OPTIMAL POINT SETS DETERMINING FEW DISTINCT ANGLES

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ABSTRACT. We characterize the largest point sets in the plane which define at most 1, 2, and 3 angles. For $P(k)$ the largest size of a point set admitting at most k angles, we prove $P(2) = 5$ and $P(3) = 5$. We also provide the general bounds of $k + 2 \leq P(k) \leq 6k$, although the upper bound may be improved pending progress toward the Weak Dirac Conjecture. Notably, it is surprising that $P(k) = \Theta(k)$ since, in the distance setting, the best known upper bound on the analogous quantity is quadratic and no lower bound is well-understood.

CONTENTS

1. INTRODUCTION

1.1. **Background.** In 1946, Erdős introduced the problem of finding asymptotic bounds on the minimum number of distinct distances among sets of n points in the plane [Er]. The Erdős distance problem, as it has become known, proved infamously difficult and was only finally (essentially) resolved by Guth and Katz in 2015 [GuKa].

The Erdős distance problem has also spawned a wide variety of related questions, including the problem of finding maximal point sets with at most k distinct distances. Characterizing the largest possible point sets satisfying a given property in this way is a classic problem in discrete geometry. As another example, Erdős introduced the problem of finding maximal point sets of all isosceles triangles in 1947 [ErKe]. Ionin

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completely answers this question in Euclidean space of dimension at most 7 [Io]. Erdős and Fishburn determine maximal planar sets with at most k distinct distances [ErFi]. Recent results by Szöllősi and Östergård classify the maximal 3-distance sets in $\mathbb{R}^4$, 4-distance sets in $\mathbb{R}^3$, and 6-distance sets in $\mathbb{R}^2$ [SzOs]. In [ELMP, BrDePaSe, BrDePaSt] point sets with a low number of distinct triangles in Euclidean space are investigated.

Along these lines, we consider the related problem of maximal planar point sets admitting at most k distinct angles in $(0, \pi)$. We ignore angles of 0 and π so as to align our convention with related research (see [PaSha92], for example), although we provide results including the 0 angle as corollaries. We completely answer this question for $k = 2$, and $k = 3$ and provide asymptotically tight linear bounds for $k > 3$. In answering this question for $k = 2$ and $k = 3$, we consider systematically consider all possible triangles in such configurations and then reduce to adding points in a finite number of positions by geometric casework. We provide linear asymptotic bounds using bounds on the related problem of the minimum number of distinct angles among n non-collinear points in the plane.

1.2. **Definitions and Results.** By convention, we only count angles of magnitude strictly between 0 and π. Our computations still answer the related optimal point configuration questions including 0 angles (see Corollaries 3.1, 4.1). We begin by introducing convenient notation:

Definition 1.1. *Let $\mathcal{P} \subset \mathbb{R}^2$. Then*

$$A(\mathcal{P}) := \#\{|\angle abc| \in (0, \pi) \,:\, a, b, c \textit{ distinct}, \, a, b, c \in \mathcal{P}\},$$

Now we define the quantity we are interested in studying.

Definition 1.2.

$$P(k) := \max\{\#\mathcal{P} \,:\, \mathcal{P} \subseteq \mathbb{R}^2, \textit{ not all points in } \mathcal{P} \textit{ are collinear}, \, A(\mathcal{P}) \leq k\}.$$

We first provide general linear lower and upper bounds for $P(k)$. In particular, we have the following theorem.

Theorem 1.3. *For all $k \geq 1$,*

$$\begin{aligned} 2k + 3 \,&\leq\, P(2k) \qquad \leq\, 12k \\ 2k + 3 \,&\leq\, P(2k + 1) \leq\, 12k + 6. \end{aligned}$$

In the distance setting, the best known upper bound on the analogous parameter is the quadratic $(2 + k)(1 + k)$, and no lower bound is well-understood [SzOs]. It is therefore interesting and surprising that we find $P(k) = \Theta(n)$ in the angle setting. We prove Theorem 1.3 in Section 2.

Furthermore, we explicitly compute $P(1), P(2)$, and $P(3)$ and exhaustively identify all maximal point configurations for each.

Proposition 1.4. *We have $P(1) = 3$, and the equilateral triangle is the unique maximal configuration.*

In order to have only a single angle, every triangle of three points in the configuration must be equilateral. As this is impossible for point configurations that are not the vertices of an equilateral triangle, $P(1) = 3$. $P(2)$ and $P(3)$ are considerably less trivial quantities. We calculate $P(2), P(3)$ via exhaustive casework, simultaneously characterizing all of the unique optimal point configurations up to rigid motion transformations and dilation about the center of the configuration. We proceed by first considering sets of three points and then search for what additional points may be added without determining too many angles. We prove Theorem 1.5 in Section 3 and Theorem 1.6 in Section 4.

Theorem 1.5. *We have $P(2) = 5$. Moreover, the unique optimal point configuration is four vertices in a square with a fifth point at its center (see A in Figure 1).*

Theorem 1.6. *We have $P(3) = 5$. There are 5 unique optimal configurations, shown in Figure 1.*

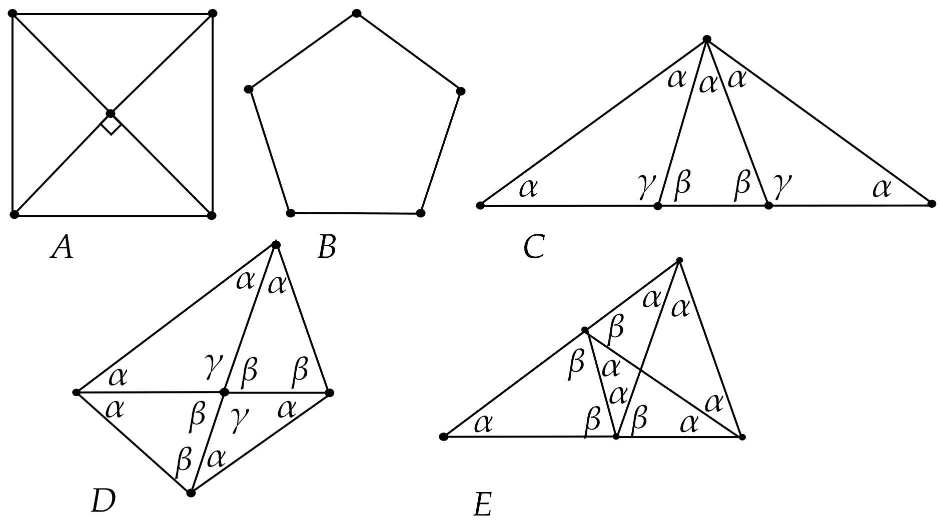

FIGURE 1. Optimal Two and Three Angle Configurations. $\alpha = \frac{\pi}{5}, \beta = \frac{2\pi}{5}, \gamma = \frac{3\pi}{5}$.

2. General Bounds

Although one may in principle calculate $P(k)$ for any k by extensive casework (as we later calculate $P(2), P(3)$), it quickly becomes overwhelming. As such, we instead provide general bounds on $P(2k)$ and $P(2k+1)$. Note that the construction of a square with a point in the center is no accident in the case of $P(2)$. Indeed, adding a point in the center of a regular $2k$-gon introduces no additional angles. This is because, in a regular $2k$-gon, the line from the center to any boundary vertex intersects an additional vertex on the other side. As such, the only additional angles that may be added from the center point are those with the center point as the center of the angle. For the other angles including it as an endpoint, choosing the point on the other end of the line through the center gives an equal angle. Moreover, the angles formed with the center point as the center of the angle are precisely $i\pi/k$ for $1 \leq i \leq k-1$, which are already achieved among the other points of the regular $2k$-gon.

So, using the regular $(2k+2)$-gon with a point added in the center yields the following lemma.

Lemma 2.1. *We have $P(2k) \geq 2k+3$.*

Moreover, in the case of $P(2k+1)$, the regular $(2k+3)$-gon and the projection of a regular $(2k+3)$-gon onto a line (via a stereographic-like projection from a cap vertex) both achieve $2k+1$ angles, providing a bound on $P(2k+1)$.

Lemma 2.2. *We have $P(2k+1) \geq 2k+3$.*

Proposition 2.3. *If we wish to also count the 0-angle, then we may not add the center to an even polygon, and in general we reach a bound of $P(k) \geq k+2$.*

We conjecture that both of these lower bounds are tight in general. Nonetheless, we provide a linear upper bound. We achieve this bound as a corollary of a lower bound on the number of distinct angles, using progress on the Weak Dirac Conjecture. In 1961, Erdős [Er] conjectured the following, based on an earlier, more difficult conjecture of Dirac:

Conjecture 2.4 (Erdős, 1961)**.** *Every set $\mathcal{P}$ of n non-collinear points in the plane contains a point incident to at least $\lceil n/2 \rceil$ lines of $\mathcal{L}(\mathcal{P})$, where $\mathcal{L}(\mathcal{P})$ is the set of lines formed by points in $\mathcal{P}$.*

While Dirac's conjecture has not been proven, significant progress has been made. Let $l(n)$ be the largest proven lower bound proven for Dirac's conjecture. I.e., every set $\mathcal{P}$ of n non-collinear points in the plane contains a point incident to at least $l(n)$ lines of $\mathcal{L}(\mathcal{P})$. We have $l(n) \geq \lceil n/3 \rceil + 1$ from [Ha]. Let $A(n)$ be the minimum number of distinct angles among n points in the plane. We have the following lemma.

Lemma 2.5. *For $n > 3$, $A(n) \geq \frac{l(n)-1}{2} \geq n/6$.*

Proof. Fix a set $\mathcal{P}$ of n non-collinear points in the plane. Let p be a point in $\mathcal{P}$ incident to at least $\ell(n)$ lines of $\mathcal{L}(\mathcal{P})$. Fix a point $q \neq p$ in $\mathcal{P}$. It shares exactly one line with p. Note that for a fixed nonzero angle $\theta < \pi$, there are exactly two possible lines which r must be on in order for $\angle qpr = \theta$. As such, since p is incident to $\ell(n) - 1$ lines without q, p is the center angle of at least $(\ell(n) - 1)/2$ distinct angles. Therefore

$$A(n) \geq \frac{\ell(n) - 1}{2}.$$

We have $\ell(n) \geq \lceil n/3 \rceil + 1$ from [Ha]. As such, we have $A(n) \geq n/6$, as desired. □

Note that such a use of the Weak Dirac Conjecture is known. See [BMP], Section 6.2.

Corollary 2.6. *We have $P(k) \leq 6k$.*

Proof. Since $A(n) \geq n/6$ by Lemma 2.5, then $P(k) \leq 6k$ as point configurations with at least $6k+1$ points define at least $k + 1$ angles. □

3. COMPUTING $P(2) = 5$

Proof. In any point configuration with at least three points, there are triangles. For any point configuration with at most two angles, all triangles must be isosceles. We divide into two cases, based on whether or not there is an equilateral triangle.

3.1. **There is an equilateral triangle.**

We consider adding a fourth point in cases (Figure 2).

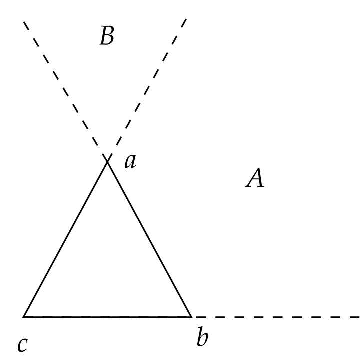

FIGURE 2. Equilateral Triangle Regions

***Case 1*:** $p \in A$.
Then $\angle acp < \pi/3$ and $\angle cap > \pi/3$, leading to more than two angles.
***Case 2*:** $p \in \overline{ab}$.
Then $\angle bcp < \pi/3$ and one of $\angle cpb$ and $\angle apc \geq \pi/2$, leading to more than two angles.
***Case 3*:** $p \in \overleftrightarrow{ac}$ to the upper-right of a.
Then $\angle cbp > \pi/3$ and $\angle cpb < \pi/3$, again leading to more than two angles.

***Case 4*:** $p \in B$.

In this case, $\angle cbp > \pi/3$ and $\angle cpb < \pi/3$, leading to more than two angles.

***Case 5*:** $p \in \triangle abc$.

In this case, one of $\angle apb, \angle bpc, \angle cpa \geq 2\pi/3$ and $\angle acp < \pi/3$, leading to more than two angles.

Up to symmetry, these cases are exhaustive. Thus if there is an equilateral triangle in the configuration, there can only be at most three points.

3.2. **There is no equilateral triangle.** Now, let a, b, and c be the vertices of an isosceles triangle with vertex angle α, base angle β, and a the apex vertex. We reduce the number of possibilities for additional points by partitioning the plane into regions A_i (Figure 3). Note that we may without loss of generality assume that

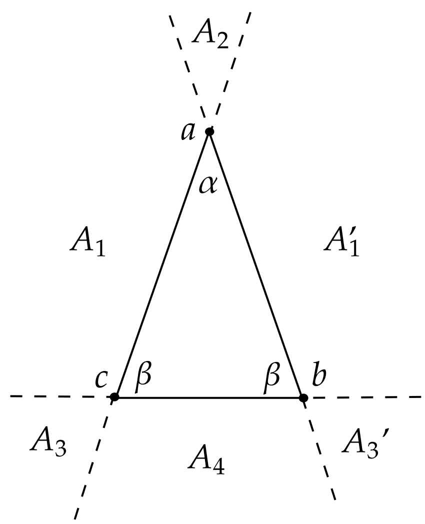

FIGURE 3. Isosceles Triangle Regions.

no fourth point is added within $\triangle abc$ as we could then choose that triangle as our initial triangle. Also note that A_1 and A_1' and A_3 and A_3' are equivalent up to symmetry.

***Case 1*:** $p \in A_1$.

In this case, $\angle pab > \alpha$ and $\angle pcb > \beta$. So, regardless of whether α or β is greater, adding p introduces an additional angle. So, no additional points can be in A_1 or A_1'.

***Case 2*:** $p \in A_2$.

In this case, $\angle pcb$ and $\angle pbc$ are greater than β, so both must be α to not add additional angles. But then $\angle cpb = \pi - 2\alpha \neq \beta$, in order to not add angles, implying $3\alpha = \pi$. But, this implies $\triangle pcb$ is an equilateral triangle. Thus no points may be added in this case.

***Case 3*:** $p \in A_3$ (or A_3' by symmetry).

In this case, $\angle bap > \alpha$ and $\angle abp > \beta$, so there is an additional angle added regardless and no additional points are possible.

***Case 4*:** $p \in A_4$.

In this case, $\angle cap, \angle bap < \alpha$, so both must equal β. Therefore, $2\beta = \alpha$, which implies $\beta = \pi/4$ and $\alpha = \pi/2$. Moreover, since $\angle acp$ and $\angle abp$ are greater than β, they must both equal $\alpha = \pi/2$. So, the only possibility for an addable point in this case is for p to be the fourth vertex of the square $acpb$.

***Case 5*:** $p \in \overleftrightarrow{bc}$.

If p is on $\overleftrightarrow{bc}$ between b and c, then $\angle cap, \angle bap < \alpha$. In order for these not to introduce additional angles, they must both be equal to β. This implies $\beta = \pi/4$ and $\alpha = \pi/2$ and p is the center of the side bc. If $p \in \overleftrightarrow{bc}$ to the left of c (or by symmetry, right of b), $\angle bap > \alpha$ and thus $\angle bap = \beta$. Since $2\beta + \alpha = \pi$, $\beta < \pi/2$. But then $\angle acp > \pi/2 > \beta > \alpha$. Thus there is exactly one point possible on line $\overleftrightarrow{bc}$, the centerpoint of the edge between b and c.

***Case 6*:** $p \in \overleftrightarrow{ac}$ (or $p \in \overleftrightarrow{ab}$).

If p is between a and c, then $\angle cbp < \beta$ and thus $\angle cbp = \alpha$. But, as before, $\beta < \pi/2$. Moreover, one of $\angle bpc$ or $\angle bpa$ is at least $\pi/2 > \beta > \alpha$. Thus there are too many angles in this case. If p is to the bottom left of c, $\angle apb < \beta$ and thus $\angle apb = \alpha$. But, again, either $\angle bca$ or $\angle bcp > \pi/2 > \beta$, creating too many angles in this case. If p is on $\overleftrightarrow{ac}$ to the upper right of a, $\angle pbc > \beta$ and thus equals α. Then $\angle pba < \alpha$ and must equal β and thus $2\beta = \alpha$. This implies $\beta = \pi/4$ and $\alpha = \pi/2$ and $\triangle cbp$ is an isosceles right triangle with b the apex vertex, p on $\overleftrightarrow{ac}$ to the upper right of a, and a at the center of side $\overline{pc}$.

As such, in order to add additional points to an isosceles triangle point configuration without adding additional angles, we must have $\alpha = \pi/2$ and $\beta = \pi/4$. The four additional possible points are marked in Figure 4.

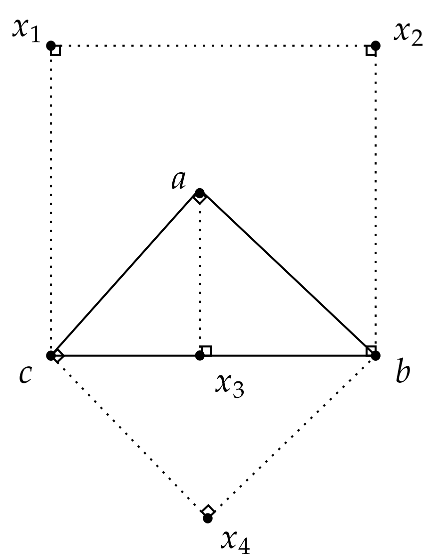

FIGURE 4. Compatible Points with the Right Triangle.

Note that $\angle x_4ax_1, \angle x_4ax_2 > \pi/2$. So, x_4 cannot be in the same point configuration as x_1 or x_2. By symmetry the same follows for x_3. However, we may have both x_1 and x_2 or both x_3 and x_4, either of which give the unique optimal configuration A in Figure 1. □

Corollary 3.1. *One might also wish to include the trivial 0-angle in our count. In this case,* $P(2) = 4$*, and the unique configuration is the square.*

Proof. The only 5-point configuration no longer holds when we count the 0-angle. Figure 4 displays all valid four point configurations which define only 2 angles excluding 0, as detailed in the proof of $P(2)$. All the shown points but x_2 define a 0-angle, so the only valid 4 point configuration is the square. □

4. COMPUTING $P(3) = 5$.

In this section we prove the surprising result that $P(3) = 5$. That is, adding an additional allowable distinct angle from two to three does not increase the maximum number of points in an optimal point configuration.

Proof. We divide the casework for this section into four parts based on the triangles exhibited in the point configuration:

(1) There is a scalene triangle.
(2) All triangles are isosceles with at least one with base angle larger than vertex angle.
(3) All triangles are isosceles with the base angle at most the vertex angle with at least one non-equilateral triangle.
(4) All triangles are equilateral.

4.1. **There is a scalene triangle.** Let a, b, and c be the vertices of a scalene triangle in the configuration. We without loss of generality assume $\alpha < \beta < \gamma$ (Figure 5). As in the proof of Theorem 1.5, we begin by

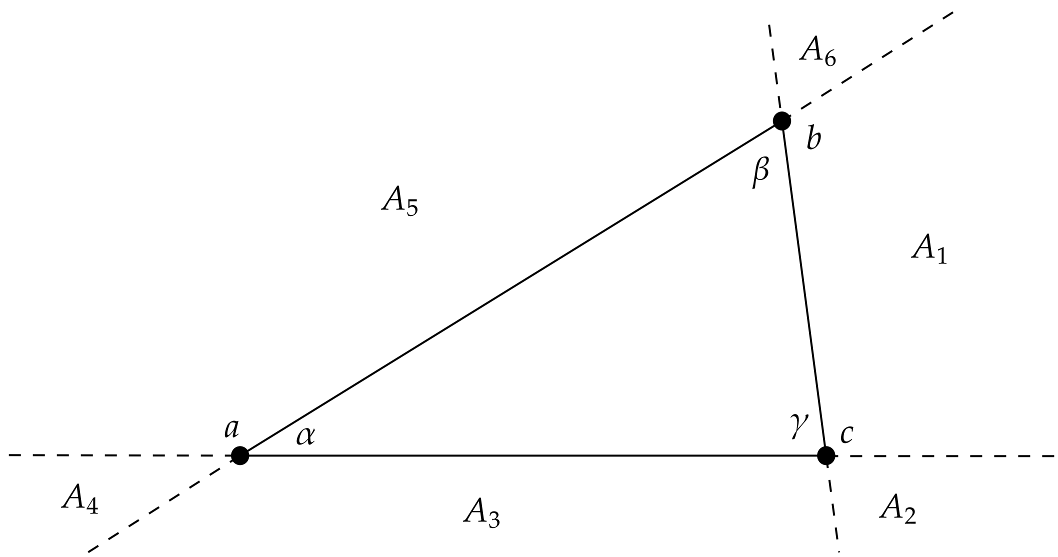

FIGURE 5. Scalene Triangle Regions.

reducing the number of possible points to a finite number by region-based casework.

***Case 1*:** $p \in A_1$.
As $\angle bap < \alpha$, no points may be added in A_1.

***Case 2*:** $p \in A_2$.
In this case, $\angle abp > \beta$ and thus must equal γ. Moreover, $\angle bap > \alpha$. If $\angle bap = \gamma$, $\angle bpa < \alpha$. Thus $\angle bap = \beta$, which implies $\angle bpa = \alpha$. But, $\angle bpc < \angle bpa = \alpha$, so we define a fourth angle. Therefore there cannot be points added in A_2.

***Case 3, Case 4*:** $p \in A_3 \cup A_4$.
As $\angle bcp > \gamma$, no points may be added in A_4.

***Case 5*:** $p \in A_5$.
In this case, $\angle cbp > \beta$ and thus $\angle cbp = \gamma$. Moreover, $\angle cap > \alpha$ and is thus β or γ (which implies $\angle pab = \alpha$ or β, respectively). If $\angle cap = \beta$, we have that $\angle pab < \beta$ and thus is equal to α. So,

$\beta = 2\alpha$. Then $\angle abp$ cannot be α because then $\angle apb > \gamma$. So, $\angle abp = \beta$. So, $2\alpha = \beta$ and $2\beta = \gamma$, so the angles are $\pi/7, 2\pi/7$, and $4\pi/7$. We then have Figure 6.

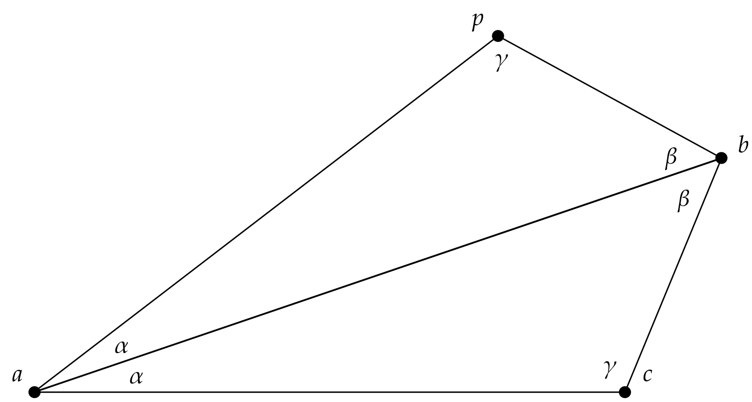

FIGURE 6. Four Point Kite Configuration. $\alpha = \dfrac{\pi}{7}$, $\beta = 2\alpha$, $\gamma = 4\alpha$.

Alternatively, we have $\angle pab = \beta$ and thus $\angle cap = \gamma$. Then $\gamma = \beta + \alpha$, so $\angle abp = \alpha$. $\gamma = \beta + \alpha$ and $\alpha + \beta + \gamma = \pi$ implies $\gamma = \pi/2$. I.e., a, b, c, p are the vertices of a rectangle. As such, we reach Figure 7.

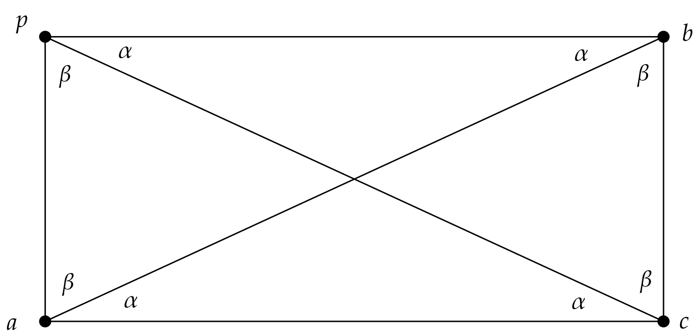

FIGURE 7. Four Point Rectangular Configuration.

Therefore, there are **exactly two possible points** to add in A_5, with each choice exactly determining the angles α, β, and γ.

Case 6**:** $p \in A_6$.

As $\angle acp > \gamma$, no points may be added in A_6.

Case 7**:** $p \in \overleftrightarrow{ab}$.

If p is to the right of b, then $\angle acp > \gamma$.

Note that $\beta, \alpha < \pi/2$. Then, if p is to the left of a, $\angle pac > \pi/2$ and must equal γ. But then α and γ are supplementary, implying $\beta = 0$.

Finally, if p is between a and b, one of $\angle cpa$ and $\angle cpb$ is at least $\pi/2$. So, $\gamma \geq \pi/2$. Moreover, as neither α nor β can be supplementary to γ, we have that the supplement of γ is γ, and hence $\gamma = \pi/2$. This yields a diagram like Figure 8.

So, **exactly one point** may be added on $\overleftrightarrow{ab}$ and it forces $\gamma = \pi/2$.

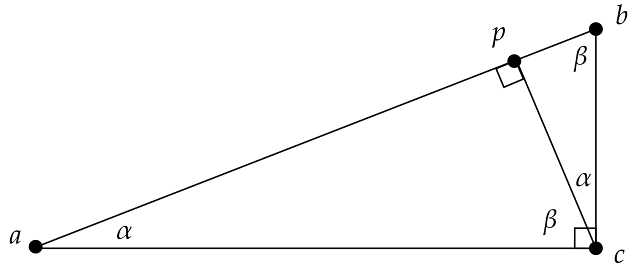

FIGURE 8. Four Point Configuration from Case 7.

***Case 8*: $p \in \overleftrightarrow{bc}$.**

If p is between b and c then $\angle bap < \alpha$.

Recall that $\beta, \alpha < \pi/2$. Then, if p is not on $\overline{bc}$ and is closest to b, $\angle abp > \pi/2$ and thus must equal γ. So, $\gamma + \beta = \pi$. But, this implies $\alpha = 0$.

Finally, if p is not on $\overline{bc}$ and is closest to c, then $\angle acp$ is supplementary to γ. Since neither β nor α can supplementary to γ be without implying the other is 0, this implies $\gamma = \pi/2$. We then have the configurations in Figure 9 since $\angle cap = \beta$ or α.

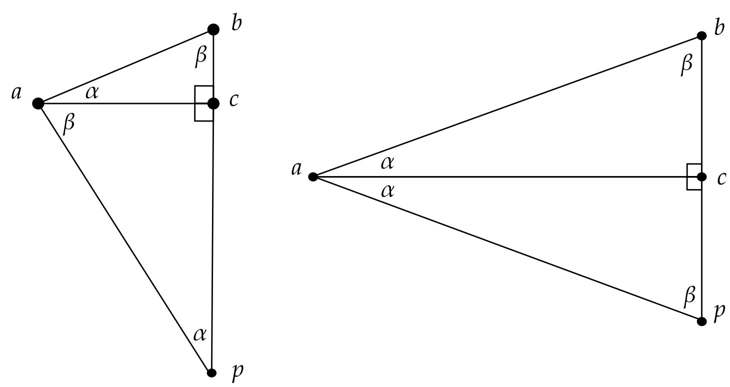

FIGURE 9. Four Point Configurations from Case 8.

So, exactly **two points** can be added on $\overleftrightarrow{bc}$ and both force $\gamma = \pi/2$.

***Case 9*: $p \in \overleftrightarrow{ac}$.**

If p is between a and c, then one of $\angle bpa$ and $\angle bpc$ are at least $\pi/2$ and must thus be γ. Since neither α nor β can be supplementary to γ, this implies $\gamma = \pi/2$. However, $\angle abp < \beta$ and thus must be α. This then yields $2\alpha = \pi/2$ from $\triangle abp$, contradicting $\alpha + \beta = \pi/2$ since $\alpha \neq \beta$.

If p is left of a, we have $\angle pab > \pi/2$ and thus must be γ. But, this implies γ and α are supplementary.

If p is right of c, then $\angle bcp$ is supplementary to γ. Since neither α nor β can be, this implies $\gamma = \pi/2$. This leads to the allowable point configuration in Figure 10.

So, exactly **one point** may be added in this case, with $\gamma = \pi/2$ being forced.

***Case 10*: p in the interior of $\triangle abc$.**

In this case $\angle pac < \alpha$, we no points may be added.

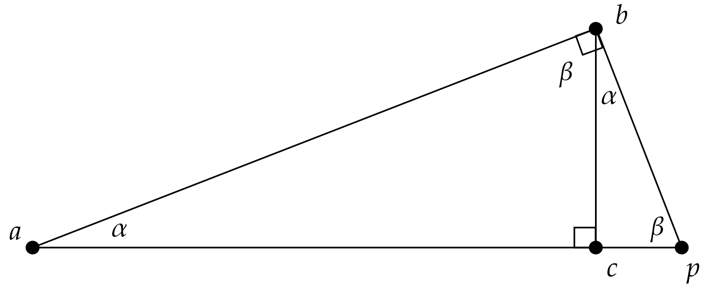

FIGURE 10. Four Point Configuration from Case 9.

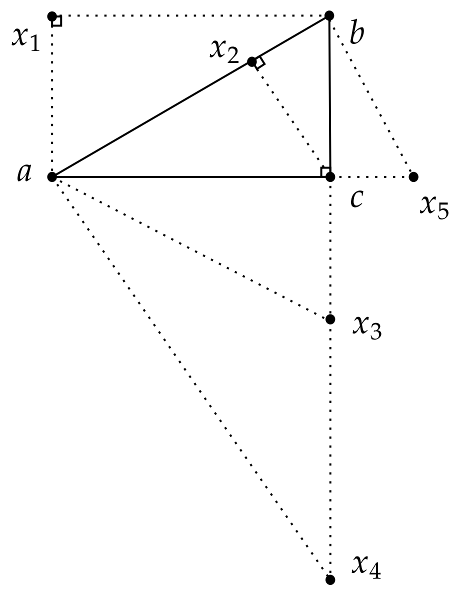

FIGURE 11. Compatible Points with the Right Triangle

It now remains to show that all six addable points are mutually incompatible. Suppose we add $p \in A_5$ as in the first case of a kite (Figure 6). As $\gamma \neq \pi/2$ as in all the other cases, no additional points may be added.

Where $\gamma = \pi/2$, we have the five point placements to consider (presented in Figure 11).

Suppose we add x_1. Adding x_2 yields an angle $\angle x_1x_2b > \pi/2$ or $\angle x_1x_2a > \pi/2$ (the diagonals cannot intersect at right angles lest $\alpha = \beta$). Adding x_3 adds $\angle x_1ax_3 > \pi/2$, and similarly for x_4. Adding x_5 adds $\angle x_1bx_5 > \pi/2$.

Suppose we add x_2. Adding x_3 yields $\angle x_2cx_3 > \pi/2$, and similarly for x_4. Adding x_5 adds $\angle ax_2x_5 > \pi/2$.

Suppose we add x_3. Adding x_4 creates $\angle ax_3x_4 > \pi/2$. Adding x_5 forces $\angle ax_3x_5 = \angle bx_5x_3 = \angle abx_5 = \pi/2$. But $\angle bax_3 = \beta < \pi/2$, so the angles in abx_5x_3 do not add to 2π.

Finally, suppose we add x_4 and x_5. In this case, $\angle cx_4x_5 < \angle cx_3x_5 = \alpha$.

So, if there is a scalene triangle in the point configuration, there can be at most four points.

4.2. **All triangles are isosceles with at least one with base angle larger than vertex angle.** Let a, b, and c be the vertices of an isosceles triangle with the base angle larger than the vertex angle (Figure 12).

Specifically, $\alpha = \angle cba < \angle abc = \angle acb = \beta$.

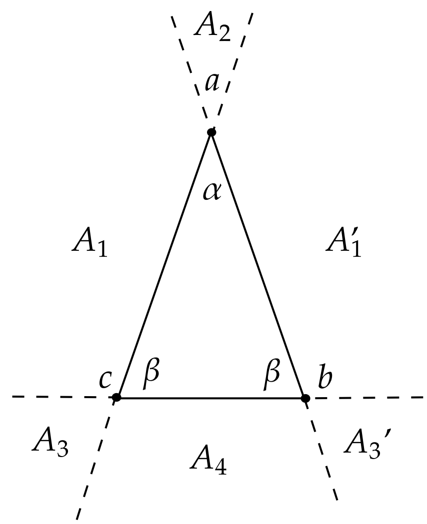

FIGURE 12. Isosceles Triangle with Small Vertex Angle.

***Case 1*:** $p \in A_1$.

Note that $\angle bcp > \beta > \alpha$. Let this new angle be γ. Then, since only three angles are admissible and since $\angle pca + \angle acb = \gamma$, then $\angle pca = \alpha$ or $\angle pca = \beta$.

Suppose $\angle pca = \beta$. Then $\angle pcb = \gamma = 2\beta$. Additionally, since $\angle pbc < \beta$, $\angle pbc = \alpha$. Since $2\beta + \alpha = \pi$, then $\angle bpc < \gamma$ lest the angles in $\triangle pbc$ be too large. Then as $\triangle pbc$ must be isosceles, $\angle bpc = \alpha$. Thus the angles in $\triangle bpc$ sum to $\gamma + 2\alpha = 2\beta + 2\alpha > 2\beta + \alpha = \pi$, a contradiction.

Thus $\angle pca = \alpha$. In this case, $\gamma = \alpha + \beta$. Observe that $\angle pbc < \beta$, and so $\angle pbc = \alpha$. Similarly, $\angle pba = \alpha$, thus giving $\angle bca = \beta = 2\alpha$. Then $\gamma = \alpha + \beta = 3\alpha$. The angles in $\triangle abc$ must add to π, so $2\beta + \alpha = 5\alpha = \pi$, which implies $\alpha = \pi/5$. Moreover, as $\triangle pbc$ must be isosceles, $\angle cpb = \alpha$. Then as $\angle bpa < \angle cpa$, we have $\angle bpa = \alpha$ or $\angle bpa = \beta$, both of which determine $\angle pac$. Thus in this case we have two points in A_1 which are contenders to give acceptable four-point configurations (Figure 13).

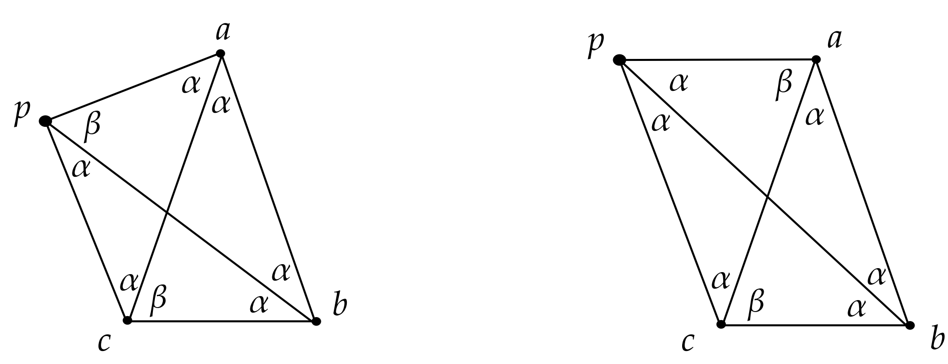

FIGURE 13. Configurations of Interest from Case 1.

Consider further the case where $\angle bpa = \alpha$. We have $pabc$ is a parallelogram. We have deduced above that $\overleftrightarrow{pb}$ bisects $\angle abc$, but this is only true if $abcp$ is a rhombus. However, this would mean that

$\triangle abc$ is equilateral, since then $|\overline{ab}| = |\overline{ac}| = |\overline{bc}|$, contradicting our original assumption that $\alpha < \beta$. Thus this case is impossible and $\angle bpa = \beta$. So, there is exactly **one point** we may add in A_1 (thus forming the left configuration of Figure 13), and, by symmetry, an additional **one point** in A_1'.

***Case 2*:** $p \in A_2$.

Note that $\angle bcp > \angle bca = \beta$, and $\angle cpb < \angle cab = \alpha$. Thus no points may be added in A_2.

***Case 3*:** $p \in A_3$.

If $p \in A_3$, then $\gamma = \angle abp > \beta$. Then, as $\angle pab > \alpha$, we have $\angle apb < \beta$ and thus $\angle apb = \alpha$ so that we may still have only three angles. But then $\angle apc < \angle apb = \alpha$, yielding a fourth angle. Thus no new angles may be added in A_3 or A_3'.

***Case 4*:** $p \in A_4$.

In this case, $\angle pac < \angle bac = \alpha < \beta = \angle bca < \angle pca$. Thus there are already four angles, and no points can be added in A_4.

***Case 5*:** p inside $\triangle abc$.

In this case, $\angle pab, \angle pac < \angle cab = \alpha$, so let $\angle pab = \angle pac = \gamma$. As $\triangle abp$ must be isosceles and $\gamma < \alpha$, γ cannot be the vertex angle of $\triangle abp$. Since $\gamma < \alpha < \beta$, this implies the vertex angle of $\triangle abp$ must be β. So, $2\gamma = \alpha$ and $2\gamma + \beta = \pi$. But, this is a contradiction as $\alpha + 2\beta = \pi$. No points are addable in this case.

***Case 6*:** $p \in \overleftrightarrow{bc}$.

First, assume p is between b and c; that is, p is located on the base of $\triangle abc$. Then, $\angle pab < \alpha$ and one of $\angle pba$ or $\angle cpa \geq \pi/2 > \beta$. Thus no points can be added on the base of $\triangle abc$.

Now suppose that p is not between b and c. By symmetry, we may assume that p is on the left (i.e., it is closer to c than to b). Now, since $\beta < \pi/2$, $\angle acp = \gamma > \pi/2$. As $\triangle acp$ must be isosceles, γ cannot be a base angle, and $\gamma > \alpha$, we have $\angle cpa = \angle pac = \alpha$. This implies $2\alpha + \gamma = 2\beta + \alpha$ and $\alpha + \gamma = 2\beta$. Along with $\pi - \gamma = \beta$, this yields $\alpha = \pi/5, \beta = 2\alpha, \gamma = 3\alpha$. So, **two points** are addable in this case (one on either side of the edge bc). See Figure 14.

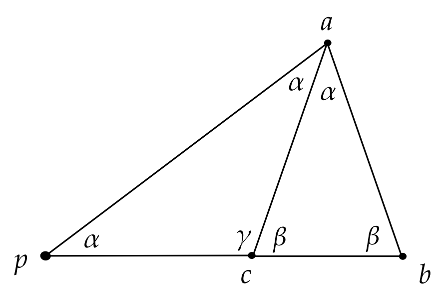

FIGURE 14. Four Point Configuration from Case 6.

***Case 7*:** $p \in \overleftrightarrow{ac}$ or $p \in \overleftrightarrow{ab}$.

By symmetry, we may assume $p \in \overleftrightarrow{ac}$.

First, assume p is between a and c; that is, p is located on a leg of $\triangle abc$. In this case, one of $\angle apb$ and $\angle bpc \geq \pi/2 > \beta$. Let this angle be γ. As $\angle cbp < \beta$ and thus must equal α, since $\triangle cbp$ must be isosceles, we have $\angle bpc = \beta$. So, $\angle bpa = \gamma$ and $\beta + \gamma = \pi$. Moreover, we have $\gamma + 2\alpha = \pi$. This again implies $\alpha = \pi/5, \beta = 2\alpha, \gamma = 3\alpha$. This is a legal configuration.

Next, assume that $p \in \overleftrightarrow{ac}$ is not on the triangle's side, and that it is closer to a than to c. In this case, $\angle bap = \pi - \alpha > \beta$ and $\angle cpb < \angle cab = \alpha$. Thus no points may be added in this case.

Now, assume that $p \in \overleftrightarrow{ac}$ is not on the triangle's side, and that it is closer to c than to a. Then we have a new angle $\gamma = \angle pba > \angle cba = \beta$. We also have $\angle pcb = \pi - \beta > \beta$ because $\beta < \pi/2$. So, to maintain only three angles, we must have $\angle pcb = \angle pba = \gamma$. Then, as $\triangle cbp$ must be isosceles and $\angle bcp$ must be the vertex angle, $\angle cbp = \angle cpb = \alpha$. As before, we conclude that $\alpha = \pi/5, \beta = 2\pi/5, \gamma = 3\pi/5$. This is a legal configuration.

So, in this case, all our angles are exactly determined and there are **two points** addable on $\overleftrightarrow{ac}$, and, by symmetry, **two points** addable on $\overleftrightarrow{ab}$.

So there are only eight addable points in the case of there being an isosceles triangle with vertex angle smaller than base angle (Figure 15). We examine each of these points up to symmetry and examine which are compatible.

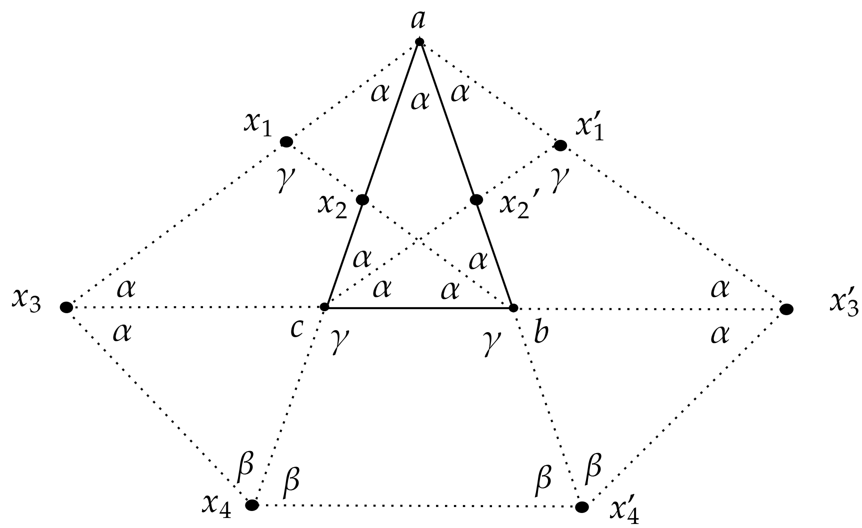

FIGURE 15. Compatible Points with the Isosceles Triangle with Small Vertex Angle

***Combination Case 1*:** Including x_1.

We cannot add x_4 as $\angle x_4x_1a > \angle cx_1a = \gamma$. Additionally, note that $\angle bx_1x_4', \angle bx_1x_2', \angle ax_3'x_1 < \alpha$ and thus none of x_4', x_3', or x_2' may be added. Each of x_1', x_2, and x_3 are individually compatible with x_1, leading to **three** valid five point configurations including x_1 (see B, D, E of Figure 1). By symmetry, x_1' is then individually compatible with x_1, x_2' and x_3'.

***Combination Case 2*:** Including x_2.

Note that $\angle x_2x_3c, \angle x_2x_3'c, ax_4'x_2 < \alpha$. So none of x_3, x_3', or x_4' are addable in this case. Adding x_2' is analogous to adding both x_1 and x_3 to $\triangle abc$, so x_2' is addable. And x_4 is addable as the projection of a regular pentagon onto a line via one of its vertices. So, there are **three** valid five point configurations including x_2 (see E, D, C of Figure 1). By symmetry, x_2' is compatible with exactly x_1', x_2 and x_4'.

***Combination Case 3*:** Including x_3.

In this case, $\angle cx_4'x_3 < \alpha$, so x_4' is not addable in this case. Adding x_3' creates a projected pentagon, and adding x_4 creates the construction of a trapezoid with a point in the middle, and both are individually compatible with x_3. So, each of x_1, x_3', and x_4 may be individually added alongside x_3, leading to **three** valid five point configurations including x_3 (see E, C, D of Figure 1). By symmetry, x_3' is compatible with x_1', x_3, and x_4'.

***Combination Case 4*:** Including x_4. x_4' is compatible with x_4. In combination with the above casework, we have x_4 is individually compatible with exactly x_2, x_3, and x_4. So, there are **three** valid five point configurations including x_4 (see C, D, E of Figure 1). By symmetry, x_4' is compatible with exactly x_2', x_3', and x_4.

At this point, we have exhaustively identified all our five point configuration for this case of $\alpha < \beta$. From our casework, we see there are no compatible, addable points which share an additional compatible point. Therefore, there are most five points in this case, with C, E, and F of Figure 1 as the possible five point configurations.

4.3. **All triangles are isosceles with the base angle at most the vertex angle with at least one non-equilateral triangle.** As before, we proceed by region-based casework. Fortunately, whenever we encounter a scalene triangle or an isosceles triangle with the vertex angle larger than the base angle, we reduce to the previous cases. Our diagram for this section is Figure 16, where $\beta < \alpha$.

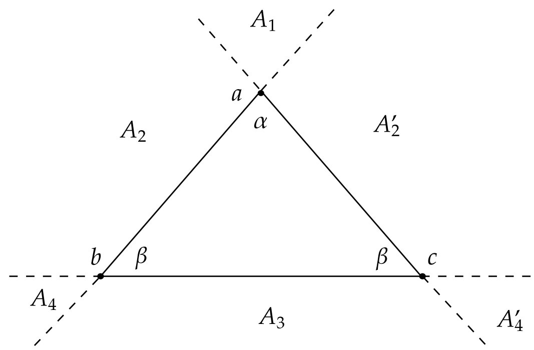

FIGURE 16. Isosceles Triangle with Large Vertex Angle.

***Case 1*:** $p \in A_1$.

In this case, $\angle bpc < \alpha$ since $\angle pcb$ and $\angle pbc > \beta$. We then have two cases.

If $\angle bpc = \beta$, then $\angle pcb$ and $\angle pbc$ cannot both be α (as $2\alpha + \beta > \pi$). Moreover, $\triangle cpb$ must be isosceles and neither $\angle pcb$ nor $\angle pbc$ can be β, so both must be γ. This implies $2\gamma + \beta = 2\beta + \alpha$ and $\beta < \gamma < \alpha$. But, $\angle bpa < \beta < \gamma$, creating more than 3 angles.

If $\angle bpc = \gamma < \alpha$, then, since $\angle cpa, \angle bpa < \gamma$, both must be β. So, $\gamma = 2\beta$. Now, $\triangle pcb$ must be isosceles. Since $\alpha + 2\beta = \alpha + \gamma = \pi$, this implies both $\angle pbc$ and $\angle pcb$ must be γ. But, then $\gamma = \pi/3$, $\beta = \pi/6$, and $\alpha = 2\pi/3$ (Figure 17).

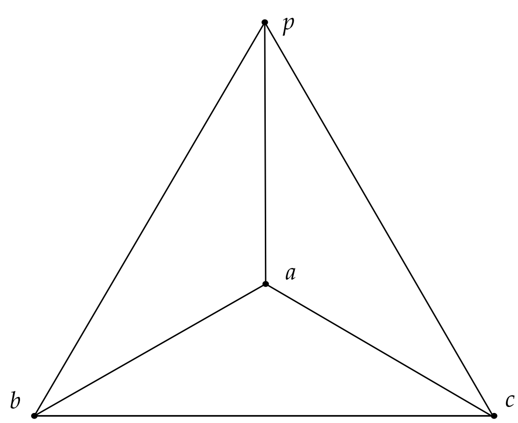

FIGURE 17. Four Point Configuration from Case 1.

As such, there is **exactly one point** addable in this case and it forces the choice of α, β, and γ.

***Case 2*:** $p \in A_2$ (or A_2').

In this case, $\angle bcp < \beta$ and $\angle cap > \alpha$, so no points may be added.

***Case 3*:** $p \in A_3$.

As both $\angle cap, \angle bap < \alpha$, we have three cases:

(1) $\angle cap = \beta = \angle bap$

(2) $\angle cap = \beta$, $\angle bap = \gamma \neq \beta$ (and swapping $\angle cap$ and $\angle bap$ by symmetry), and

(3) $\angle bap, \angle cap = \gamma \neq \beta$

Case (1) implies $\alpha = 2\beta$, and hence $\beta = \pi/4, \alpha = \pi/2$. Moreover, $\triangle bcp$ must be isosceles and must include one of α or β. Note $\triangle bcp$ cannot be equilateral as then $\angle acp = 7\pi/12$, and we have four angles. Note that $\pi/2$ cannot be a base angle in $\triangle bcp$ and β being the vertex angle reduces to the previous casework in section 4.2. Thus the only option is $\angle cbp = \angle bcp = \pi/4$ and $\angle bpc = \pi/2$, yielding a **valid** configuration, the square $abpc$.

Case (2) implies $\beta + \gamma = \alpha$. This is illustrated in the Figure 18.

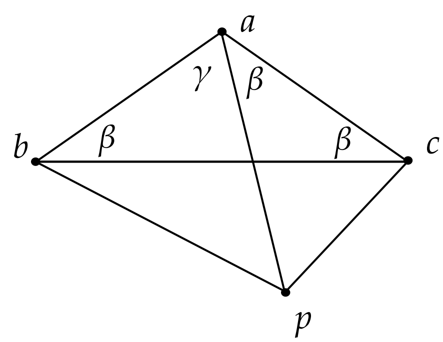

FIGURE 18. Four Point Configuration from Case 3.

First, we show that $\gamma > \beta$. If $\gamma < \beta$, then $\alpha = \angle abp > \beta$, which implies $\angle cbp = \gamma$. We similarly have $\angle bcp = \gamma$. But then, $\angle bpc = \pi - 2\gamma > \alpha$, yielding four angles $\gamma < \beta < \alpha < \pi - 2\gamma$. Thus $\gamma > \beta$.

We then have two sub-subcases to consider:

(2i) $\angle abp = \alpha$.

(2ii) $\angle abp = \gamma \neq \alpha$.

In case $(2i)$, note that $\angle cbp = \gamma$. Since $\triangle abp$ must be isosceles with largest angle the vertex angle, $\angle apb = \gamma$. Then, we have $\alpha + 2\gamma = \alpha + 2\beta$, contradicting the assumption of case (2) that $\gamma \neq \beta$.

In case $(2ii)$ we have $\beta < \gamma < \alpha$. Since $\angle cbp < \gamma$, it must equal β. So, $\gamma = 2\beta$. This implies $\alpha = 3\beta$. Thus $\beta = \pi/5$, $\gamma = 2\pi/5$, and $\alpha = 3\pi/5$. Since $\triangle acp$ must be isosceles with smaller base angle, $\angle apc = \beta$. Completing $\triangle abp$, we have $\angle apb = \beta$. But then $\triangle abp$ is isosceles with smaller vertex angle. Thus this case reduces to the previous casework.

In case (3), we have $2\gamma = \alpha$. Now, $\angle cbp$ cannot be γ since $\beta + \gamma \neq \alpha$. As $\gamma < \alpha$, it cannot be α either. Thus it must be β. Since $\beta \neq \gamma$, we must then have $2\beta = \gamma$. So, we have $\alpha = 4\beta$. Thus $\beta = \pi/6, \gamma = \pi/3, \alpha = 2\pi/3$. This yields the **valid** point configuration in Figure 19. So, there are exactly **two** addable points $p \in A_3$ and each forces a choice of α, β, and γ.

***Case 4*:** $p \in A_4$ (or A_4').

In this case, $\angle cap > \alpha$ and $\angle apc < \beta$. Thus no points are addable in this case.

***Case 5*:** $p \in \overleftrightarrow{ac}$ (or $\overleftrightarrow{ab}$ by symmetry).

If p is to the left of a, $\angle pab = \pi - \alpha \neq \beta$. We then have two subcases:

(1) $\angle pab = \alpha$

(2) $\angle pab = \gamma \neq \alpha$

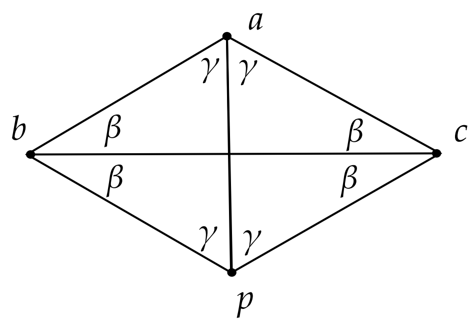

FIGURE 19. Four Point Configuration from Case 3.

In case (1), $\angle pab = \alpha$ implies $\alpha = \pi/2$ and $\beta = \pi/4$. As $\triangle pab$ must be isosceles, $\angle bpa = \angle pba = \pi/4$. So, we get the **valid** configuration in Figure 20.

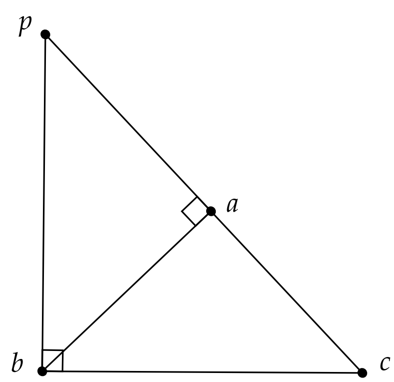

FIGURE 20. Four Point Configuration from Case 5.

In case (2), we have $\angle pab = \gamma = \pi - \alpha > \beta$. Note $\triangle pbc$ needs to be isosceles with the vertex angle at least as large as the base angle, and that $\angle pcb = \beta$. Thus since $\angle pbc > \beta$, $\angle pbc = \alpha$ and $\angle bpc = \beta$. Then $\angle abp = \gamma$, since $\triangle pab$ must be isosceles and $\angle pab \neq \alpha$. So, $2\gamma + \beta = \pi$, $2\beta + \alpha = \pi$, and $\alpha + \gamma = \pi$. This implies $\beta = \pi/5, \gamma = 2\pi/5$, and $\alpha = 3\pi/5$. But then $\triangle pba$ has angles $2\pi/5, 2\pi/5, \pi/5$, and so the vertex angle is smaller than the base angles. Thus this case reduces to the previous section.

Now, if p is between a and c then $\gamma = \angle abp = \angle cbp < \beta$. Furthermore, $\angle bpc > \alpha$, giving four angles, so no points can be added in this subcase.

Finally, if p is to the bottom right of c then $\gamma = \angle bcp = \pi - \beta > \alpha$. But, $\triangle bcp$ must be isosceles with largest angle not repeated. Thus $\angle cbp < \beta$.

So, exactly **two points** are addable in this case and they exactly determine α and β (the second by symmetry).

***Case 6*: $p \in \overleftrightarrow{bc}$.**

In this case, left of b is equivalent to right of c by symmetry. So, we without loss of generality consider p left of b. In this case, $\angle pba = \pi - \beta > \alpha$. Then, $\triangle pba$ must be isosceles with largest angle non-repeated. So, $\angle pab < \beta$. So, no points are addable in this subcase.

It remains to consider p between b and c. In this case, $\triangle acp$ and $\triangle abp$ are isosceles triangles including β. However, β cannot be the vertex angle, so another of their angles must be β and the third α. Since $\angle cap$, $\angle pab < \alpha$, we must then have $2\beta = \alpha$. Thus we have $\beta = \pi/4$ and $\alpha = \pi/2$. The resulting **valid** point configuration is displayed in Figure 21.

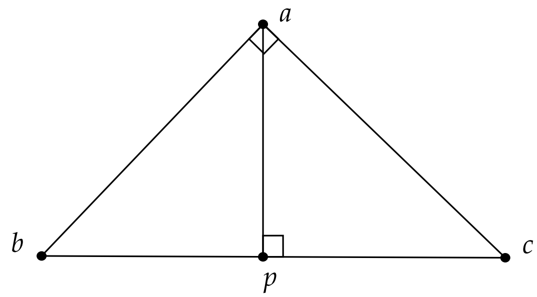

FIGURE 21. Four Point Configuration from Case 6.

So, there is a **single addable** point in this case and it forces the choice of α and β.

***Case 7*:** p in the interior of $\triangle abc$.

In this case, $\angle pbc$, $\angle pcb < \beta$ and thus $\angle cpb$ is greater than α. Thus no points are addable in this case.

***Combinations*:** Adding more than one point.

Now we determine which of the six addable points are mutually compatible. As exactly two force $\alpha = 2\pi/3, \beta = \pi/6$, and $\gamma = \pi/3$, those two (from Case 1 and Case 3) are at most compatible with each other. This is displayed in Figure 22.

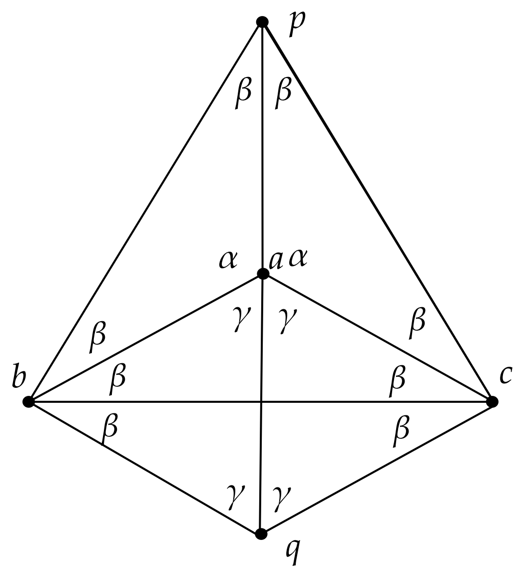

FIGURE 22. Attempting to Combine Cases 1 and 3.

However, this adds an additional angle, $\angle qcp = 3\beta = \pi/2$.

So, we only consider the addable points which force $\alpha = \pi/2$ and $\beta = \pi/4$. Such addable points are shown in Figure 23.

Both pairs x_1, x_2 and x_3, x_4 are compatible, and both yield a square with a point in the center (see A of Figure 1).

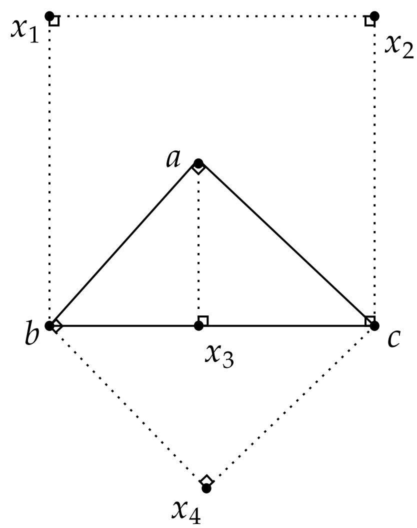

FIGURE 23. Compatible Points with the Right Triangle

However, x_3 is not compatible with either of x_1, x_2. For x_1, then $\triangle x_1x_3b$ is not isosceles, and similarly for x_2.

Also, x_4 is not compatible with either of x_1, x_2 as then we have angles $\alpha, \beta, \alpha + \beta, \angle bx_1x_4 < \beta$ (or $\angle bx_2x_4 < \beta$).

Therefore, the only 5-point configurations are $\{a, b, c, x_1, x_2\}, \{a, b, c, x_3, x_4\}$. Therefore, only 5 points are allowed in this case, and the only acceptable 5-point configuration of the square with its center (see A in Figure 1).

4.4. **There are only equilateral triangles.** Since an equilateral triangle has all equal side lengths, every distance between two points in the configuration must be equal. That is, we need a 1-distance set. The largest 1-distance set in the plane is the equilateral triangle. Thus no configuration of four or more points can exist defining only equilateral triangles. The maximal number of points in this case is thus three.

Then across all cases, we find that the largest configurations of points on the plane which define at most three angles contain exactly five points. As such, $P(3) = 5$, and the complete list of configurations is shown in Figure 1. □

Corollary 4.1. *One might also wish to include the trivial 0-angle in our count. In this case,* $P(3) = 5$*, but the square with the center-point and the pentagon are now the only valid configurations.*

Proof. The set of valid five-point configurations when we count the 0-angle must be a subset of the valid five-point configurations we identified above. By direct inspection, the square with the center-point and the pentagon are the only of the five in Figure 1 which define only three angles. All the others define three angles greater than zero and also the 0-angle by collinearity. □

5. FUTURE WORK

While it seems possible to compute $P(k)$ by exhaustive casework for higher values of k, the casework quickly becomes overwhelming. Additionally, while it is potentially possible to repeat such methods in higher dimensions, the visualization of the proofs played a crucial role in this analysis. In combination with the added degrees of freedom from adding dimensions, this would make this method of computation quickly intolerable.

Future work may tighten our upper bound on $P(k)$. However, we make the following conjecture.

Conjecture 5.1. *The lower bound on $P(k)$ in Theorem 1.3 is tight. Namely, $P(2k) = 2k + 3$ and $P(2k + 1) = 2k + 3$ for all $k \geq 1$.*

Therefore, we believe that future work should improve the upper bound of $P(n) \leq 6n$, either via progress towards the Weak Dirac Conjecture (which would still fall short of our conjecture) or by some other means. Alternatively, future research may find a more efficient method of constructing viable point sets without the need for the exhaustive search we perform.

We propose the related problem of characterizing optimal point sets in higher dimensions with a low number of *solid angles.*

Definition 5.2 (Solid Angles)**.** *Given $d+1$ points in $\mathbb{R}^d$, fix one of the points p. Let S be a unit d-dimensional hypersphere about p. Project the remaining d points onto the surface of the sphere along the lines connecting them to p. The* ***solid angle*** *formed by the $d + 1$ points with center p is the surface area of S enclosed by the projections of the other points onto S and connected via geodesics.*

Solid angles have applications to physics and have not been extensively studied in the context of discrete geometry. They provide an exciting new avenue for angle-related problems.

They also motivate the following problem. For a fixed $d \geq 3$, what is the maximum number of non-coplanar points in a configuration yielding at most k solid angles?

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