TINKERING WITH LATTICES: A NEW TAKE ON THE ERDOS DISTANCE PROBLEM

ELZBIETA BOLDYRIEW, ELENA KIM, STEVEN J. MILLER, EYVINDUR PALSSON, SEAN SOVINE, FERNANDO TREJOS SUÁREZ, JASON ZHAO

Abstract. The Erdős distance problem concerns the least number of distinct distances that can be determined by N points in the plane. The integer lattice with N points is known as *near-optimal*, as it spans around $O(N/\sqrt{\log(N)})$ distinct distances which is the lower bound for a set of N points (Erdős, 1946). The only previous non-asymptotic work relating to the Erdős distance problem that has been done was carried out for $N \leq 13$. We take a new non-asymptotic approach to this problem, studying the distance distribution, or in other words, the plot of frequencies of each distance of the $N \times N$ integer lattice. In order to fully characterize this distribution and determine its most common and least common distances, we adapt previous number-theoretic results from Fermat and Erdős, in order to relate the frequency of a given distance on the lattice to the sum-of-squares formula, which determines the number of ways in which a positive integer may be written as the sum of two squares.

In order to apply our work on the lattice to the distance problem, we study the distance distributions of all its possible subsets; although this is a restricted case, we find that the structure of the integer lattice allows for the existence of subsets which can be chosen so that their distance distributions have certain properties, such as emulating the distribution of randomly distributed sets of points for certain small subsets, or that of the larger lattice itself. We define an error which compares the distance distribution of a subset with that of the full lattice. The structure of the integer lattice allows us to take subsets with certain geometric properties in order to maximize error, by exploiting the potential for sub-structure in the integer lattice. We show these geometric constructions explicitly; further, we calculate explicit upper bounds for the error for when the number of points in the subset is 4, 5, 9 or $\lceil N^2/2 \rceil$ and prove a lower bound for more general numbers of points.

Contents

Date: September 26, 2020.

1. Introduction

In 1946, Paul Erdős proposed the now famous Erdős distinct distance problem: Given n points in a plane, what is the minimum number of distinct distances, $f(n)$, they can determine? He accompanied this question with the first bounds on $f(n)$,

$$\sqrt{n-\frac{3}{4}}-\frac{1}{2} \leq f(n) \leq \frac{cn}{\sqrt{\log n}},$$

and further conjectured that the upper bound was tight—to this day, nobody has found evidence to contradict this conjecture. However, since 1946, incidence theory and algebraic geometry have provided a series of improvements on the original lower bound, culminating with Guth and Katz's seminal result in 2015, which proved a lower bound of $\Omega(n/\log n)$.

Since Erdős's original upper bound, coming from an estimate for the number of distinct distances on the $\sqrt{n} \times \sqrt{n}$ integer lattice, has not been improved upon to this day, any set with $O(n/\sqrt{\log n})$ distinct distances is known as *near-optimal*. Erdős further conjectured in 1996 that any near-optimal set would have lattice structure, although the truth of this conjecture remains an open problem for large values of n.

In addition to the Erdős distinct distance problem, a significant amount of work has been published on related problems which analyze aspects of distributions of distinct distances on planar point sets. The unit distance problem, for instance, focuses on the number of times a single given distance—often, the unit distance—can appear in a planar set of n points. However, most of the work done on these subjects has been asymptotic, and previous non-asymptotic work was carried out for $n \leq 13$.

In this paper we take a novel approach and examine the whole distance distribution for the lattice and its subsets in a non-asymptotic setting. Although working in $\mathbb{Z}^2$ is a simplification, our work is complicated by considering its whole distance distribution—namely, taking into account the frequency with which each distance appears on the lattice—rather than working asymptotically with only the number of distinct distances. In particular, we first examine the distance distribution for the lattice, characterizing its behavior and applying number-theoretic methods to determine an upper bound for the frequency of its most common distance. Our work results in a value that matches previous work on the Erdős unit distance problem. We then turn to the distance distributions of subsets of the lattice and compare them to the distance distribution of the lattice itself. Although they are subsets of a highly regular set, the behavior of the distance distributions for these sets can vary widely. Some subsets have distance distributions that highly mimic that of the full lattice, while others have distance distributions that are similar to that of a random set. We devise an error that measures how similar or different a subset's distance distribution is from that of the lattice itself; the details of this error are given in the following section.

For the upper bounds, we were able to find specific configurations of p that maximize the error and calculate their error. For the lower bounds, as there was less of a discernible pattern to the subsets that maximize error, we take a more

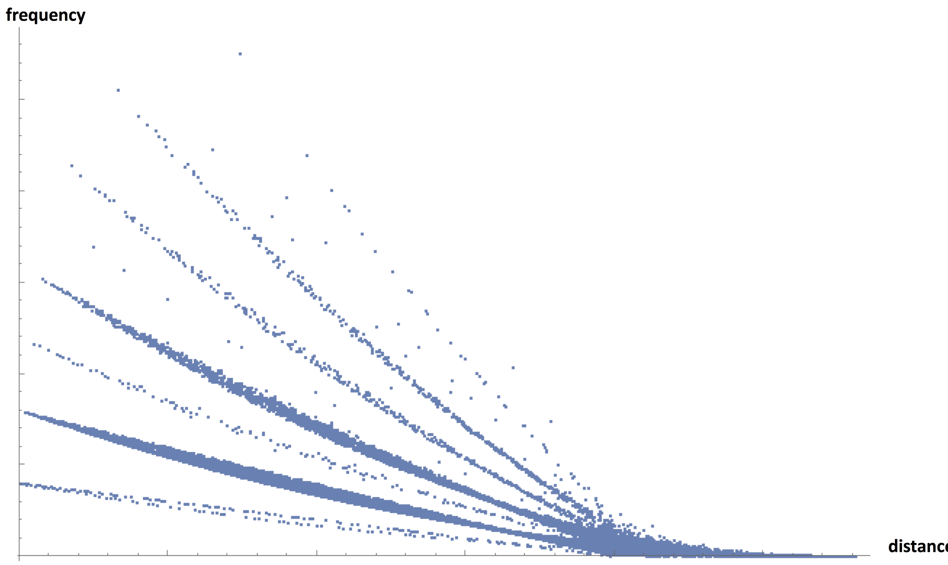

FIGURE 1. The distance distribution for the 200×200 lattice.

theoretical approach and construct theoretical optimal distance distributions, ones that cannot necessarily be realized by an actual subset of the lattice. We then bound their error from below to create a lower bound.

Thus in this paper we seek to highlight preliminary results on this new perspective on the Erdős distinct distances problem. We first begin with some definitions.

2. Introducing Distance Distributions

Throughout this section, we characterize the distance distribution of the $N \times N$ integer lattice, as seen in Figure 1.

Definition 2.1. For a fixed N, we denote by $\mathcal{L}_N \subset \mathbb{Z}^2$ the $N \times N$ integer lattice, where

$$\mathcal{L}_N = \{(x, y) \in \mathbb{Z}^2 \mid 0 \leq x \leq N-1,\ 0 \leq y \leq N-1\}. \tag{2.1}$$

Definition 2.2. For any $d \geq 1$, denote by $L_{\sqrt{d}}$ the number of times that the distance $\sqrt{d}$ appears on $\mathcal{L}_N$.

Definition 2.3. For any $d \geq 1$, denote by $S_{\sqrt{d}}$ the number of times that the distance $\sqrt{d}$ appears in a subset $S \subseteq \mathcal{L}_N$.

Lemma 2.4. For any $d \geq 1$,

$$L_{\sqrt{d}} = \sum_{\substack{a^2+b^2=d \\ a \geq 1,\ b \geq 0}} 2\,(N-a)\,(N-b)\,, \tag{2.2}$$

where the sum is taken over all ordered pairs of integers (a, b) with $a \geq 1,\ b \geq 0$ which satisfy $a^2 + b^2 = d$.

Proof. We first note that each occurrence of the distance $\sqrt{d}$ can be related uniquely to a pair of integers $(w-y)$ and $(x-z)$ whose sum of squares is precisely d.

Let (a,b) be such an ordered pair, and suppose both $a,b>0$. Notice that for any point (w,x) on the integer lattice, $(w+a,x+b)$ is in $\mathcal{L}_N$ if and only if $0\le w\le N-a-1$, $0\le x\le N-b-1$; hence there are $(N-a)(N-b)$ such points (w,x). Similarly, for a point (w,x) on the integer lattice, $(w+a,x-b)$ is in $\mathcal{L}_N$ if and only if $0\le w\le N-a-1$, $b\le x\le N-1$, and so there are $(N-a)(N-b)$ such points (w,x). Adding these values, there are precisely $2(N-a)(N-b)$ pairs of points separated by an x-distance of a and a y-distance of b.

In the event where $a\neq b>0$, we can count the ordered pairs $(a,b),(b,a)$ separately. This gives a total of $4(N-a)(N-b)$ distances characterized by the (un-ordered) pair $a,b>0$. Otherwise, if $a=b$, the two are synonymous, and we have $2(N-a)(N-b)$ such distances.

In the case where $a>b=0$, since we assumed that $a=0$, we only have the ordered pair (a,b). This gives a total of $2N(N-a)$ distances.

As this accounts for any possible occurrence of $\sqrt{d}$ on $\mathcal{L}_N$, we are done. As a final check, we note that the well-known identity

$$\sum_{a=1}^{N-1}\sum_{b=0}^{N-1}2(N-a)(N-b)=\frac{N^2\left(N^2-1\right)}{2} \tag{2.3}$$

confirms that we have counted all $\binom{N^2}{2}$ distances on the lattice. □

As seen in Figure 1, the frequencies of distances are arranged in distinct curves. Clearly, which curve $L_{\sqrt{d}}$ falls on is closely tied to the distinct ways it can be written as the sum of two squares; if d has m representations as the sum of two squares, then $L_{\sqrt{d}}$ falls on the mth highest curve. In fact, this is a subject which has been studied in some detail, which we summarize below.

Definition 2.5. For any $n\in\mathbb{Z}$, let $r_2(n)$ be the number of ordered pairs $(a,b)\in\mathbb{Z}^2$ such that $a^2+b^2=n$.

We state the following classical result due to Fermat without proof:

Theorem 2.6 (Fermat)**.** If d is a positive integer with prime factorization $d=2^f p_1^{g_1}\cdots p_m^{g_m} q_1^{h_1}\cdots q_n^{h_n}$, where for any i, $p_i\equiv 1 \pmod 4$ and $q_i\equiv 3 \pmod 4$, then

$$r_2(n)=\begin{cases}4\left(g_1+1\right)\cdots\left(g_m+1\right) & \text{all the } h_i \text{ are even}\\ 0 & \text{otherwise.}\end{cases} \tag{2.4}$$

Remark 1. The pairs $(a,b)\in\mathbb{Z}^2$ which are counted in the above sum may be negative; this contradicts our original condition for ordered pairs (a,b), as in Lemma 2.4, for which we only required $a\ge 1$, $b\ge 0$. This is intentional, as both quantities are calculated through different methods, and are used for different purposes. In particular, because of the way these are counted, we see that for a distance $\sqrt{d}$ on the m-th curve, $r_2(d)=4m$.

We use these resuls to find the most common distance on the integer lattice, which proves to be useful later.

Definition 2.7. For any $k \geq 1$, let $\sqrt{n_k}$ be the smallest distance on the k-th curve, i.e., n_k is the smallest positive integer such that $r_2(n_k) = 4k$.

Recalling the original lattice distance distribution as seen in Figure 1, we note that the most common distance on each individual curve is the leftmost, i.e., the smallest. This can be proven rigorously using methods similar to those of Fermat; in particular, it may be shown that the frequency of distances on each curve is strictly decreasing as the distance grows larger.

In our search for the most common distance on the lattice, we thus may narrow our focus to the set of integers $n_1, n_2, \ldots$. We determine the explicit size of the integers in this sequence.

Lemma 2.8. Let $5 = p_1 < p_2 < \cdots$ be the primes satisfying $p_i \equiv 1 \pmod 4$, listed in increasing order. Suppose $k = q_1^{a_1} \cdots q_m^{a_m}$, where $q_1, \ldots, q_n$ are any m distinct primes, and $q_1 > q_2 > \cdots > q_m$. Then n_k is precisely

$$\Bigg(\underbrace{p_1 \cdots p_{a_1}}_{a_1 \text{ primes}}\Bigg)^{q_1 - 1} \Bigg(\underbrace{p_{a_1+1} \cdots p_{a_1+a_2}}_{a_2 \text{ primes}}\Bigg)^{q_2 - 1} \cdots \Bigg(\underbrace{p_{a_1+\cdots+a_{m-1}+1} \cdots p_{a_1+\cdots+p_{a_1+\cdots+a_m}}}_{a_m \text{ primes}}\Bigg)^{q_m - 1} \tag{2.5}$$

Proof. Omitted for brevity. □

Note that the sequence $n_1, n_2, \ldots$ is not strictly increasing. Evaluating some specific values, we see that for $k = 2^m$, $n_k = p_1 \cdots p_m$; additionally, for k prime, $n_k = 5^{k-1}$. These may easily be seen to be the extremal values for n_k, from which we have the following corollary.

Corollary 2.9. For any $k \geq 1$,

$$\prod_{i=1}^{\lfloor \log_2(k) \rfloor} p_i \leq n_k \leq 5^{k-1}. \tag{2.6}$$

We wish to find a more explicit lower bound for n_k, which is equivalent to estimating the product of the first k primes $p_1 < \cdots < p_k$ which are congruent to $1 \pmod 4$. While this quantity has never before been studied, we may adapt existing work on bounding the product of the first k primes $q_1, \ldots, q_k$ of any class (mod 4), denoted $q_k\#$.

The best general estimate is

$$q_k\# = \prod_{i=1}^{k} q_i = e^{(1+o(1))k \log k}. \tag{2.7}$$

We can then approximate our quantity as

$$\prod_{i=1}^{k} p_i = \left(\prod_{i=1}^{2k} q_i\right)^{1/2} = e^{\frac{1}{2}(1+o(1))2k \log 2k} = e^{(1+o(1))k \log 2k}. \tag{2.8}$$

Given the equal spread of the primes in any arithmetic progression, we know this must converge towards the real quantity; to determine the speed at which this occurs is a more complex problem, and is outside the scope of this paper.

Using this quantity, we have a general lower bound for n_k, namely

$$n_k \gg e^{\frac{1}{2}(1+o(1))\log_2(2k)\log\log_2(2k)}. \tag{2.9}$$

Now, given that a distance $\sqrt{d}$ is on the k-th curve of the distance distribution, can maximize each summand in Lemma 2.4 to find the upper bound

$$L_{\sqrt{d}} \leq 2kN\left(N - \sqrt{d}\right). \tag{2.10}$$

As this assumes that all k pairs of integers (a, b) with $a^2 + b^2 = d$ are identically $(\sqrt{d}, 0)$, we know that for $k > 1$, it is indeed a strict upper bound for this quantity.

Putting these pieces together, we can determine

$$L_{\sqrt{n_k}} \ll 2kN\left(N - e^{\frac{1}{4}(1+o(1))\log_2(2k)\log\log_2(2k)}\right). \tag{2.11}$$

In particular, for large N, maximizing this quantity in terms of k gives us a strict upper bound for the most common distance on the $N \times N$ lattice. Interestingly, taken in terms of N, this maximum quantity is $\mathcal{O}(N^2)$; this agrees with existing work on the Erdős unit distance problem, and thus tells us that the frequency of the most common distance of the lattice follows the same behavior as the most common distance on any set of N^2 points.

3. Error Estimates

With an understanding of the distance distribution for the lattice, one then may ask about the behavior of the distance distributions for subsets of the lattice. Although we know that the lattice is a near-optimal set, as previously discussed, the behavior of the distance distributions of its subsets can vary widely. Thus, we aim to examine how different and similar the distance distributions for subsets of the lattice can be to that of the lattice, an analogous version of the Erdős distinct distance problem on subsets of the lattice.

We now define our method of calculating the difference between the lattice's distance distribution and one of its subset. We call this difference the error, ε.

Recall that $L_{\sqrt{d}}$ is the frequency of a distance $\sqrt{d}$ in the lattice and $S_{\sqrt{d}}$ is its frequency in a particular subset S. We note that the $N \times N$ lattice has $N^2(N^2 - 1)/2 \approx N^4/2$ total distances and a subset with p points has $p(p-1)/2 \approx p^2/2$ total distances. Thus we scale up each $S_{\sqrt{d}}$ by N^4/p^2 to have a distance distribution with about the same total number of frequencies as that of the lattice. Finally, we sum $\left|(N^4/p^2)S_{\sqrt{d}} - L_{\sqrt{d}}\right|$ over all d

More explicitly,

$$\varepsilon = \sum_{d=1}^{\sqrt{2}N} \left|\frac{N^4}{p^2} S_{\sqrt{d}} - L_{\sqrt{d}}\right|.$$

In many of our calculations, instead of working with the actual distances, we work with individual pairs (a, b) as sole representatives of the distance $\sqrt{a^2 + b^2}$.

This gives exact values for the contribution to the error of a distance $\sqrt{d}$ only for distances on the first and second curves; in all other cases, this is a simplification. Thus we introduce some new notation. Let $L_{a,b}$ denote the number of times $\sqrt{a^2+b^2}$ appears in the full lattice and let $S_{a,b}$ denote the number of times $\sqrt{a^2+b^2}$ appears in the the subset of the lattice. Finally, let $\varepsilon_{a,b} = |(N^4/p^2)S_{a,b} - L_{a,b}|$. It may be shown that counting repeat distances as distinct either strictly increases error, or has no effect on it. If $\sqrt{a^2+b^2} = \sqrt{c^2+d^2}$ for $\{a,b\} \neq \{c,d\}$, we then see that the total contribution to error for this distance is $|\frac{N^2}{p^2}S_{a,b} - L_{a,b} + \frac{N^2}{p^2}S_{c,d} - L_{c,d}|$, whereas counting them as distinct gives a total contribution to error $|\frac{N^2}{p^2}S_{a,b} - L_{a,b}| + |\frac{N^2}{p^2}S_{c,d} - L_{c,d}|$. Counting each pair as distinct thus gives an upper bound for total error. Furthermore, these two quantities are identical if and only if $\frac{N^2}{p^2}S_{a,b} - L_{a,b}$ and $\frac{N^2}{p^2}S_{c,d} - L_{c,d}$ have the same sign, which we suspect is true of most subsets which maximize error.

Recall from the previous calculations we made on the frequency of distances on the lattice, we know that when $b = 0$ or $a = b$, $L_{a,b} = 2(N-a)(N-b)$. Otherwise, $L_{a,b} = 4(N-a)(N-b)$. For later error calculations, we need the average values of $2(N-a)(N-b)$ and $4(N-a)(N-b)$. The average value of $2(N-a)(N-b)$ is $(N(5N-1))/6$ and the average value of $4(N-a)(N-b)$ is $N(3N-1)/3$.

Additionally, we have calculated that the fraction of $L_{a,b}$ that are of the form $2(N-a)(N-b)$ to be $4/N+2$ and the fraction of $L_{a,b}$ that are of the form $4(N-a)(N-b)$ to be $N-2/N+2$.

Similarly to the original Erdős distinct distances problem, we are interested in finding upper and lower bounds for the behavior we are studying. For the upper bounds, we are able to find patterns of configurations that maximize the error. We can then calculate the error for these specific configurations. For the lower bound, we construct a theoretical optimal distance distributions and calculate a lower bound on their error.

4. Upper Bounds

A configuration of p points that maximizes error needs to have as different a distance distribution as possible from the original full lattice when scaled up. Thus, the ratio of each distance's frequency to the total number of distances, including repeated distances, must be as different as possible from that in full lattice. Specifically, we want to have a subset that has many distances that were infrequent in the full lattice and minimal number of distances that were very frequent in the full lattice.

For certain values of p, we know the configuration that maximizes error. See Figures 2, 3, 4, 5, 6, 7.

For $p = 4$, the maximal error subset is all four corners of the lattice. To transition to $p = 5$, the middle point is added in. For $p = 9$, the maximal subset is a 3×3 lattice stretched to the size of the $N\times N$ lattice. To transition from $p = 5$ to $p = 9$, the points that are in $p = 9$ but not in $p = 5$ are added in one by one. We then know for $p = 4(N-1)$, the maximal error subset is the perimeter of the lattice, with no other points. For $p = \sum_{i=1}^{m} 4(N-(2i-1))$, the maximal error subset is

FIGURE 2. $p = 4$.

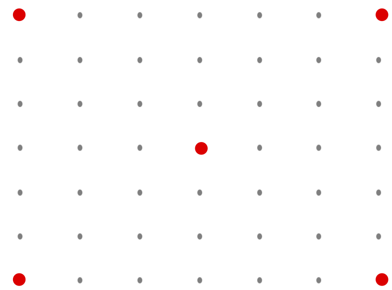

FIGURE 3. $p = 5$.

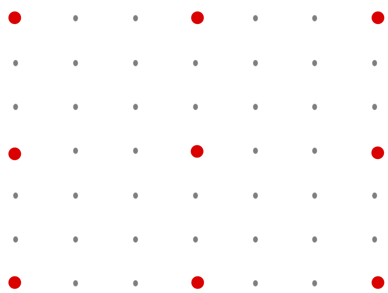

FIGURE 4. $p = 9$.

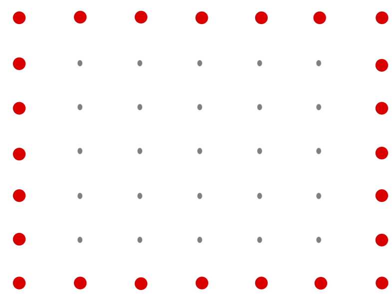

FIGURE 5. $p = 4(N-1)$.

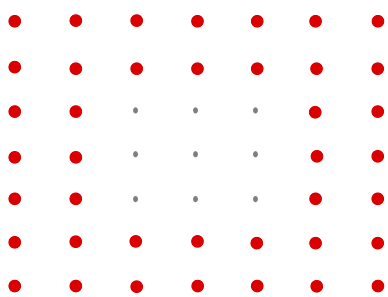

FIGURE 6. $p = 4(N-1) + 4(N-3)$.

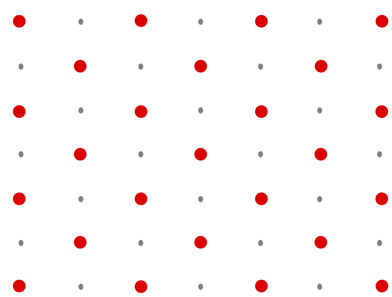

FIGURE 7. $p = \left\lceil \frac{N^2}{2} \right\rceil$.

the filled-in perimeter with a depth of i points. For example, Figure 6 is a filled-in perimeter with a depth of 2 points. To transition from $p = \sum_{i=1}^{m} 4(N-(2i-1))$ to $p = \sum_{i=1}^{m+1} 4(N-(2i-1))$, the points only present in the latter configuration are filled-in one by one. The final maximal error configuration we found is for $p = \lceil N^2/2 \rceil$, where every other point is filled-in in a configuration we refer to as a checkerboard lattice. One can note that depending on the value of N, $\lceil N^2/2 \rceil$, is less than $p = \sum_{i=1}^{m} 4(N-(2i-1))$ for different values of m. As a result, there is a transition between these two types of configuration and then a transition back.

We then calculate error estimates for some of these configurations. For these calculations, we use the simplification of working with $\sqrt{a^2+b^2}$ where $0 \leq b \leq N-1$ and $b \leq a \leq N-1$, excluding $a = b = 0$, instead of distinct distances.

We begin with $p = 4$, where the maximal error configuration is a point in each of the corners of the lattice. We first note that the scaling constant is $N^4/p^2 = N^4/16$. We also have just two distinct distances: $\sqrt{2}(N-1) = \sqrt{(N-1)^2 + (N-1)^2}$ which has $L_{N-1,N-1} = 2$ and $S_{N-1,N-1} = 2$ and $N - 1 = \sqrt{(N-1)^2 + 0^2}$ which has $L_{N-1,0} = 2N$ and $S_{N-1,0} = 4$ in the new configuration. To find $\varepsilon_{N-1,N-1}$ and $\varepsilon_{N-1,0}$, we have to scale $S_{N-1,N-1}$ and $S_{N-1,0}$ by $N^4/16$ and subtract $L_{N-1,N-1}$ and $L_{N-1,0}$, respectively. This is $N^4/8 - 2$ and $N^4/4 - 2N$. Thus the total error contribution from these two distances is $3N^4/8 - 2 - 2N$.

Both $L_{N-1,N-1}$ and $L_{N-1,0}$ are of the form $2(N-a)(N-b)$, so we have to update the average value of $2(N-a)(N-b)$ and the fraction of the time $L_{a,b}$ is of the form $2(N-a)(N-b)$, to exclude $L_{N-1,N-1}$ and $L_{N-1,0}$. The new average is $(5N^2 + 4N + 3)/6$ and the new fraction is $(4N - 8)/(N^2 + N - 2)$. The fraction of $L_{a,b}$ that are $L_{N-1,N-1}$ and $L_{N-1,0}$ is $(4)/(N^2 + N - 2)$. We note that for $\{a, b\} \neq \{N-1, N-1\}$ and $\neq \{N-1, 0\}$, $S_{a,b} = 0$. Thus the average error contribution for these distances is their average frequency in the lattice.

We can then put everything together to get our error estimate:

$$\begin{aligned}
\text{Error} &= \frac{4}{N^2+N-2}\left(\frac{3N^4}{8} - 2 - 2N\right) + \frac{4N-8}{N^2+N-2}\left(\frac{5N^2+4N+3}{6}\right) \\
&\quad + \frac{N-2}{N+2}\left(\frac{N(3N-1)}{3}\right) \\
&= \frac{5N^2}{2} - \frac{5N}{2} - \frac{15}{2(N-1)} - \frac{16}{N+2} + \frac{13}{2}.
\end{aligned}$$

This error estimate is an overestimate of the error when N is small because of the fact that we are looking at $\sqrt{a^2 + b^2}$ instead of distinct distances. However, the only distances that this way of estimating the error affects is the two distances present on the lattice, $\sqrt{2}(N-1)$ and $N - 1$. However, as $N \to \infty$, the fraction of the total distances that these distances represent goes to zero. Thus, this error estimate converges to the actual error.

Similarly, we can calculate the error for when $p = 5$. Recall, for this value of p, that the subset configuration that maximizes error is the four corners and the middle point of the lattice.

This configuration has 3 distinct distances: $N - 1 = \sqrt{(N-1)^2 + 0^2}$ for which we have $S_{N-1,0} = 4$ and $L_{N-1,0} = 2N$, $\sqrt{2}(N-1) = \sqrt{(N-1)^2 + (N-1)^2}$ for which we have $S_{N-1,N-1} = 2$ and $L_{N-1,N-1} = 2$, and $\sqrt{2}(N-1)/2$ $\sqrt{((N-1)/2)^2 + ((N-1)/2)^2}$ for which we have $S_{(N-1)/2,0} = 4$ and $L_{(N-1)/2,0} = N + 1$. To calculate $\varepsilon_{N-1,0}$, $\varepsilon_{N-1,N-1}$ and $\varepsilon_{(N-1)/2,0}$, we need to multiply $S_{a,b}$ by $\frac{N^4}{25}$ and subtract $L_{a,b}$. Thus, $\varepsilon_{N-1,0} = 4N^4/25 - 2n$, $\varepsilon_{N-1,N-1} = 2N^4/25 - 2$ and $\varepsilon_{(N-1)/2,0} = 4N^4/25 - (N+1)$. $L_{N-1,0}$, $L_{N-1,N-1}$ and $L_{(N-1)/2,0}$ are of the form $2(N-a)(N-b)$, so we need to edit the average value of $2(N-a)(N-b)$ and the fraction of the time $L_{a,b}$ is of the form $2(N-a)(N-b)$ to no longer include the three distances listed above.

The new average value is $(5N^3 - 6N62 - 8N - 9)/(3(2N-5))$ and the fraction of the time La, b is of the form $2(N-a)(N-b)$ is $6/(N+2) - 2/(N-1)$. The fraction of the total distances that are the three distances listed above is $2/(N-1) - 2/(N+2)$.

We then put this all together to calculate the error:

$$\begin{aligned}\text{Error} &= \left(\frac{2}{N-1} - \frac{2}{N+2}\right)\left[\left(\frac{4N^4}{25} - 2N\right) + \left(\frac{2N^4}{25} - 2\right) + \left(\frac{4N^4}{25} - (N+1)\right)\right] \\ &+ \left(\frac{6}{N+2} - \frac{2}{N-1}\right)\left[\frac{5N^3 - 6N^2 - 8N - 9}{3(2N-5)}\right] + \frac{N-2}{N+2}\left[\frac{N(3N-1)}{3}\right] \\ &= \frac{17N^2}{5} - \frac{17N}{5} - \frac{6}{N-2} - \frac{56}{5(N-1)} - \frac{124}{5(N+2)} - \frac{31}{3(2N-5)} + \frac{113}{15}.\end{aligned}$$

Similarly to $p = 4$, this error estimate overestimates the error when N is small because of the fact that we are looking at $\sqrt{a^2 + b^2}$ instead of distinct distances. However, as $N \to \infty$, this error estimate converges to the actual error.

We can then examine what happens when $p = 9$. The 9 point configuration that maximizes error is a 3×3 lattice that has been stretched to the size of the $N \times N$ lattice.

We first note that $S_{a,b} > 0$ if and only if $a, b \equiv 0 \mod (N-1)/2$. Thus the fraction of $S_{a,b}$ such that $S_{a,b} \neq 0$ for $b \neq 0$ and $a > b$ is $(2/(N-1))^2 = 4/(N-1)^2$ and the fraction of $S_{a,b}$ such that $S_{a,b} \neq 0$ for $b = 0$ or $a = b$ is $\frac{2}{N-1}$. The scaling constant is $N^4/81$. We estimate the error from above by assuming that if $S_{a,b} \neq 0$, then $S_{a,b} = L_{a,b}$. This assumption does not increase the error estimate by an unreasonable amount because for large enough N, the fraction of total distances that are represented in this configuration is very low. We can then use our previous averages of $L_{a,b}$ to calculate error:

$$\begin{aligned}\text{Error} &< \frac{4}{N+2}\left[\frac{2}{N-1}\left(\frac{N^4}{81}\left(\frac{N(5N-1)}{6}\right) - \frac{N(5N-1)}{6}\right)\right. \\ &\quad \left. + \left(1 - \frac{2}{N-1}\right)\left(\frac{N(5N-1)}{6}\right)\right] \\ &\quad + \frac{N-2}{N+2}\left[\frac{4}{(N-1)^2}\left(\frac{N^4}{81}\left(\frac{N(3N-1)}{3}\right) - \frac{N(3N-1)}{3}\right)\right. \\ &\quad \left. + \left(1 - \frac{4}{(N-1)^2}\right)\left(\frac{N(3N-1)}{3}\right)\right] \\ &= \frac{32N^4}{243} - \frac{52N^3}{243} + \frac{4N^2}{9} - \frac{220N}{243} - \frac{23044}{2187(N-1)} - \frac{14000}{2187(N+2)} \\ &\quad - \frac{6200}{729(N-1)^2} + \frac{112}{27(N-1)^3} + \frac{32}{9(N-1)^4} + \frac{428}{243}. \qquad (4.1)\end{aligned}$$

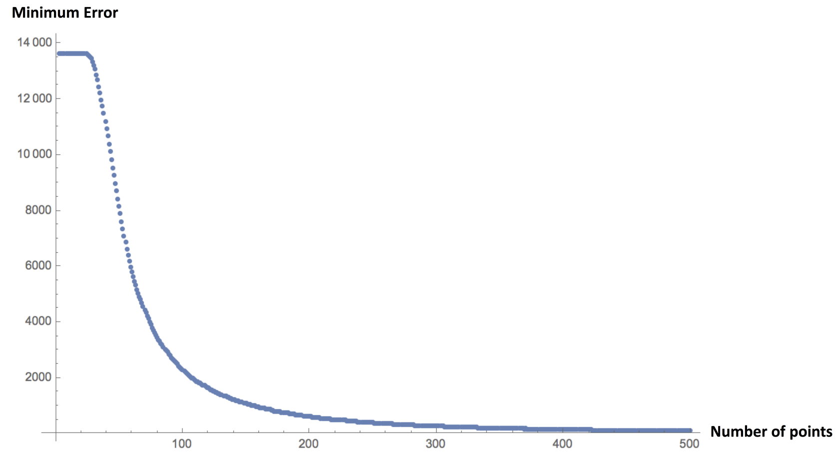

FIGURE 8. Computer generated data for the error of the optimal distance distribution in the 100×100 lattice.

Finally, we examine the error for the configuration for $p = \lceil N^2/2 \rceil$. This configuration is the "checkerboard lattice," a subset that is missing every other point from the full lattice and resembles a checkerboard, as seen in Figure 7.

To provide some intuition, the checkerboard lattice is a reasonable configuration for maximizing error because it very strongly prioritizes distances where $b = 0$ or $a = b$. These distances have $L_{a,b} = 2(N - a)(N - b)$, which tend to be smaller than other frequencies where $L_{a,b} = 4(N - a)(N - b)$. Thus this configuration has many frequencies which are not very common in the full lattice.

In Appendix A, we calculate the frequency of some distances on the checkerboard lattice explicitly.

5. Lower Bounds

To minimize error, we want to preserve the same ratio of total distances, including repeated distances, to frequency that appeared in the $N \times N$ lattice for each unique distance. So to calculate a lower bound, we create an "optimal" distribution of frequencies for p points. Clearly this optimal distribution cannot always be achieved, as not every distance distribution is realizable by a certain configuration. To come up with the optimal distribution, we scale each $L_{a,b}$ by N^4/p^2 and round this number to the nearest integer. We then find the error for this optimal distribution in the same way as before. Code can easily be written to find the optimal distance distribution and calculate the error. For example, Figure 8 demonstrates what this error is for a 100×100 lattice. Note that this figure does not include all possible values of p, rather it includes enough values to capture to behavior of the error.

We then construct a lower bound for this minimal error:

$$\text{Error} \geq \begin{cases} \binom{N^2}{2} & \text{if } p \leq \frac{\log_5(N)}{5}\left(11 - 2\sqrt{10}\right), \\ \frac{N^4}{8p^2} & \text{if } p \text{ sufficiently large.} \end{cases} \tag{5.1}$$

We begin with discussing the first part of the piecewise function:

$$\binom{N^2}{2} \text{ if } p \leq \frac{\log_5(N)}{5}\left(11 - 2\sqrt{10}\right). \tag{5.2}$$

Here, we note that $\binom{N^2}{2}$ is the precise value for error for the empty subset of the lattice, i.e., the sum of distance frequencies of the lattice itself. The idea is that the distance distribution of a small enough subset of the lattice, after rescaling, has a greater error than the empty subset, as its few distances are overrepresented as to increase overall error.

As we use a scaling factor of N^4/p^2, we notice that if p is such that

$$\frac{N^4}{p^2} > 2L_{\sqrt{d}} \tag{5.3}$$

for any distance $\sqrt{d}$ on the integer lattice, then the error of any subset of size p is strictly greater than that of the empty subset, as for any $\sqrt{d}$,

$$\left|\frac{N^4}{p^2} S_{\sqrt{d}} - L_{\sqrt{d}}\right| \geq \left|\frac{N^4}{p^2} - L_{\sqrt{n_k}}\right| \geq L_{\sqrt{n_k}},$$

where, as we previously defined, $\sqrt{n_k}$ is the most common overall distance on the $N \times N$ lattice.

Using our earlier estimates for the frequency of n_m, we may find a trivial bound for equation 2.11 in order to determine that

$$p \leq \frac{\log_5(N)}{5}\left(11 - 2\sqrt{10}\right) \tag{5.4}$$

is sufficient.

We then discuss the second part of the piecewise function:

$$\frac{N^4}{8p^2} \text{ if } p \text{ sufficiently large.} \tag{5.5}$$

Once $L_{a,b}$'s are scaled down by p^2/N^4, rounded, and scaled up by N^4/p^2, they are a multiple of N^4/p^2. That means that the largest $\varepsilon_{a,b}$ can be is $N^4/2p^2$ and the smallest $\varepsilon_{a,b}$ can be is 0. One might expect the average contribution to error to be $N^4/4p^2$. However, for small p, many frequencies in the $N \times N$ lattice are much closer to 0 than $N^4/2p^2$ as $N^4/2p^2$ is quite large. Thus the average is smaller than $N^4/4p^2$. For large enough p, we know that the average is at least larger than $N^4/8p^2$.

6. Future Work

There are several ways to improve and extend our work. We have already done some characterizations of the subsets that maximize error and how the subsets transition from one configuration to another. We know we have a checkerboard configuration when $p = \lceil N^2/2 \rceil$ and we have a filled-in perimeter when

$p = \sum_{i=1}^{m} 4(N-(2i-1))$ for different values of m. However the transition between these two configurations has still yet to be characterized.

Additionally, we hope to improve our lower bound work. Work can be done to find a characterization of the sets that minimize error. Furthermore, we hope to refine our lower bound formula by finding a rigorous lower bound the values of p where $N^4/8p^2$ holds.

Finally, as Erdős conjectured that all near-optimal sets have lattice structure, it is natural to extend results to other lattice structures. We expect that the error is similar.

Appendix A. Additional Calculations on the Lattice

In a checkerboard, $\sqrt{a^2+b^2}$ only appears as a distance if either a and b are both odd or both even. We note that is this causes about $1/2$ of $S_{a,b}$ to be zero for $a > b$ and $b \neq 0$. Additionally, we note that this causes about $1/4$ of $S_{a,b}$ to be zero for $a = b$ or $b = 0$. We use the simplifying assumption that $S_{a,b} = La, b$ if $S_{a,b} \neq 0$. This ultimately increases our error estimate. We then have the following error calculation:

$$\frac{4}{N+2}\left[\frac{3}{4}\left(4\left(\frac{N(5N-1)}{6}\right)-\frac{N(5N-1)}{6}\right)+\frac{1}{4}\left(\frac{N(5N-1)}{6}\right)\right]$$
$$+\frac{N-2}{N+2}\left[\frac{1}{2}\left(4\left(\frac{N(3N-1)}{3}\right)-\frac{N(3N-1)}{3}\right)+\frac{1}{2}\left(\frac{N(3N-1)}{3}\right)\right]$$
$$= 2N^2 - \frac{N}{3} - \frac{2}{3(N+2)} + \frac{1}{3}. \tag{A.1}$$

If wanted, we can increase the accuracy of the error estimate for the checkerboard configuration by precisely calculating $S_{a,b}$ for $S_{a,b} \neq 0$. First, we may look at distances of the form $\sqrt{a^2}$, i.e., those for which $b = 0$. We note these only appear on the checkerboard lattice when a is even.

We count the total number of times that two points on the lattice are separated by a horizontal distance of a—this number matches the number of pairs which are separated by a vertical distance of a, by rotational symmetry. In the case where N is even, each row of the lattice contains $N/2$ points, of which there are $\frac{N}{2} - \frac{a}{2}$ pairs at a distance a. We thus see that

$$S_{a,0} = 2N\left(\frac{N}{2} - \frac{a}{2}\right) = N(N-a). \tag{A.2}$$

In the case where N is odd, we assume that the checkerboard lattice is chosen in such a way that the four corners are included. In this case, we see that $\frac{N+1}{2}$ of the rows contain $\frac{N+1}{2}$ points, whereas $\frac{N-1}{2}$ rows contain $\frac{N-1}{2}$ points. Thus,

$$S_{a,0} = 2\left(\frac{N+1}{2}\left(\frac{N+1}{2} - \frac{a}{2}\right) + \frac{N-1}{2}\left(\frac{N-1}{2} - \frac{a}{2}\right)\right) = N(N-a)+1. \tag{A.3}$$

We now look at distances of the form $\sqrt{2a^2}$, i.e., those formed by values $a = b$. We assume N is odd.

In the case where N is odd, we notice that on the $N \times N$ checkerboard lattices, all diagonals at 45° contain an odd number of points. Moreover, for any $1 \leq n \leq \frac{N-1}{2}$, there are four diagonals with $2n-1$ points; additionally, there are two diagonals with N points. Finally, the distance $\sqrt{2a^2} = \sqrt{2}a$ appears on a diagonal with $2n-1$ points precisely $2n-1-a$ times; in the case where $n \leq a$, the distance is not present.

In the case where a is odd, we see $\sqrt{2}a$ is present on all diagonals with at least $a+2$ points. We have

$$\begin{aligned}
S_{a,a} &= 4\left((a+2)-a+(a+4)-a+\cdots+(N-2)-a\right)+2(N-a)\\
&= 4\left(2+4+\cdots+N-a-2\right)+2\left(N-a\right)\\
&= 8\left(1+\cdots+\frac{N-a-2}{2}\right)+2(N-a)\\
&= 8\left(\frac{\frac{N-a-2}{2}\frac{N-a}{2}}{2}\right)+2(N-a)\\
&= (N-a-2)(N-a)+2(N-a)\\
&= (N-a)^2.
\end{aligned}$$

In the case where a is even, we see $\sqrt{2}a$ is present on all diagonals with at least $a+1$ points. Thus,

$$\begin{aligned}
C_{a,a} &= 4\left((a+1)-a+(a+3)-a+\cdots+(N-2)-a\right)+2(N-a)\\
&= 4\left(1+3+\cdots+N-2-a\right)+2\left(N-a\right)\\
&= 4\left(2+4+\cdots+N-1-a\right)-4\left(\frac{N-a-1}{2}\right)+2(N-a)\\
&= 8\left(\frac{\frac{N-a-1}{2}\frac{N-a+1}{2}}{2}\right)-2(N-a-1)+2(N-a)\\
&= (N-a-1)(N-a+1)+2\\
&= (N-a)^2-1+2\\
&= (N-a)^2+1.
\end{aligned}$$

In the case where N is even, the calculation is slightly more complicated, as the only possible arrangement of $N^2/2$ points in a checkerboard pattern results in a loss of 4-fold rotational symmetry. Without loss of generality, we see that the bottom-left to top-right diagonals each contain an even number of points, whereas the top-left to bottom-right diagonals each contain an odd number of points. In particular, the cardinalities of the bottom-left to top-right diagonals are

$$2,\ 4,\ 6,\ldots,N-2,\ N,\ N-2,\ldots,6,\ 4,\ 2.$$

The top-left to bottom-right diagonals instead have cardinalities

$$1,\ 3,\ 5,\ldots,N-3,\ N-1,\ N-1,\ N-3,\ ,\ldots,5,\ 3,\ 1.$$

For a even, the distance $\sqrt{2}a$ is on any diagonal with cardinality $n > a$, with frequency $n - a$. Thus we calculate

$$\begin{aligned}
S_{a,a} &= 2 \sum_{n=a/2}^{(N-2)/2} (2n + 1 - a) + 2 \sum_{n=1+a/2}^{(N-2)/2} (2n - a) + (N - a) \\
&= 2\,(1 + 3 + \cdots + N - 1 - a) + 2\,(2 + 4 + \cdots + N - 2 - a) + N - a \\
&= 2 \left(\frac{(N - a - 1)\,(N - a)}{2} \right) + N - a \\
&= (N - a - 1)\,(N - a) + N - a \\
&= (N - a)^2 .
\end{aligned}$$

For a odd, the calculation is nearly identical, although $a + 1$ becomes the first even diagonal cardinality on which a distance a appears, and $a + 2$ the first odd diagonal cardinality. Hence we see

$$\begin{aligned}
S_{a,a} &= 2 \sum_{n=(a+1)/2}^{(N-2)/2} (2n + 1 - a) + 2 \sum_{n=(a+1)/2}^{(N-2)/2} (2n - a) + (N - a) \\
&= 2\,(2 + 4 + \cdots + N - 1 - a) + 2\,(1 + 3 + \cdots + N - 2 - a) + N - a \\
&= 2 \left(\frac{(N - a - 1)\,(N - a)}{2} \right) + N - a \\
&= (N - a - 1)\,(N - a) + N - a \\
&= (N - a)^2 .
\end{aligned}$$

References

[BMO] D. Brink, P. Moree, R. Osburn, *Principal Forms $X^2 + nY^2$ Representing Many Integers*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg (March, 2010), 129-139

[HP] M. Herman, J. Parkianathan, *A note on the unit distance problem for planar configurations with $\mathbb{Q}$-independent direction set*, Turkish Journal of Mathematics **39** (June 2014)

[Sh] P. Erdős, P. Fishburn, *Discrete Mathematics* **160** (1996) 115-125

[Sh] A. Sheffer, *Distinct Distances: Open Problems and Current Bounds*, preprint 2014. https://arxiv.org/abs/1406.1949

[Na] U. P. Nair, *Elementary Results on the Binary Quadratic Form $a^2 + ab + b^2$*, preprint 2004, https://arxiv.org/abs/math/0408107

[Er] P. Erdős, *On Sets of Distances of n Points*, American Mathematical Monthly, **53** (May, 1946), 248-250

[Co] K. Conrad, *Sums of Two Squares and Lattices*, https://kconrad.math.uconn.edu/blurbs/ugradnumthy/Picksumofsq.pdf