

Experimental Mathematics: Discovering the Product Rule

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Abstract: Similar to the physical sciences, mathematics can have an experimental component, though it's often ignored when people are first learning the subject. Using the Product Rule as a motivating example, we showcase methods to sniff out this complicated, initially unintuitive formula. In the course of doing so, we see not only the right answer, but a suggestion of how to prove it and how to predict answers for other problems.

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1 INTRODUCTION

Scientists often make discoveries by beginning with simple observations and experiments, and then drawing tentative conclusions from the patterns they observe. Interestingly, we can view mathematics as an experimental science as well! We can start by studying simple examples, try to observe patterns and gain insights, and then conjecture a formula for the general case based on our observations; of course, as mathematicians, we don't stop here, and we also try to *prove* our conjectured formulas!

Frequently, lectures in introductory classes skip this stage of gathering data, proceeding immediately to a correct but uninspired derivation. Each step is understandable, as are the transitions from one line to the next, but what is lost is any insight as to how someone thought to take this path. This is a pity, as it makes it significantly harder for the student to attack a new problem. The goal of this article is to try to remedy this lost opportunity, and describe how to collect good data that can reveal a path towards a proof. Our motivating example is the Product Rule from Calculus.

It's straightforward to show that the standard limit laws imply the derivative of a sum is the sum of the derivatives (and similarly for the derivative of a difference):

$$(f(x) + g(x))' = f'(x) + g'(x). \quad (1)$$

For generations, calculus students have hoped that products behave in a similarly simple way, namely that the derivative of a product is the product of the derivatives:

$$(f(x)g(x))' =?? f'(x)g'(x). \quad (2)$$

This hope, however, isn't true¹, and students quickly learn the correct formula:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x). \quad (3)$$

¹Sometimes we are lucky and our hopes are realized. A great example is the famous *Sophomore's Dream*: Johann Bernoulli [1, 5] proved that $\sum_{n=1}^{\infty} n^{-n} = \int_0^1 x^{-x} dx$.

We describe how to experimentally derive the correct formula for the derivative of a product. We hope the principles we describe below help the reader as they encounter new problems in the course of their studies and research.

2 A Simple Conjecture

Although the Product Rule is a basic result in Calculus whose form is known to all (successful) students, discovering this formula would have been far from obvious if we were the first to attempt to state and then derive it. For example, let's put ourselves in the position of the discoverers of Calculus.² Let's imagine that we've worked hard and know the derivatives of what we'll call "elemental" functions³:

$$(x^n)' = nx^{n-1}, \quad (\sin(x))' = \cos(x), \quad (\cos(x))' = -\sin(x), \quad (e^x)' = e^x. \quad (4)$$

Using those functions, it's natural to explore the derivative of their combinations: for example, $\cos(x) + x^n$, $\sin(x)e^x$, $\sin(x)/x^n$.

Very quickly, as mentioned above, we find the derivative of a sum (or difference) is the sum (or difference) of the derivatives. Similarly we get the constant and multiple rules, but we don't have the power, chain, inverse and of course we don't have the product or quotient rules. Thus in addition to the derivatives of the basic functions in (4) we assume we also know:

- *Sum/Difference Rule:* $(f(x) \pm g(x))' = f'(x) \pm g'(x)$,
- *Constant Rule:* $(cf(x))' = cf'(x)$, and
- *Multiple Rule:* $(f(cx))' = cf'(cx)$.

What about the derivative for the product of two elemental functions? Based on the derivative for the sum $(f(x) + g(x))' = f'(x) + g'(x)$, as remarked we might guess the following.

Conjecture 1. *The derivative of a product is the product of the derivatives.⁴ Thus if f and g are differentiable functions, then*

$$(f(x)g(x))' = f'(x)g'(x). \quad (5)$$

²Although the modern consensus holds that Isaac Newton and Gottfried Wilhelm Leibniz developed Calculus independently, many of its foundational ideas were already known before their work. For example, Pierre de Fermat made significant advances in methods for finding tangents and extrema, which anticipated later differential techniques. Likewise, early forms of the Fundamental Theorem of Calculus appeared in the work of predecessors such as Isaac Barrow. Nevertheless, it was Newton and Leibniz who synthesized these scattered results into a coherent and systematic framework, thereby giving rise to the modern formulation of Calculus. For further reading on the history of Calculus, see [2, 3].

³While finding the derivative of a polynomial is straightforward if the power is an integer, it requires a bit of work if it's real, in particular the exponential and logarithmic functions and the chain rule! Similarly there are challenges in finding the derivatives of trigonometric functions; these formulas only hold if their argument is in radians. Finally, the exponential function notation hides a lot of math; $e^x e^y = e^{x+y}$ is not true by laws of exponents, but is actually a statement that the product of two infinite sums is a third infinite sum! See for example Appendix B.5 of [4].

⁴In fact, Leibniz entertained a similar conjecture when he first attempted to derive the Product Rule. He initially speculated that $(f(x)g(x))' = f'(x)g'(x)$, but he quickly dismissed this idea after observing a counterexample, for instance when $f(x) = g(x) = x$. Likewise, in his search for the Quotient Rule, he briefly considered whether $(f(x)/g(x))' = f'(x)/g'(x)$, before again rejecting the conjecture upon finding counterexamples. A detailed historical account of Leibniz's discovery of the Product Rule and related results in Calculus can be found in Chapter 9 of [3].

After formulating a conjecture, we should pause and ask: how reasonable is this conjecture? To test it, we first identify some natural properties that any valid Product Rule should satisfy. Since we are finding a formula for the product of two functions, the formula should, at a minimum, satisfy the following properties.

- *Symmetry Property*: As multiplication is commutative, we need $(f(x)g(x))' = (g(x)f(x))'$.
- *Identity Property*: It reduces to $f'(x)$ or $g'(x)$ when one of the functions is the constant function 1. For example, if $g(x)$ is the constant function 1, then the Product Rule must give $(f(x) \cdot 1)' = f'(x)$.

Observe that these two properties reflect the basic behavior we expect from ordinary multiplication of functions. We see that (5) easily satisfies the *Symmetry Property*, since

$$(f(x)g(x))' = f'(x)g'(x) = g'(x)f'(x) = (g(x)f(x))'. \quad (6)$$

Now, we try whether it satisfies our *Identity Property*. Let $g(x) = 1$. We would expect the Product Rule gives us $f'(x)$. Using the conjectured formula (5), however, we find that

$$f'(x) = (f(x) \cdot 1)' = f'(x) \cdot 0 = 0, \quad (7)$$

regardless of what $f(x)$ is. This offers a counterexample to our conjecture. We see that not only does our conjecture fail, but if it happened to be true then there could be no motion at all!

Clearly, we need to refine our conjecture. Reflecting on what we have so far, we know that the Product Rule cannot be a simple product like the one we conjectured. Later in §4, we'll see how these two simple properties and good special cases help us deduce the formula for the Product Rule. For now, we'll just try a direct approach, namely choosing some functions where we know their derivatives and the derivative of their product, and hope that is enough to lead us to the right answer.

3 Gathering Data

3.1 Monomials

We now attempt to sniff out the correct formula by experimenting with simple products. Our strategy is to gather data from examples we already understand well. We do not yet know how to differentiate the product of two arbitrary functions, so we restrict our attention to functions for which we can compute the derivative of their product directly. One nice choice is monomials: polynomials with just one term. These are perhaps the easiest functions to study, as the product of two monomials is again a monomial, and we already know how to differentiate such functions.

Which monomials should we use? We could start with $f(x) = x$ and $g(x) = x^2$, then maybe try the pair x and x^3 , and then x^2 and x^3 perhaps and so on. However, rather than doing a lot of special cases one at a time, we can consider infinitely many monomials by fixing positive integers n, m and considering the functions

$$f(x) = x^n, \quad g(x) = x^m \quad (8)$$

with derivatives

$$f'(x) = nx^{n-1}, \quad g'(x) = mx^{m-1}. \quad (9)$$

Since the product $f(x)g(x)$ is just the monomial x^{n+m} , we can compute its derivative directly:

$$(f(x)g(x))' = (n+m)x^{n+m-1}. \quad (10)$$

Based on the above expression, it's natural to guess that the derivative of $f(x)g(x)$ is some combination of $f(x)$, $f'(x)$, $g(x)$, and $g'(x)$:

$$\begin{aligned} f(x) &= x^n, & f'(x) &= nx^{n-1} \\ g(x) &= x^m, & g'(x) &= mx^{m-1}. \end{aligned} \quad (11)$$

To deduce the Product Rule from (11), we can first expand (10) as

$$(n+m)x^{n+m-1} = nx^{n+m-1} + mx^{n+m-1}, \quad (12)$$

and we can try to match this expression with the functions in (11). We can now ask: how can we get x^{n+m-1} from the functions given in (11)? After some algebra, we see that there are two ways that work for all n and m : $f'(x)g(x) = nx^{m+n-1}$ and $f(x)g'(x) = mx^{m+n-1}$. While we could also try some exotic combinations⁵ that work for specific n and m , these are the only natural choices (and show the advantage of using free parameters n and m instead of specific values).

Notice that the addition of $f'(x)g(x)$ and $f(x)g'(x)$ naturally gives us

$$(x^n x^m)' = (n+m)x^{n+m-1} = \underbrace{nx^{n+m-1}}_{f'(x)g(x)} + \underbrace{mx^{n+m-1}}_{f(x)g'(x)}. \quad (13)$$

Recalling that $f(x) = x^n$ and $g(x) = x^m$, we observe the following relationship for the case of monomials

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x). \quad (14)$$

This looks like another promising candidate for the Product Rule. Now, we have found a new formula and we should ask whether this formula satisfies the *Symmetry Property* and *Identity Property* that we discussed before. We see that this formula is clearly symmetric, since

$$\begin{aligned} (f(x)g(x))' &= f'(x)g(x) + f(x)g'(x) \\ &= g'(x)f(x) + g(x)f'(x) \\ &= (g(x)f(x))'. \end{aligned} \quad (15)$$

It also satisfies the *Identity Property* easily. If $g(x) = 1$, then

$$\begin{aligned} (f(x)g(x))' &= (f(x) \cdot 1)' \\ &= f'(x) \cdot 1 + f(x) \cdot 0 \\ &= f'(x), \end{aligned} \quad (16)$$

giving the desired result. We omit the other case here, since it uses the same argument.

⁵If $n = 5$ and $m = 4$ note $x^{n+m-1} = x^8 = (x^5/x^4)^8 = f(x)^8/g(x)^8$, but we don't expect something like this to be true in general.

3.2 Trigonometric Functions

Of course, our guess for the Product Rule, (14), might only hold for monomials. Perhaps we don't always have a sum as we thought. It's possible our choices were too simple, and we're not seeing the full richness of behavior. Thus, to fortify our confidence that the derivative of the product is given by (14), it would help to explore different functions.

Since we know that it holds for monomials, one natural guess is that it should also hold for polynomials. It turns out that this is true, and we state this result formally.

Theorem 2. *Let n, m be positive integers. If*

$$f(x) = \sum_{i=0}^n a_i x^i \quad \text{and} \quad g(x) = \sum_{j=0}^m b_j x^j, \quad (17)$$

then the derivative of their product equals

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x). \quad (18)$$

We leave the proof as an exercise for you, and we strongly urge you to do this calculation. *The above is a great example of an important general principle: do a simple case (here two monomials), and then massively generalize!*

Building on our success with polynomials, can we test our conjecture with other classes of functions? These new choices have to be such that not only do we know their derivatives, but we also know the derivative of their product. Are there such candidates? Yes, if we remember some identities! Consider the trigonometric functions

$$f(x) = \sin(x) \quad \text{and} \quad g(x) = \cos(x). \quad (19)$$

Remember, this is a good choice only if we can directly compute the derivative of the product

$$f(x)g(x) = \sin(x)\cos(x). \quad (20)$$

Can we compute this derivative without the Product Rule? Fortunately, we can compute this derivative if we recall the double-angle identity

$$2\sin(x)\cos(x) = \sin(2x), \quad (21)$$

as well as the Multiple Rule

$$(f(cx))' = cf'(cx). \quad (22)$$

While the Multiple Rule is a special case of the Chain Rule, which is a bit of work to prove, we can prove this case directly.⁶

Using the double-angle identity, we can rewrite the product (20) as

$$f(x)g(x) = \sin(x)\cos(x) = \frac{1}{2} \cdot 2\sin(x)\cos(x) = \frac{1}{2}\sin(2x), \quad (23)$$

⁶*Hint:* We can view $f(cx)$ as changing the passage of time by a factor of c , which then changes our speed by the same factor. With the function $f(x)$ it took 1 unit of time to cover the distance $f(1) - f(0)$, but with the function $f(cx)$ we cover that same distance in $1/c$ units of time. As the derivative is the limit of the distance traveled over the time elapsed, as the time tends to zero, we see that scaling the time by $1/c$ is equivalent to multiplying the derivative by c , proving the Multiple Rule.

where we have applied the trick of multiplying by 1 in the middle. We've transformed the product into a simple trigonometric function, and we can easily compute its derivative⁷ as

$$(f(x)g(x))' = \frac{1}{2}(\sin(2x))' = \frac{1}{2} \cdot 2\cos(2x) = \cos(2x), \quad (24)$$

where we used the Multiple Rule to compute the derivative of $\sin(2x)$. This gives the left-hand side of our conjecture, (14).

To verify our conjecture, we now compute the right-hand side. Simple algebra shows that

$$\begin{aligned} f'(x)g(x) + f(x)g'(x) &= \cos(x)\cos(x) - \sin(x)\sin(x) \\ &= \cos(2x), \end{aligned} \quad (25)$$

where we used the double-angle formula for $\cos(2x)$ in the last step. We see that our proposed formula, (14), also holds for this pair of trigonometric functions.

Not only is this another vindication of our proposed Product Rule, but as these are far more complicated functions⁸ than the monomials x^n and x^m , we should have significantly more confidence that we're on the right path! For those wishing more evidence, you can try the product of two exponential functions, though sadly we cannot do a monomial times an exponential.

You could also try other products of trigonometric functions. For example, recall that

$$\cos(2x) = \cos(x)\cos(x) - \sin(x)\sin(x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1, \quad (26)$$

where we used $\cos^2(x) + \sin^2(x) = 1$. This then gives $\cos^2(x) = \frac{1}{2}\cos(2x) + \frac{1}{2}$. We can take the derivative of all functions in sight; for example, we can test (14) for $f(x) = g(x) = \cos(x)$ or $f(x) = g(x) = \sin(x)$.

4 Symmetry and Special Case Arguments

In §2, we have identified two simple properties that a valid Product Rule should satisfy: the *Identity Property* and the *Symmetry Property*. In §3.1, we used these two properties to verify whether our conjectured formula (14) is reasonable. It turns out that we can also use these two properties to deduce the Product Rule.

We know the formula for the derivative of a product has to satisfy the *Identity Property*: if $f(x) = 1$ the derivative of $f(x)g(x)$ must be $g'(x)$, while if $g(x) = 1$ then it is $f'(x)$. This suggests that perhaps the answer has the following shape:

$$(f(x)g(x))' = A(f, g)f'(x) + B(f, g)g'(x), \quad (27)$$

where $A(f, g)$ and $B(f, g)$ can be very complicated expressions involving f, g and their derivatives, *but* if we let $\mathbf{1}$ represent the constant function 1, then

$$A(f, \mathbf{1}) = 1 \quad \text{and} \quad B(\mathbf{1}, g) = 1. \quad (28)$$

⁷We need to use the Constant Rule, which is another easily established Calculus result: the derivative of $cf(x)$ is just $cf'(x)$.

⁸We could also try to argue that the result for these trigonometric functions follows from the result for polynomials but using the infinite Taylor expansions for $\sin(x)$ and $\cos(x)$; however, that would require us to know Taylor series before proving the product formula!

This requirement on $A(f, g)$ and $B(f, g)$ would allow (27) to satisfy the *Identity Property*. To see this, we let $g(x) = 1$ and $f(x)$ be a non-constant function. This then gives

$$\begin{aligned}(f(x)g(x))' &= A(f, 1)f'(x) + B(f, 1) \cdot 0 \\ &= f'(x),\end{aligned}\tag{29}$$

where $g'(x) = 0$ and the derivative of the product is just $f'(x)$. The other case can be verified using the same argument.

Our formula (27) satisfying the *Identity Property* is not enough. We also need it to satisfy the *Symmetry Property*: the derivative of $f(x)g(x)$ is the same as the derivative of $g(x)f(x)$. Formally, we need

$$\begin{aligned}(f(x)g(x))' &= (g(x)f(x))' \\ A(f, g)f'(x) + B(f, g)g'(x) &= A(g, f)g'(x) + B(g, f)f'(x).\end{aligned}\tag{30}$$

But this equality further implies the following constraint on $A(f, g)$ and $B(f, g)$:

$$A(f, g) = B(g, f) \quad \text{and} \quad A(g, f) = B(f, g),\tag{31}$$

The two requirements on $A(f, g)$ and $B(f, g)$, (28) and (31), can help us derive the Product Rule. To see this, we need to determine explicit expressions for $A(f, g)$ and $B(f, g)$, and we can do so by computing the derivative of some products (again, the idea of gathering data).

Let's go back to the case of monomials and consider $f(x) = x^n$ and $g(x) = x^m$. As we suspect the derivative of a product is of the form

$$(f(x)g(x))' = A(f, g)f'(x) + B(f, g)g'(x),\tag{32}$$

substituting what we know for the derivatives of $f(x)$, $g(x)$ and $f(x)g(x)$ yields

$$(n + m)x^{n+m-1} = A(x^n, x^m)nx^{n-1} + B(x^n, x^m)mx^{m-1}.\tag{33}$$

Using (31), we see that $A(x^n, x^m) = x^m$ and $B(x^n, x^m) = x^n$. This means that, for this pair of functions, we have $A(f, g) = g(x)$ and $B(f, g) = f(x)$. As a nice check, note that substituting $A(f, g) = g(x)$ and $B(f, g) = f(x)$ into (32) gives the correct Product Rule. Just like what we have done in §3, we can test whether (27) gives us the Product Rule for other pairs of functions, and we leave this as an exercise for you.

We end our tour with a quick lesson on how symmetry and good special cases can help us discover the right formula. Consider the simple case: $f(x) = g(x) = x^n$. Note we are not assuming we know the Power Rule, we are again choosing functions where we can differentiate the product. We find that the derivative of the product $f(x)g(x)$ is equal to

$$(x^n x^n)' = (x^{2n})' = 2nx^{2n-1}.\tag{34}$$

Since $f'(x) = nx^{n-1}$, we can re-write it as

$$\begin{aligned}2nx^{2n-1} &= nx^{2n-1} + nx^{2n-1} \\ &= nx^{n-1} \cdot x^n + x^n \cdot nx^{n-1} \\ &= f'(x)f(x) + f(x)f'(x).\end{aligned}\tag{35}$$

In this process, we exploited symmetry again by splitting $2nx^{2n-1}$ into two equal pieces. This once more gave us the form of the Product Rule. While not a proof, the above is highly suggestive that our conjectured Product Rule is on the right track.

5 Proving the Product Rule

Our analysis of monomials, polynomials, and some special trigonometric functions suggest the Product Rule is given by (14). For completeness, we give the standard proof.

Theorem 3 (Product Rule). *If f and g are differentiable functions, then*

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x). \quad (36)$$

Proof. From the definition of the derivative, we have

$$(f(x)g(x))' = \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}. \quad (37)$$

Our previous investigations suggest how to proceed. We believe the answer is $f'(x)g(x) + f(x)g'(x)$; thus, somehow, we need to perform some algebraic operations inside the limit so this quantity emerges.

Our first term is $f(x+h)g(x+h)/h$; the part $f(x+h)/h$ is a good start towards getting an $f'(x)$ if we send h to zero. To get that, we need of course an $f(x)$. This suggests we add zero in the form

$$0 = f(x)g(x+h) - f(x)g(x+h) \quad (38)$$

and then group accordingly. Thus we have

$$\begin{aligned} & f(x+h)g(x+h) - f(x)g(x) \\ &= f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x) \\ &= [f(x+h) - f(x)]g(x+h) + f(x)[g(x+h) - g(x)]. \end{aligned} \quad (39)$$

Substituting the above into (37), and using the limit of a sum or product is the sum or product of the limits (so long as at least one of the limits is finite) yields

$$\begin{aligned} (f(x)g(x))' &= \lim_{h \rightarrow 0} \frac{[f(x+h) - f(x)]g(x+h) + f(x)[g(x+h) - g(x)]}{h} \\ &= \lim_{h \rightarrow 0} \frac{[f(x+h) - f(x)]g(x+h)}{h} + \lim_{h \rightarrow 0} \frac{f(x)[g(x+h) - g(x)]}{h} \\ &= \left(\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \right) \left(\lim_{h \rightarrow 0} g(x+h) \right) + \left(\lim_{h \rightarrow 0} f(x) \right) \left(\lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \right) \\ &= f'(x)g(x) + f(x)g'(x), \end{aligned} \quad (40)$$

completing the proof. \square

We conclude with an alternative proof of the Product Rule using a geometric argument. To many students, this is a more intuitive proof. We can view $f(x)g(x)$ as representing the area of a rectangle with side lengths $f(x)$ and $g(x)$, and interpret $(f(x)g(x))'$ as the instantaneous rate of change of this area with respect to x . Assume $f(x)$ and $g(x)$ are positive so that the rectangle is well-defined, though we can handle the other possibilities by a case analysis. If we increase x by h the side lengths grow from $f(x)$ and $g(x)$ to $f(x+h)$ and $g(x+h)$.

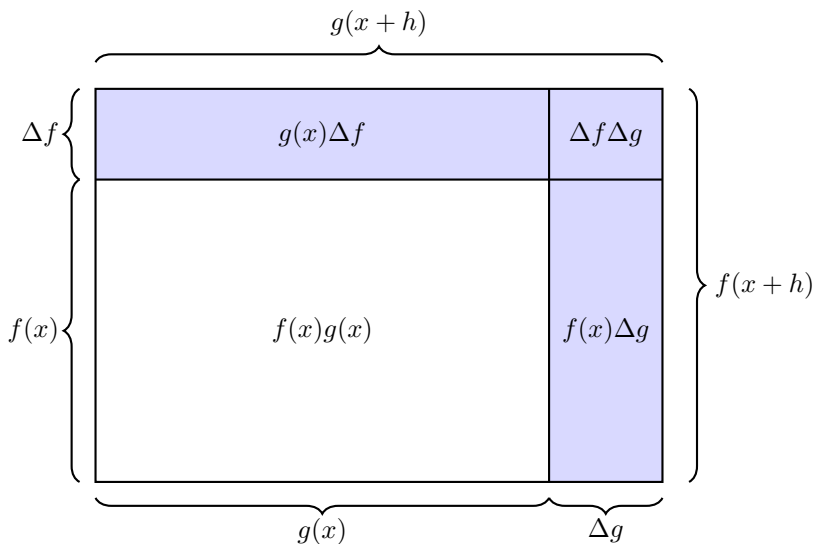


Figure 1. Geometric illustration of the Product Rule.

Thus the increases in side lengths of

$$\Delta f = f(x+h) - f(x) \quad \text{and} \quad \Delta g = g(x+h) - g(x) \quad (41)$$

lead to an increase in area, represented as the shaded regions in Figure 1.

So $\Delta(fg)$, the change in the area of rectangle due to an increase of x by h , is

$$\begin{aligned} \Delta(fg) &= f(x)\Delta g + g(x)\Delta f + \Delta f\Delta g \\ &= f(x)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)] \\ &\quad + [f(x+h) - f(x)][g(x+h) - g(x)]. \end{aligned} \quad (42)$$

If we divide both sides by h and take $h \rightarrow 0$, then we see the derivatives of f and g at x emerge and obtain the instantaneous rate of change of the area with respect to x :

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x), \quad (43)$$

which is the Product Rule! Of course, this is only a sketch of the proof (remember we assumed $f(x)$ and $g(x)$ are positive, did we also assume $f(x+h) > f(x)$ for $h > 0$?), and a rigorous proof requires more details. We leave this as an exercise for you.

REFERENCES

- [1] J. Bernoulli, *Opera omnia* (1697), vol. **3**, 376–381.
- [2] Carl B. Boyer, *The History of the Calculus and Its Conceptual Development*, Dover Publications, 1959.
- [3] C. H. Edwards, Jr., *The Historical Development of the Calculus*, Springer-Verlag, New York, 1979.
- [4] S. J. Miller, *The Probability Lifesaver*, Princeton University Press, 2017.
- [5] Wikipedia, *Sophomore's Dream*, https://en.wikipedia.org/wiki/Sophomore%27s_dream.