OPTIMAL POINT SETS DETERMINING FEW DISTINCT TRIANGLES

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ABSTRACT. We generalize work of Erdős and Fishburn to study the structure of finite point sets that determine few distinct triangles. Specifically, we ask for a given $t$, what is the maximum number of points that can be placed in the plane to determine exactly $t$ distinct triangles? Denoting this quantity by $F(t)$, we show that $F(1) = 4$, $F(2) = 5$, and $F(t) < 48(t + 1)$ for all $t$. We also completely characterize the optimal configurations for $t = 1, 2$.

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1. INTRODUCTION

1.1. Background. Finite point configurations are a central object of study in discrete geometry. Perhaps the most well-known problem is the Erdős distinct distances conjecture, which states that any set of $n$ points in the plane determines at least $\Omega(n/\sqrt{\log n})$ distinct distances between points. This problem, first proposed by Erdős in 1946 [Er], remains open; however, in 2010, Guth and Katz mostly resolved it by proving that $n$ points determined at least $\Omega(n/\log n)$ distinct distances [GK]. A closely related question is: given a fixed positive integer $k$, what is the maximum number of points that can be placed in the plane to determine exactly $k$ distances? Furthermore, can the optimal configurations be completely characterized? Erdős and Fishburn [EF] introduced this
question in 1996 and characterized the optimal configurations for $1 \leq k \leq 4$. Shinohara [Sh] and Wei [We] have characterized the optimal configurations for $k = 5$ and $k = 6$, respectively.

As a distance is just a pair of points, distances can be phrased as the set of 2-point configurations determined by a set. In an analogous manner, we can study the set of 3-point configurations (i.e., triangles) determined by a set. Much less is known about distinct triangles determined by a finite set. The analogue of the Erdős distinct distance problem would ask for the minimum number of distinct triangles determined by $n$ points in the plane. In general, the current best known bound is that $n$ points determine at least $\Omega(n^{5/3})$ distinct triangles, a fact that follows from the best-known bound for the number of unit distances determined by a point set in the plane (see, for example, [Sz]). Greenleaf and Iosevich [GI] show that this can be improved to $\Omega(n^{12/7 - \epsilon})$ if some restrictions are imposed on the set $P$.

In this paper, we adopt a slightly different definition of triangle than that of [GI]. Loosely (this is made precise in the next section), we define a triangle as a triple of noncollinear points, and two triangles are considered distinct if and only if they are not congruent. In [GI], degenerate triangles (i.e., collinear triples of points) are allowed, but here they are not.

We study the following related question: given a fixed $t$, what is the maximum number of points that can be placed in the plane to determine exactly $t$ distinct triangles? Our main results are the following.

**Theorem 1.1.** Let $F(t)$ denote the maximum number of points that can be placed in the plane to determine exactly $t$ distinct triangles. Then

1. $F(1) = 4$ and the only configuration that achieves this is a rectangle,
2. $F(2) = 5$ and the only configurations that achieve this are a square with its center and a regular pentagon, and
3. $F(t) < 48(t + 1)$ for all $t$.

**1.2. Definitions and setup.** We make precise the notion of distinct triangles.

**Definition 1.2.** Given a finite point set $P \subset \mathbb{R}^2$, we say two triples $(a, b, c), (a', b', c') \in P^3$ are equivalent if there is an isometry mapping one to the other, and we denote this as $(a, b, c) \sim (a', b', c')$.

**Definition 1.3.** Given a finite point set $P \subset \mathbb{R}^2$, we denote by $P_{nc}^3$ the set of noncollinear triples $(a, b, c) \in P^3$.

**Definition 1.4.** Given a finite point set $P \subset \mathbb{R}^2$, we define the set of distinct triangles determined by $P$ as

$$T(P) := P_{nc}^3 / \sim.$$  

We prove the first two parts of Theorem 1.1 by enumerating cases and disposing of them one by one via elementary geometry. We prove the last part by providing a self-contained lower bound for the minimum number of distinct triangles. We then conclude with some conjectures and ideas for future work.

In the proofs of the first two parts, we also use the following lemma, which we prove in Section 6.
Lemma 1.5. For a set of four noncollinear points in the plane, exactly one of the following holds.

1. The four points are not in convex position.
2. The four points are in convex position.
   (a) Three of the points are collinear.
   (b) The determined quadrilateral has four distinct side lengths.
   (c) The determined quadrilateral has exactly one pair of congruent sides.
      (i) The congruent sides are adjacent.
      (ii) The congruent sides are opposite.
   (d) The determined quadrilateral has two distinct pairs of congruent sides.
      (i) The congruent sides are adjacent to each other (a kite).
      (ii) The congruent sides are opposite each other (a parallelogram).
   (e) Three sides are congruent and the fourth is distinct.
   (f) All four sides are congruent (a rhombus).

Cases 2b, 2(c)i, 2(c)ii, and 2(d)i determine at least three distinct triangles. Cases 1, 2a, and 2e determine at least two distinct triangles.

2. Classifying optimal 1-triangle sets

In this section, we prove part (1) of Theorem 1.1. We show that the only four-point configuration that determines exactly one triangle is a rectangle. This proves that $F(1) = 4$ because there is no five-point configuration such that every four-point subconfiguration is a rectangle.

By Lemma 1.5, we only need to consider the cases 2(d)ii and 2f because all of the other cases trivially lead to at least three triangles. We consider first the case 2(d)ii, when there are two pairs of congruent sides opposite each other.

Proof of case 2(d)ii: two pairs of opposite congruent sides. Since two pairs of opposite sides are congruent, the quadrilateral must be a parallelogram (Figure 1). We claim $\triangle ABC$ and $\triangle BCD$ are congruent if and only if $ABCD$ is a rectangle. They share side $BC$ and $AB = CD$, so $\triangle ABC \cong \triangle BCD$ if and only if $BD = AC$, which happens if and only if $ABCD$ is a rectangle. □

![Figure 1](image-url)

Figure 1. A quadrilateral with two pairs of opposite congruent sides. If $ABCD$ is a rectangle, then it determines only one triangle, but if $ABCD$ is not a rectangle, then $\triangle ABC$ and $\triangle BCD$ are distinct.

Proof of case 2f: four congruent sides. Any quadrilateral with four sides congruent is a rhombus, and a rhombus is a parallelogram. So, by the argument in case 2(d)ii, a
rhombus determines two distinct triangles if and only if it is not a square. Thus, we have shown that the only four-point configuration that determines one triangle is a rectangle. This completes the proof of part (1) of Theorem 1.1.

3. Classifying Optimal 2-Triangle Sets

In this section, we prove part (2) of Theorem 1.1. As in the proof of part (1), we show that the only possible configurations determining exactly two triangles are the square with its center and the regular pentagon. We consider the possible four-point configurations enumerated in Lemma 1.5 and we show that the addition of a fifth point to any of them (unless it creates one of the two claimed configurations) necessarily determines a third triangle. Moreover, adding a sixth point to either of the demonstrated optimal configurations also must determine a third triangle. By Lemma 1.5, the only cases we need to consider are 1, 2a, 2(d)ii, 2e, and 2f because the other four-point configurations already contain more than two distinct triangles.

Proof of case 1: not in convex position. Using the notation of Figure 2, if \( \triangle ABC \) is not equilateral, or if \( \triangle ABC \) is equilateral but \( D \) is not the center of \( \triangle ABC \), then there are already three distinct triangles, so no more work is needed.

If \( \triangle ABC \) is equilateral and \( D \) is its center, we show that the addition of a fifth point anywhere necessarily determines a new triangle. When we add a fifth point \( E \), it will necessarily determine a triangle with \( AB \) (Figure 2). If \( \triangle EAB \) is not congruent to \( \triangle ABC \) or \( \triangle ABD \), we’re done, so assume it’s congruent to one of those. Either way, \( \triangle ECB \) will be distinct from the other two, so we have three distinct triangles, so this case is done.

![Figure 2](image)

**Figure 2.** Possibilities for adding a fifth point to a non-convex set.

Proof of case 2a: three collinear points. With the notation of Figure 3, if \( D \) does not lie on the perpendicular bisector of \( AB \), then \( \triangle ACD, \triangle BCD, \) and \( \triangle ABD \) are all distinct, so no more work is needed. Also note that if a fifth point \( E \) is added to the interior of \( \triangle ABD \), it creates a non-convex four-point subconfiguration, so the previous
case applies to show that there are at least 3 distinct triangles. Thus we assume the fifth point $E$ is added outside $\triangle ABD$.

If $D$ lies on the perpendicular bisector of $AB$ but $DC \neq AB$, the addition of a fifth point $E$ will create a triangle with $AC$. Triangle $\triangle EAC$ can’t be congruent to $\triangle ABD$ because $AC$ is shorter than any side of $\triangle ABD$, so to avoid a third triangle we must have $\triangle EAC \cong \triangle ACD$. There are three choices for $E$ that satisfy this (Figure 3), but either way, $\triangle EAC$, $\triangle EAB$, and $\triangle EDB$ are all distinct.

If $D$ lies on the perpendicular bisector of $AB$ and $DC = AB$, then the same argument from above still applies; however, in this case, choosing $E$ to form the square $ABDE$ leaves us with only two triangles, but the other two choices for $E$ give us three (see Figure 3), so this case is done.

![Figure 3](image-url)

**Figure 3.** Addition of a fifth point when three points are collinear. If $DC \neq AC$, then any choice of $E$ forces a third triangle. If, on the other hand, $DC = AC$, then choosing $E$ creates a square with its center but $E'$ and $E''$ still generate a third triangle.

**Proof of case 2(d)ii: two pairs of opposite congruent sides.** This case has two subcases.

**Subcase A: non-rectangle:** Using the notation of Figure 4 if we add a fifth point $E$ on line $AB$, then we have five points with three collinear, so we have 3 distinct triangles by case 3. So assume $E$ does not lie on line $AB$. Then $\triangle EAB$ will be created. If $\triangle EAB$ is distinct from both $\triangle ABC$ and $\triangle ABD$, then we also have three distinct triangles, so assume otherwise. The only ways this can happen are enumerated in Figure 4. In Figure 4a, point $E$ creates three collinear points ($EAD$), point $E'$ creates a non-convex subconfiguration ($ACBE''$), and point $E''$ creates three collinear points ($CDE''$). Thus in any case there will be three distinct triangles. In Figure 4b, point $E'$ creates three collinear points ($CBE$) and point $E''$ also creates three collinear points ($DE''C$). Point $E$ creates a kite $ABDE$ if $AD \neq DB$, and if $AD = DB$, then $CBE$ must be collinear, so in this case also, we have three distinct triangles no matter what.

**Subcase B: non-square rectangle:** If the fifth point is added inside the rectangle, then we get either a non-convex configuration or a configuration with three collinear points (Figure 5a). So assume that the fifth point is added outside the rectangle. Using the notation of Figure 5b, to add a fifth point $E$ without creating three distinct triangles there are three potential possibilities.

1. $\triangle EAB \cong \triangle ABC$. In this case, we get three collinear points, so we have three triangles.

2. $\triangle E'AD \cong \triangle ABC$. Here, $DCE$ are collinear, so we have three triangles.
(3) $\triangle E''DC \cong \triangle E''CB \ncong \triangle ABC$. In this case, $E''DAB$ will form a kite, so we have three triangles.

(A) Possibilities for $E$ so that $\triangle EAB \cong \triangle ABC$. Any one of these choices creates a 4-point subconfiguration determining at least 3 distinct triangles.

(B) Possibilities for $E$ so that $\triangle EAB \cong \triangle ABD$. Here also, any choice creates a bad 4-point subconfiguration.

FIGURE 4. Possible additions of a fifth point when two pairs of opposite sides are congruent.

(A) Any way to place a fifth point inside a rectangle results in at least 3 distinct triangles.

(B) Any way to place a fifth point outside a rectangle also results in at least 3 distinct triangles.

FIGURE 5. Any way to add a fifth point to a rectangle results in at least 3 distinct triangles.

So we see both subcases yield at least three triangles, so the proof of case 2(d) is complete.

Proof of case 2(e) three congruent sides. Using the notation of Figure 6, if the quadrilateral $ABCD$ is not a trapezoid, then in particular $AC \neq BD$. Then we claim $\triangle ABD$, $\triangle BDC$, and $\triangle ABC$ are all distinct. Triangle $\triangle ABC \ncong \triangle ABD$ because $AC \neq BD$. 

□
If $\triangle ABC \cong \triangle BDC$, then $AB = BD$ and $CD = AC$, but this is impossible because then there would be two isosceles triangles based on $AD$.

So we can assume $ABCD$ is a trapezoid. When we add a fifth point $E$, $\triangle EAD$ is created (Figure 6). As in case 2(d)ii, we must have $\triangle EAD \cong \triangle ABD$ or $\triangle EAD \cong \triangle ACD$. Suppose $\triangle EAD \cong \triangle ABD$ (Figure 6a). In the figure, point $E$ creates a non-convex configuration $EABD$ and point $E'$ creates three collinear points $E'DC$. For point $E''$, if $E''C = AC$, then $E''DAC$ is a kite, so we have three triangles. If $E''C = BC$, then $ABCE''D$ is a regular pentagon, and this is one of our claimed optimal configurations.

Now suppose that $\triangle EAD \cong \triangle ACD$ (Figure 6b). Point $E$ in the figure makes $EACD$ either a kite, a non-convex configuration, or a configuration with three collinear points, depending on the length of $DC$. In any case, we have at least three triangles. Point $E'$ makes three collinear points $E'AB$. For point $E''$, if $E''C$ is a new distance, we have a new triangle. If $E''C = AD$, then $ADE''C$ is a non-rhombus parallelogram, so we have three triangles. If $E''C = AC$, then $DE''C$ is a new triangle. Finally, if $E''C = DC$, then $DE''C$ is also a new triangle. This shows that the only way to add a fifth point to a trapezoid configuration without generating a third triangle is to create a regular pentagon, which concludes the proof of case 2e.

**Figure 6.** Possible additions of a fifth point when three sides are congruent.

*Proof of case 2f* four congruent sides. There are two subcases: the four points either form a non-square rhombus or a square.

If the four points form a non-square rhombus, then the argument presented in case 2(d)ii for a non-rectangle parallelogram also applies to show that the addition of a fifth point anywhere generates a third triangle (see Figure 4).
If the four points form a square, we must show that the addition of a fifth point anywhere but the center results in a configuration determining at least three triangles. If the fifth point is on the interior of the square but not in the center, then it creates a non-convex configuration (Figure 7a).

If the fifth point $E$ is added outside the square, to avoid three distinct triangles, we must place it so that either $\triangle EBC \cong \triangle BCD$ or $\triangle EBC \cong \triangle EBA$ (see Figure 7b). If $\triangle EBC \cong \triangle BCD$, then $ECD$ are collinear, so there are at least three triangles. If $\triangle EBC \cong \triangle EBA$, then we have a non-convex configuration, so there are at least three distinct triangles in this case also.

This shows that the addition of a fifth point to a square anywhere but the center generates at least three distinct triangles, and this completes the proof of case 2f. □

Figure 7. Options for adding a fifth point to a square. Any choice except for the center of the square will result in a configuration with at least three distinct triangles.

4. Upper bound for $F(t)$

In this section, we prove part (3) of Theorem 1.1 by proving the following.

**Theorem 4.1.** Any set of $n$ noncollinear points in the plane determines at least $n/48$ distinct triangles.

For the sake of clarity later, we introduce a quick definition.

**Definition 4.2.** Let $S \subset \mathbb{R}^2$. We say $S$ is an $(n, k)$ set if $|S| = n$ and $k$ is the maximum number of points of $S$ that are collinear.

Now we can prove Theorem 4.1.

**Proof.** We use the following standard pigeonhole philosophy. Let $X(n, k)$ denote the minimum over all $(n, k)$ sets of the number of noncollinear triples of points. Let $Y(n, k)$
denote the maximum over all \((n,k)\) sets of the number of determined unit distances. Let \(T(n,k)\) denote the minimum over all \((n,k)\) sets of the number of distinct triangles determined by the set. Since a given triangle can only contain a given line segment at most four times, we have by the pigeonhole principle that

\[
T(n,k) \geq \frac{X(n,k)}{4 \cdot Y(n,k)}.
\] (4.1)

First we claim that \(Y(n,k) \leq \binom{n}{2} - \binom{k}{2} + (k - 1)\). This is because the \(k\) collinear points give at most \(k - 1\) unit distances, and the number of pairs of points that are not both among the \(k\) is \(\binom{n}{2} - \binom{k}{2}\).

Now we claim that \(X(n,k) \geq n(n-1)(n-k)/6\). This is because to pick a non-collinear triple, we can pick any two points to start, and there are guaranteed to be at least \(n-k\) points noncollinear with the first two.

Thus, we have that

\[
T(n,k) \geq \frac{\frac{1}{6}n(n-1)(n-k)}{\frac{1}{4}n(n-1) - \frac{1}{2}k(k-1) + (k-1)}
\]

\[
= \frac{n(n-1)(n-k)}{12n(n-1) - 12k(k-1) + 24(k-1)}.
\] (4.2)

Any set of \(n\) points is guaranteed to determine at least \(\min_{k\in[2,n-1]} T(n,k)\) distinct triangles, so we finish the proof by showing that

\[
\min_{k\in[2,n-1]} T(n,k) \geq \frac{n}{48}.
\] (4.3)

We hold \(n\) fixed and take the derivative with respect to \(k\) we get

\[
T'(n,k) = (-1) \frac{n(n-1)}{12(12n(n-1) - 12k(k-1) + 24(k-1))^2} \cdot (k^2 - 2nk + (n^2 + 2n - 2)).
\] (4.4)

This quantity is negative for all \(k \in [2,n-1]\), so \(T(n,k)\) is minimized at \(k = n - 1\).

We compute

\[
T(n,n-1) = \frac{n(n-1)}{12n(n-1) - 12(n-1)(n-2) + 24(n-2)}
\]

\[
= \frac{n(n-1)}{48n - 24 - 48}
\]

\[
\geq \frac{n(n-1)}{48n - 48} = \frac{n}{48};
\] (4.5)

and this completes the proof. \(\square\)

As a corollary, we have the following upper bound for \(F(t)\).

**Corollary 4.3.** For a given \(t\), any set of \(48(t+1)\) noncollinear points determines at least \(t+1\) distinct triangles; thus \(F(t) < 48(t+1)\).

**Remark 4.4.** It may be noticed that in computing \(Y(n,k)\) above, we have not bounded the number of unit distances very carefully. One might think that our bound can be improved significantly by counting distances more carefully, but in fact this is not the
case. To see this, consider a configuration with \( n - 1 \) collinear points, scaled so that the smallest distance is the unit distance, and the \( n \)th point placed at a unit distance from two other points (see Figure 8). There are \( n \) unit distances and \( \binom{n-1}{2} \) noncollinear triples, so the pigeonhole principle only gives us on the order of \( n \) distinct triangles. This reveals that the significant loss of information occurs not when we count the distances imprecisely, but when we reduce counting triangles to counting distances.

![Figure 8](image)

**Figure 8.** The configuration described in Remark 4.4 with \( n = 10 \).

5. CONJECTURES AND QUESTIONS FOR FURTHER STUDY

In this section, we present some conjectures and investigate their consequences.

**Conjecture 5.1.** Any set of seven points in the plane determines at least four distinct triangles; thus \( F(3) = 6 \).

In Figure 9 we see that the vertices of a regular hexagon determine exactly three distinct triangles, so we know \( F(3) \geq 6 \).

![Figure 9](image)

**Figure 9.** A regular hexagon determines three distinct triangles.

Another interesting question to ask concerns the general structure of the optimal configurations. For example, are regular polygons always optimal? What about regular polygons with their centers? In [EF], it is conjectured that optimal configurations for distinct distances are subsets of the triangular lattice. In this triangle setting, we make a different conjecture.

**Conjecture 5.2.** The regular \( n \)-gon minimizes (not necessarily uniquely) the number of distinct triangles determined by an \( n \)-point set.

Conditional on Conjecture 5.2, we are able to prove better bounds for both \( F(t) \) and the minimum number of distinct triangles determined by an \( n \) point set.
Theorem 5.3. Assuming Conjecture 5.2 any set of \( n \) noncollinear points determines at least \( n^2/12 - 1 \) distinct triangles.

Proof. We show that the vertices of a regular \( n \)-gon determine at least \( \left\lfloor \frac{n^2}{12} \right\rfloor \) distinct triangles, where \( \left\lfloor \cdot \right\rfloor \) denotes the nearest integer function. Conditional on Conjecture 5.2, this completes the proof. Label the vertices of a regular \( n \)-gon \( \{P_0, \ldots, P_{n-1}\} \). By the symmetry of the configuration, every congruence class of a triangle has a member with \( P_0 \) as a vertex, so when counting triangles we can just count triangles incident on \( P_0 \).

To form a triangle, we just have to pick two other vertices, \( P_a \) and \( P_b \), and we can assume \( a < b \). By symmetry, \( \triangle P_0 P_a P_b \) will be distinct from \( \triangle P_0 P_a' P_b' \) if and only if \( \{a-0, b-a, n-b\} \) and \( \{a'-0, b'-a', n-b'\} \) are not the same set (see Figure 10). Thus there is a bijection between distinct triangles determined by the regular \( n \)-gon and ways to write \( n \) as a sum of three positive integers. Using a result from the theory of integer partitions (see [Ho]), this quantity is equal to \( \left\lfloor \frac{n^2}{12} \right\rfloor \), so this completes the proof. A self-contained proof that this quantity is asymptotic to \( n^2/12 \) is also given in Appendix A.

Remark 5.4. If true, this bound is better than the unrestricted bound of \( \Omega\left(\frac{n^{5/3}}{3}\right) \) [Sz] and the restricted bound \( \Omega\left(\frac{n^{12/7}}{\epsilon}\right) \) [GI], despite the fact that these bounds count degenerate triangles and ours does not.

We can also phrase Theorem 5.3 in terms of our function \( F(t) \).

Corollary 5.5. Assuming Conjecture 5.2 we have \( F(t) \leq \left\lceil \sqrt{12(t+1)} \right\rceil \).

Proof. Fix \( t \) and let \( n_t = \left\lceil \sqrt{12(t+1)} \right\rceil \). The regular \( n_t \)-gon will determine at least \( t+1 \) triangles, so by Conjecture 5.2 any set of \( n_t \) points will determine at least \( t+1 \) triangles. Thus \( F(t) < n_t \). □

6. Proof of Lemma 1.5

Proof of case 1: not in convex position. In this case, the four points form a triangle with one point in the interior (Figure 11). Triangle \( \triangle ABD \) is contained in \( \triangle ABC \), so they must be distinct. □

Proof of case 2a: three collinear points. Say point \( C \) lies on \( AB \) and \( D \) does not (Figure 12). Then \( \triangle ACD \) is contained in \( \triangle ABD \), so they are distinct. □

Proof of case 2b: no congruent sides. Say the four points form quadrilateral \( ABCD \) (Figure 13). We have \( \triangle ABD \not\sim \triangle CBD \) because \( AB, AD, BC, \) and \( CD \) are all distinct. We claim \( \triangle ABC \) is distinct from both of these. Triangle \( \triangle ABC \) shares \( AB \) with \( \triangle ABD \), and \( BC \not\sim AD \), so if they are congruent then we must have \( BC = BD \) and \( AC = AD \). This is impossible because then \( \triangle CBD \) and \( \triangle CAD \) would both be isosceles triangles with \( CD \) as base, which is impossible unless one contains the other, which is not the case here. Thus \( \triangle ABC \not\sim \triangle ABD \). A similar argument shows that \( \triangle ABC \not\sim \triangle CBD \), so we have three distinct triangles. □
**Figure 10.** Illustrating the bijection described in the proof of Theorem 5.3 with \( n = 9 \). Note that triangles \( \triangle P_0P_4P_7 \) and \( \triangle P_0P_3P_5 \) represent the same partition of 9 \( \{4 - 0, 7 - 4, 9 - 7\} = \{3 - 0, 5 - 3, 9 - 5\} = \{4, 3, 2\} \). Thus they are congruent; however, \( \triangle P_0P_6P_8 \) represents a different partition \( \{6 - 0, 8 - 6, 9 - 8\} = \{6, 2, 1\} \), so it is a different triangle.

**Figure 11.** Four points not in convex position; \( \triangle ABC \) and \( \triangle ABD \) are distinct.

**Figure 12.** Four points containing three collinear points; \( \triangle ACD \) and \( \triangle ABD \) are distinct.

**Proof of case 2(c)i:** one pair of adjacent congruent sides. Let the points form quadrilateral \( ABCD \) and suppose \( AB = AD \) (Figure 14). Triangle \( \triangle ABD \not\sim \triangle BCD \) because \( \triangle ABD \) is isosceles but \( \triangle BCD \) is not. Also, by the same argument as in part 2b, we see that \( \triangle ABC \) is distinct from both of these, so there are at least three distinct triangles. \( \square \)

**Proof of case 2(c)ii:** one pair of opposite congruent sides. Suppose \( AB = CD \) (Figure 15). Triangle \( \triangle ABC \not\sim \triangle DBC \) because they have two sides congruent to each other and the third is not. We now claim that \( \triangle ACD \) is distinct from both of these.
Triangle $\triangle ACD \not\sim \triangle BCD$ by the same isosceles triangle argument from parts 2b and 2(c). If $\triangle ACD \cong \triangle ABC$, then $BC$ must equal $AD$. But that would force $AB$ to be parallel to $CD$, which would force $AC = BD$, a contradiction. Thus there are at least three distinct triangles. □

Proof of case 2(d): two pairs of adjacent congruent sides. Say $AB = AD$ and $BC = CD$ and assume without loss of generality that $AC > BD$ (Figure 16). Triangle $\triangle ABD \not\sim \triangle BCD$ because $AB \neq BC$. We claim that there is another triangle distinct from both of these. First note that it is impossible to have both $AC = CD = BC$ and $BD = AD = AB$. Because of this, the triangles $\triangle ABD$, $\triangle BCD$, and $\triangle ACD$ are necessarily distinct, so there are at least three distinct triangles. □

Proof of case 2(e): three congruent sides. Say $AD = AB = BC$ (Figure 17). Triangle $\triangle ABC \not\sim \triangle ADC$ because they have two sides congruent with each other and one side not congruent, thus there are at least two distinct triangles. □
APPENDIX A. NUMBER OF DISTINCT TRIANGLES DETERMINED BY A REGULAR $n$-GON

We give a self-contained proof that the number of distinct triangles determined by a regular $n$-gon is asymptotic to $n^2/12$. In the proof of Theorem \[5.3\] we establish that this is equal to the number of ways to write $n$ as a sum of three positive integers. Denote this quantity by $p(n, 3)$. Since the order of a partition doesn’t matter, we view this quantity as the number of ways to pick two elements $k < l$ from $\{1, \ldots, n\}$ such that $k \geq l - k \geq n - l > 0$. Note that $k$ can be any of the elements $[n/3], \ldots, n - 2$. Once $k$ is chosen, $l$ can be any of the elements $k + [(n - k)/2], \ldots, \min(2k, n - 1)$. Note $2k$ is the minimum when $k \leq \lceil n/2 \rceil$, and $n - 1$ is the minimum otherwise. Thus the
number of choices is given by

\[
p(n, 3) = \sum_{k=\lceil n/3 \rceil}^{\lfloor n/2 \rfloor} \sum_{l=k+(n-k)/2}^{2k} 1 + \sum_{k=\lfloor n/2 \rfloor+1}^{n-2} \sum_{l=k+(n-k)/2}^{n-1} 1
\]

\[
= \sum_{k=\lfloor n/3 \rfloor}^{n/2} \sum_{l=k+(n-k)/2}^{2k} 1 + \sum_{k=\lfloor n/2 \rfloor+1}^{n-2} \sum_{l=k+(n-k)/2}^{n-1} 1 + O(n)
\]

\[
= \sum_{k=\lfloor n/3 \rfloor}^{n/2} (3k/2 - n/2 + 1) + \sum_{k=\lfloor n/2 \rfloor+1}^{n-2} (n/2 - k/2) + O(n)
\]

\[
= \frac{3}{4} \left( \frac{n^2}{4} - \frac{n^2}{9} \right) - \frac{n^2}{12} + \frac{n^2}{4} - \frac{1}{4} \left( \frac{n^2}{4} - \frac{n^2}{4} \right) + O(n)
\]

\[
= \frac{n^2}{12} + O(n),
\]

(A.1)

and this completes the proof. □

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