2 - F_{2m}^2 - (F_{2m+1} - F_{2m-1})F_{2m+1}) + 3F_{2m}^2 - 3F_{2m+1}^2 + 1 \\
&= 3F_{2m}F_{2m+1}(-F_{2m}^2 + F_{2m-1}F_{2m+1}) + 3F_{2m}^2 - 3F_{2m+1}^2 + 1 \\
&= 3F_{2m}F_{2m+1} + 3F_{2m}^2 - 3F_{2m+1}^2 + 1 \\
&= 3((F_{2m+1} - F_{2m-1})F_{2m+1} + F_{2m}^2 - F_{2m+1}^2) + 1 \\
&= 3(F_{2m}^2 - F_{2m-1}F_{2m+1}) + 1 = -2,
\end{aligned}$$

as desired. \square

4. ANOTHER PAIR OF DIOPHANTINE EQUATIONS

Up until now, we have been working with Equations (1.1) and (1.2) given in Theorem 1.1, which admit a unique nonnegative, integral solution. We now turn our attention to the question of the existence of a distinct set of equations that instead admits a unique positive, integral solution. Such a system can be found trivially by shifting the equations in Theorem 1.1 by $a + b$.

Proposition 4.1. *For relatively prime $a, b \in \mathbb{N}$, exactly one of the following equations has a nonnegative, integral solution:*

$$ax + by = \frac{(a-1)(b-1)}{2} + (a+b), \quad (4.1)$$

$$ax + by + 1 = \frac{(a-1)(b-1)}{2} + (a+b). \quad (4.2)$$

Furthermore, the solution is unique.

Our next theorem presents a system that is not a linear shift of the equations in Theorem 1.1.

Theorem 4.2. *Let $a, b \in \mathbb{N}$ satisfy $(a, b) = 1$, $b \geq 2$, and a is odd. Consider the two following equations:*

$$ax + by = \frac{(a+1)b}{2} + 1, \quad (4.3)$$

$$ax + by = \frac{(a+1)b}{2} - 1. \quad (4.4)$$

Exactly one of the two equations has a positive, integral solution, and the solution is unique.

The following lemma shall be used in due course.

Lemma 4.3. *For integers n, x, y, a, b with a, b positive and $(a, b) = 1$, we consider the equation $xa + yb = n$. If there is a solution $(x, y) = (r, s) \in \mathbb{Z}^2$ with $r < b$ and $s \leq 0$, then there are no solutions with x, y both positive.*

Proof. All integral solutions are of the form $(x, y) = (r + tb, s - ta)$ for some $t \in \mathbb{Z}$. To get $y > 0$, we must have $t < 0$, but that implies $x < 0$. \square

Proof of Theorem 4.2. Let $k = (a+1)b/2$. Let $1 \leq r_1 \leq b-1$ be chosen such that $ar_1 \equiv k+1 \pmod{b}$ and $s_1 := (k+1 - ar_1)/b$. Let $1 \leq r_2 \leq b-1$ be chosen such that $ar_2 \equiv k-1 \pmod{b}$ and $s_2 := (k-1 - ar_2)/b$. Observe that

$$a(r_1 + r_2) = ar_1 + ar_2 \equiv 2k = (a+1)b \equiv 0 \pmod{b}.$$

It follows from $(a, b) = 1$ that $b \mid (r_1 + r_2)$. Since $2 \leq r_1 + r_2 \leq 2b - 2$, we know that $r_1 + r_2 = b$. Hence,

$$s_1 + s_2 = \frac{(k+1 - ar_1) + (k-1 - ar_2)}{b} = \frac{2k - a(r_1 + r_2)}{b} = \frac{(a+1)b - ab}{b} = 1.$$

Thus, exactly one of s_1, s_2 is positive. By definition, $r_1a + s_1b = k+1$ and $r_2a + s_2b = k-1$. Suppose, without loss of generality, that (4.3) has a positive solution. We know that (4.4) has no positive solution due to Lemma 4.3.

It remains to show that (4.3) has at most one positive solution. Let (x_1, y_1) and (x_2, y_2) be two positive solutions of (4.3). Observe that

$$x_1, x_2 \leq \frac{(a+1)b}{2} - 1,$$

which gives

$$x_1, x_2 \leq \frac{b}{2} + \frac{b}{2a} - \frac{1}{a} < b.$$

Hence, $1 \leq x_1, x_2 \leq b-1$, and so, $|x_1 - x_2| \leq b-2$. Furthermore, $x_1a + y_1b = x_2a + y_2b$ implies that

$$(x_1 - x_2)a = -(y_1 - y_2)b.$$

Since $(a, b) = 1$, b divides $x_1 - x_2$, which, in combination with $|x_1 - x_2| \leq b-2$, implies that $x_1 = x_2$. As a result, $(x_1, y_1) = (x_2, y_2)$. Therefore, (4.3) has at most one positive solution. \square

5. PROBLEMS FOR FURTHER INVESTIGATION

A natural question would be, for every fixed $i \geq 4$, whether there is a formula (similar to the ones in [5, Theorem 1.6] and Theorems 1.2 and 1.5) for the unique nonnegative, integral solution (x, y) of the pair

$$F_n^i x_n + F_{n+1}^i y_n = \frac{(F_n^i - 1)(F_{n+1}^i - 1)}{2}, \quad (5.1)$$

$$1 + F_n^i x_n + F_{n+1}^i y_n = \frac{(F_n^i - 1)(F_{n+1}^i - 1)}{2}. \quad (5.2)$$

We can follow the same process that we use to obtain Theorems 1.2 and 1.5: collecting data and then looking for any pattern. The drawback is that when i is big, collecting data for $\Gamma(F_n^i, F_{n+1}^i)$ becomes more difficult, which limits our ability to observe a pattern (if any) for $\Gamma(F_n^i, F_{n+1}^i)$. For instance, in collecting data for $\Gamma(F_n^4, F_{n+1}^4)$, we face a memory error at $\Gamma(F_{12}^4, F_{13}^4)$. While there are some patterns in Table 3 that may hint at a possible formula for the solution (x_n, y_n) such as

$$y_4 - x_3 = 1, y_5 - x_4 = -1, y_7 - x_6 = 1, y_8 - x_7 = -1, y_{10} - x_9 = 1, y_{11} - x_{10} = -1,$$

there is no clear periodicity for $\Gamma(F_n^4, F_{n+1}^4)$, at least from the first 10 values in Table 3.

n	F_n^4	F_{n+1}^4	x_n	y_n	$\Gamma(F_n^4, F_{n+1}^4)$
2	1	16	0	0	1
3	16	81	2	7	2
4	81	625	285	3	1
5	625	4096	183	284	2
6	4096	28561	1286	1863	2
7	28561	194481	88473	1287	1
8	194481	1336336	60247	88472	2
9	1336336	9150625	412554	607919	2
10	9150625	62742241	28542389	412555	1
11	62742241	429981696	19385711	28542388	1

Table 3. Data for $\Gamma(F_n^4, F_{n+1}^4)$.

Another unexpected feature of Table 3 is the non-monotonicity of x_n and y_n , unlike what we have in Tables 1 and 2.

A more general problem is to find a formula for the solution of $\Gamma(F_n^i, F_{n+1}^j)$ for arbitrary i and j . As an example, we collect the data for $\Gamma(F_n^2, F_{n+1}^3)$:

n	F_n^2	F_{n+1}^3	x_n	y_n	$\Gamma(F_n^2, F_{n+1}^3)$
2	1	8	0	0	1
3	4	27	3	1	1
4	9	125	55	0	2
5	25	512	20	11	1
6	64	2197	51	30	1
7	169	9261	4493	2	2
8	441	39304	356	216	1
9	1156	166375	935	571	1
10	3025	704969	350037	10	2

Table 4. Data for $\Gamma(F_n^2, F_{n+1}^3)$.

Table 4 suggests that $\Gamma(F_n^2, F_{n+1}^3)$ follows the same pattern as $\Gamma(F_n^2, F_n^2)$.

Finally, all of the above questions can be asked for the pair in Theorem 4.2.

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