

Black Hole Zeckendorf Games

Caroline Cashman^a, Steven J. Miller^b, Jenna Shuffelton^c, and Daeyoung Son^a,

^aCollege of William & Mary, Williamsburg, VA; ^bWilliams College, Williamstown, MA; ^cWilliams College, Williamstown, MA; ^dWilliams College, Williamstown, MA

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ABSTRACT

Zeckendorf proved that every positive integer can be written as a decomposition of non-adjacent Fibonacci numbers. Baird-Smith, Epstein, Flint, and Miller converted the process of decomposing an integer n into a 2-player game, using the moves of $F_i + F_{i-1} = F_{i+1}$ and $2F_i = F_{i+1} + F_{i-2}$, where F_i is the i th Fibonacci number. They showed non-constructively that for $n \neq 2$, Player 2 has a winning strategy: a constructive solution remains unknown.

We expand on this by investigating “black hole” variants of this game. The F_m Black Hole Zeckendorf game is played with any n but solely in columns F_i for $i < m$. Gameplay is similar to the original Zeckendorf game, except any piece that would be placed on F_i for $i \geq m$ is locked out in a “black hole” and removed from play. With these constraints, we analyze the games with black holes on F_3 and F_4 and construct a solution for specific configurations, using a non-constructive proof to lead to a constructive one. We also examine a pre-game in which players take turns placing down n pieces in the outermost columns before the decomposition phase, and find constructive solutions for any n .

KEYWORDS

Zeckendorf decompositions; Fibonacci numbers; game theory; recreational mathematics

1. Introduction

1.1. Background

The beauty of the Fibonacci numbers is undeniable: a simple sequence, recursively defined by the sum of the two previous numbers, that has the tendency to show up in both natural and surprising places. Indexing so that $F_1 = 1$, $F_2 = 2$ and $F_{k+1} = F_k + F_{k-1}$, Zeckendorf proved a particularly interesting fact about the Fibonacci numbers, namely that any positive integer n can be written as the sum of non-adjacent Fibonacci numbers, known as the number’s Zeckendorf decomposition [14]. Baird-Smith, Epstein, Flint, and Miller [2, 3] created a game from the process of converting a positive integer into its Zeckendorf decomposition using the moves of $F_i + F_{i-1} = F_{i+1}$ and $2F_i = F_{i+1} + F_{i-2}$, where F_i is the i th Fibonacci number. We outline the rules to the original Zeckendorf game as follows.

- (1) **Setup:** The game is played on a board with columns corresponding to each of the Fibonacci numbers, indexing so that the first column corresponds with $F_1 = 1$, the second

CONTACT Caroline Cashman. Email: cecashman@wm.edu

CONTACT Steven J. Miller. Email: sjm1@williams.edu

CONTACT Jenna Shuffelton. Email: jms13@williams.edu

CONTACT Daeyoung Son. Email: ds15@williams.edu

column corresponds with $F_2 = 2$ and the m th column corresponds with F_m , the m th Fibonacci number. All n pieces begin in the first column.

- (2) **Gameplay:** Players alternate, selecting their moves from the following.
- (a) Adding consecutive terms: If the board contains pieces in both F_i and F_{i-1} columns, players can remove one piece from each column to add as one piece in the F_{i+1} column.
 - (b) Merging 1's: If the board contains more than one piece in the F_1 column, players can remove two pieces from the F_1 column to merge as one piece in the F_2 column.
 - (c) Splitting: If the board contains more than one piece in the F_2 column, players can split two pieces from the F_2 column to place one piece in each F_1 and F_3 . For $i \geq 3$, players can split two pieces in the F_i column to place one in each F_{i-2} and F_{i+1} .
- (3) **Winning:** The last player to move wins.

They proved that the game is playable, meaning it always ends in finite time, and that the final board placed down will be equal to the Zeckendorf decomposition of n . Moreover, they showed that for all $n \neq 2$, Player 2 has a winning strategy. Notably, this is not a constructive winning strategy, and instead relies on a parity stealing argument. If one assumes that Player 1 has a winning strategy, Player 2 later has the opportunity to steal it, therefore Player 2 must have a winning strategy. With increasing n , the number of possible game positions grows exponentially, making the construction of a winning solution for Player 2 challenging. Multiple variations of this game have been studied; see [6, 4, 9, 10, 8, 11, 12, 13]. In order to develop a greater understanding of the original Zeckendorf game, we consider a variation occurring on a smaller board.

1.2. Main Results

We consider an “ F_m Black Hole” variation of the Zeckendorf game, where once a piece is placed on some F_i for $i \geq m$, it falls into the “Zeckendorf Black Hole” and is permanently removed from game play. This variant reduces the number of possible moves a player has, making the game easier to analyze. We combine this Black Hole variation with an Empty Board variation, where the game begins with an empty board, and players take turns placing down pieces in the outermost columns until the weighted sum equals the starting value n . The last player to place down a piece moves second in the decomposition phase of the game, assuming the role of Player 2 from the original Zeckendorf game. Combining these two variations is interesting for a variety of reasons. The solution to the F_m Black Hole Zeckendorf game is heavily based on modular arithmetic, while the solution to the Empty Board Zeckendorf game uses a move mirroring strategy common in combinatorial game theory.

We quickly consider the Empty Board F_m Black Hole Zeckendorf game with black holes on F_1 , which is not playable, and on F_2 , which is deterministic. We then shift our attention to the Empty Board F_m Black Hole Zeckendorf game with black holes on F_3 and then on F_4 . We determine which player has a winning strategy for any positive integer n , and provide a constructive solution.

1.2.1. Terminology

We first clarify some terminology. We refer to any column corresponding with the i th Fibonacci number as the F_i column. The number of pieces in a column at any given game state is a for the F_1 column, b for the F_2 column, and c for the F_3 column, resulting in a game state (a, b, c) .

Because our solutions are based on modular arithmetic, we also describe game states in terms of α, β, γ and k_1, k_2, k_3 , where α and k_1 correspond with the F_1 column, β and k_2 correspond with the F_2 column, and γ and k_3 with the F_3 column.

To further clarify which player wins at a given position, we use the terms Player 1 and Player 2 only when talking about the game as a whole. For a given position within the game, we describe it as either P or N , which is standard notation within combinatorial game theory; see [1], [5], [7]. A certain game state is a P position if the previous player who has placed it has a winning strategy. A game state is an N position if the next player to play has a winning strategy. If a given position is P , all following possibilities must be N positions, and if a given position is N , at least one of the following possibilities must be a P position. When it is necessary to explain our game trees, we assume that Red is the first to move from a given position.

1.2.2. F_3 Results

With a black hole on F_3 , pieces can only be placed in the F_1 and F_2 columns. The winner of all possible games can be determined based on the value of a and b modulo 3. We describe a board setup (a, b) as $(3\alpha + k_1, 3\beta + k_2)$, where $\alpha, \beta, k_1, k_2 \in \mathbb{Z}^{\geq 0}$, $0 \leq k_1, k_2 \leq 2$. We find that (a, b) is a P position for all $a \equiv b \equiv 0$, $a \equiv 0, b \equiv 1$ or $a \equiv 1, b \equiv 0$. Any other setup is an N position. It follows that when placing n pieces on an empty board, Player 1 has a constructive winning strategy for $n \equiv 1, 2, 3, 6, 8 \pmod{9}$ and Player 2 has a constructive winning strategy when $n \equiv 0, 4, 5, 7 \pmod{9}$.

1.2.3. F_4 Results

When the black hole moves to F_4 , the game immediately becomes more interesting. The game is relatively straightforward for $(a, 0, 0)$, which is a P position for all $a \neq 2$. The game is also relatively straightforward for $(0, 0, c)$, which is an N position for all $c \neq 0, 1, 5$.

However, once the board is in the position $(a, 0, c)$, the minor exceptions from the $(a, 0, 0)$ and the $(0, 0, c)$ cases become incredibly influential in the game reduction. Still, we are able to simplify the game, and determine winners as outlined in Figure 1. Then, we describe the board setup of $(a, 0, c)$ as $(3\alpha + k_1, 0, 4\gamma + k_3)$, where $\alpha, \gamma, k_1, k_3 \in \mathbb{Z}^{\geq 0}$, $0 \leq k_1 \leq 2$, and $0 \leq k_3 \leq 3$. We show the winners based on the values of α and γ below, denoting P positions in bold blue and N positions in red.

	$a \equiv 0 \pmod{3}$	$a \equiv 1 \pmod{3}$	$a \equiv 2 \pmod{3}$
$c \equiv 0 \pmod{4}$	$\alpha \geq \gamma$ $\alpha \leq \gamma - 1$	$\forall \alpha, \gamma$	$\alpha \geq \gamma + 1$ $\alpha \leq \gamma$
$c \equiv 1 \pmod{4}$	$\alpha \geq \gamma - 1$ $\alpha \leq \gamma - 2$	$\forall \alpha, \gamma$	$\alpha \geq \gamma$ $\alpha \leq \gamma - 1$
$c \equiv 2 \pmod{4}$	$\forall \alpha, \gamma$	$\alpha \geq \gamma + 1$ $\alpha \leq \gamma$	$\forall \alpha, \gamma$
$c \equiv 3 \pmod{4}$	$\forall \alpha, \gamma$	$\alpha \geq \gamma$ $\alpha \leq \gamma - 1$	$\forall \alpha, \gamma$

Figure 1: Winners for board setups $(a, 0, c)$ in an F_4 Black Hole Zeckendorf game. P positions are depicted in bold blue, and N positions are depicted in red.

With this knowledge, we continue onto the Empty Board F_4 Black Hole Zeckendorf game, where players may only place in the outermost columns. We define the game in this way as the

general case gives players far more options, which poses a variety of challenges for determining a constructive solution. We find that Player 1 has a constructive winning solution for all $n \equiv 1, 3, 5, 7, 8, 10, 12, 14, 15 \pmod{16}$ such that $n \neq 17, 47$, for which Player 2 has a winning solution. Player 2 has a constructive winning solution for all $n \equiv 0, 2, 4, 6, 9, 11, 13 \pmod{16}$ such that $n \neq 2, 32$, for which Player 1 has a winning solution. For large values of n , $\alpha \geq \gamma + 1$, so it is possible to quickly determine winners by Figure 1. For smaller values of n , it is necessary to explicitly consider certain values of n , as the winner is then dependent on the values of k_1 and k_3 , which can lead to exceptions, as with $n = 2, 17, 47, 32$.

Moving the black hole to any F_m such that $m \geq 5$ limits both players' ability to reduce down to a game with a known solution, meaning that a constructive solution is no longer immediately apparent. The fact that there exists no obvious constructive solution to a Zeckendorf game in as few as four columns supports the conjectured complexity of the original Zeckendorf game.

2. Rules

We define an Empty Board F_m Black Hole Zeckendorf game as follows.

- (1) **Setup:** The game begins on an empty board with $m - 1$ columns. The F_i column is weighted to correspond with the i th Fibonacci number, indexing so that $F_1 = 1$ and $F_2 = 2$. Players start the game with any positive integer n pieces.
- (2) **Placing the pieces:** Players take turns placing one piece in the outermost columns of the board, namely F_1 and F_{m-1} , until the weighted sum equals n . For us, this is equivalent to placing in columns F_1 for an initial board setup (a) , F_1 and F_2 for an initial board setup (a, b) , and in columns F_1 and F_3 for an initial board setup $(a, 0, c)$ ¹. Placing one piece in the F_i column removes F_i pieces from the pile of n , so players can only place in columns such that F_i is less than or equal to the number of pieces left. This stage ends when there are no pieces left to be placed. The last player to place down a piece moves second in the decomposition phase of the game. Thus, it is the goal of both players to either set the board as a P position, or to force their opponent to set the board as an N position.
- (3) **Decomposition:** Players now begin the decomposition phase of the game. Players alternate, selecting their moves from the following.
 - (a) *Add:* Add one piece from each F_i and F_{i+1} to combine as one piece on F_{i+2} .
 - (b) *Merge:* Merge two pieces from F_1 into one piece in column F_2 .
 - (c) *Split:* Split two pieces in column F_2 into one in each column F_1 and F_3 or, for $i \geq 3$, split two pieces in column F_i into one in each column F_{i-2} and column F_{i+1} .
- (4) **Black Hole:** Note that in the moves above, it is possible for pieces to be placed in F_m . In this situation, they become trapped in the "Zeckendorf Black Hole," where they are permanently removed from the board.
- (5) **Winning:** The last player to move wins the game. In combinatorial game theory, this is commonly referred to as normal play.

Theorem 2.1. *The Empty Board Black Hole Zeckendorf game is playable and always ends at the Zeckendorf decomposition of $n \pmod{F_m}$.*

Proof. The Empty Board portion of the game does not affect whether the game is playable, as

¹We are not able to provide a solution for the general case, as during the decomposition phase neither player is able to reduce the value of the board without giving the other player the option to do so first.

there are a finite number of pieces to place, by definition always resulting in a setup with some integer number of pieces in the F_1 and F_{m-1} columns.

Since the Zeckendorf game is playable and always results in the Zeckendorf decomposition of n , it is sufficient to show that any Black Hole Zeckendorf game with n pieces and a black hole on F_m reduces to a board such that the weighted sum of pieces is $n \pmod{F_m}$, as that is equivalent to a board in the original Zeckendorf game. From here, the game proceeds as the general Zeckendorf game does, so it will reduce to the Zeckendorf decomposition of $n \pmod{F_m}$.

Both the add and merge options combine two pieces into one piece in a column corresponding to a greater F_m , therefore closer to the black hole. The split option moves one piece towards the black hole and one piece away from it. Since there are a finite number of pieces, all moves shift at least the same amount of pieces towards the black hole as they do away from it, meaning that pieces must eventually be placed into the black hole if the weighted sum of pieces is greater than F_m . Every time a piece is placed in the black hole, it decreases the value of the board by F_m . Thus, the game must eventually be reduced to a board such that the weighted sum of all pieces is $n \pmod{F_m}$. \square

Note that the weighted sum of all pieces on the board is a non-increasing monovariant, as pieces must eventually be placed into the black hole and every time a piece is placed in the black hole, it reduces the value of the board by F_m . This is different from the original Zeckendorf game, in which the number of pieces on the board is a non-increasing monovariant, but the weighted sum of the board itself is constant.

We consider the games with black holes on F_2 , F_3 , and F_4 . For the game with a black hole on F_2 , possible outcomes are (0) and (1). For the game with a black hole on F_3 , possible outcomes are (0, 0), (1, 0), and (0, 1). For the game with a black hole on F_4 , possible outcomes are (0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1) and (1, 0, 1).

To play through games, we draw game trees, denoting the player to place the setup with bold blue text, and the player to move next with red text. We label each edge with the corresponding move, labeling as follows:

- (1) M for merging,
- (2) A_1 for adding from columns F_1 and F_2 ,
- (3) A_2 for adding from columns F_2 and F_3 ,
- (4) S_2 for splitting from column F_2 , and
- (5) S_3 for splitting from column F_3 .

We organize our trees so that options 1 through 5 are considered from left to right. Note that not all moves are always possible. In the game with a black hole on F_2 , the only possible move is to merge, and in the game with a black hole on F_3 , only the moves M , A_1 and S_1 are possible. For game states that we reference later in the proof, we label them by the color that placed it, the round it was placed, and which move was used to place it, noting both if two moves can be used to reach that state. Again, we assume that Red is the next player to move from a given position. See Figure 2 as an example of the possible game moves from the setup (a, b, c) .

3. Game with a Black hole on F_1 or F_2

We first consider the F_1 and F_2 Black Hole Zeckendorf games.

As we define it, the F_1 Black Hole Zeckendorf game is not possible, as every piece is immediately trapped in the black hole. There is also no Empty Board Black Hole Zeckendorf game to

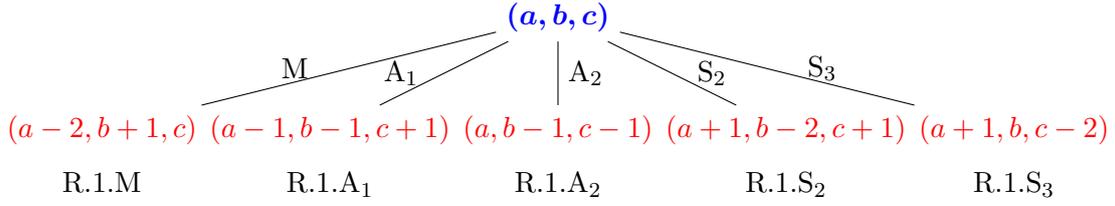


Figure 2: Example Game Tree for a Setup (a, b, c)

play, as by definition, players cannot place on the black hole.

In an F_2 Black Hole Zeckendorf game, the only possible move is to merge pieces into the black hole and there is only one column to place in for the Empty Board game, meaning it is deterministic. We outline the winners for the F_2 Black Hole Zeckendorf game and the Empty Board F_2 Black Hole Zeckendorf game below.

Theorem 3.1. *Let (a) be a setup for an F_2 Black Hole Zeckendorf game. This is a P position for $a \equiv 0, 1 \pmod{4}$ and an N position for $a \equiv 2, 3 \pmod{4}$. For the Empty Board F_2 Black Hole Zeckendorf game with n pieces, Player 1 wins for all $n \equiv 1, 2 \pmod{4}$ and Player 2 wins for all $n \equiv 0, 3 \pmod{4}$.*

Proof. We proceed by induction on a . As base cases, see that there are no moves from (0) or (1) , so it is trivially a P position. Then, for induction's sake, assume (a) is a P position for some $a \equiv 0, 1 \pmod{4}$ and consider $a + 4 \equiv 0, 1 \pmod{4}$. The following move must be $(a + 2)$, and the move after must be $(a) \equiv 0, 1 \pmod{4}$, which is a P position by assumption. Thus, (a) is a P position for all $a \equiv 0, 1 \pmod{4}$. It follows that (a) is an N position for all $a \equiv 2, 3 \pmod{4}$, as the following player places some $(a - 2) \equiv 0, 1 \pmod{4}$, which we showed is a P position.

For the Empty Board F_2 Black Hole Zeckendorf game, players can only place pieces in the F_1 column, which has weight 1. Player 2 concludes the empty board phase when n is even, so they win if they set a P position, and lose if they set an N position. Therefore, Player 2 wins for all $n \equiv 0 \pmod{4}$, and Player 1 wins for all $n \equiv 2 \pmod{4}$.

When n is odd, Player 1 concludes the empty board phase, and likewise, they win if they set a P position and lose if they set an N position. Therefore, Player 1 wins for all $n \equiv 1 \pmod{4}$, and Player 2 wins for all $n \equiv 3 \pmod{4}$. \square

4. Game with a Black Hole on F_3

We now consider the game with a black hole on $F_3 = 3$. Here players can choose how they move, making the game more interesting.

4.1. Single Column Winning Board Setups

We first consider which setups are P positions. We start by considering all pieces in one column.

Theorem 4.1. *Let (a, b) be a setup for an F_3 Black Hole Zeckendorf game. When $a \equiv 0, 1 \pmod{3}$ and $b = 0$, this is a P position and likewise this is a P position when $a = 0$ and $b \equiv 0, 1 \pmod{3}$.*

Proof. For games with $a \equiv 0, 1 \pmod{3}$ and $b = 0$, we proceed by induction on α . If the game

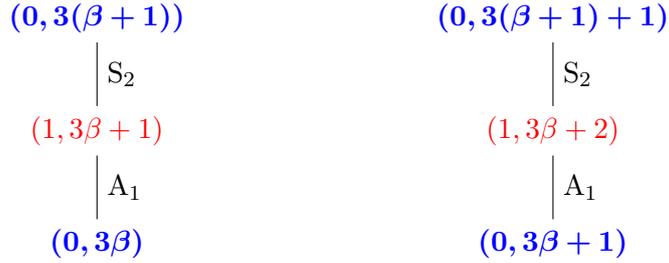
starts with 0 or 1 pieces, in either the F_1 or F_2 column, then (a, b) is trivially a P position.

As our inductive hypothesis, assume that $(3\alpha + k_1, 0)$ is a P position for a fixed α with $k_1 = 0, 1$. Then, consider the corresponding game trees below, with $k_1 = 0$ on the left and $k_1 = 1$ on the right.



In both trees, it is possible to reduce down to a setup that wins by the inductive hypothesis, proving that it is indeed a P position.

For games with $b \equiv 0, 1 \pmod{3}$ and $a = 0$, we proceed by induction on β . Assume as an inductive hypothesis that $(0, 3\beta + k_2)$ is a P position for a fixed β with $k_2 = 0, 1$. Consider the corresponding game trees below, with $k_2 = 0$ on the left and $k_2 = 1$ on the right.



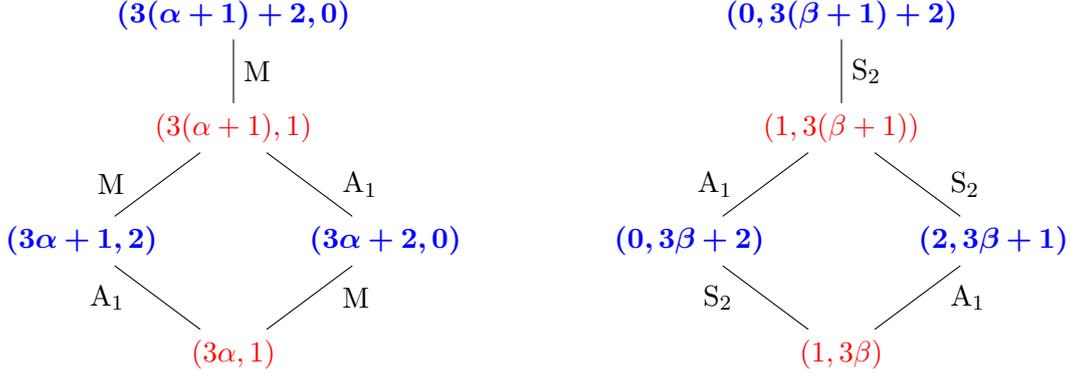
Again, it is possible to reduce to a setup that wins by the inductive hypothesis, proving the claim.

Therefore, for an F_3 Black Hole Zeckendorf game, (a, b) is a P position for all board setups with a or $b \equiv 0, 1 \pmod{3}$, and the other equal to 0. \square

Theorem 4.2. *Let (a, b) be a setup for an F_3 Black Hole Zeckendorf game. If $a \equiv 2 \pmod{3}$ and $b = 0$ or starts with $a = 0$ and $b \equiv 2 \pmod{3}$, then (a, b) is an N position.*

Proof. We proceed by induction. If the game starts with 2 pieces in the F_1 column and none in the other, the next player to move can merge to place $(0, 1)$ which wins. If the game starts with 2 pieces in the F_2 column, and none in the F_1 column, the next player to move can split to place $(1, 0)$ which wins.

For induction's sake, suppose the setups $(3\alpha + 2, 0)$ and $(0, 3\beta + 2)$ are N positions. Since the only possible moves from here are $(3\alpha, 1)$ and $(1, 3\beta)$, respectively, this assumption also implies that $(3\alpha, 1)$ and $(1, 3\beta)$ are P positions. Then, consider the game trees below, inducting on α in the left tree and β in the right tree.



Since we assumed $(3\alpha, 1)$ and $(1, 3\beta)$ are P positions, and showed that it is possible to reduce there from $(3(\alpha+1), 1)$ and $(1, 3(\beta+1))$, respectively, it follows that boards of the form $(3\alpha+2, 0)$ or $(0, 3\beta+2)$ are both N positions. \square

Corollary 4.3. *Let (a, b) be a setup for an F_3 Black Hole Zeckendorf game. $(3\alpha, 1)$ and $(1, 3\beta)$ are both P positions.*

Proof. As base cases, $(1, 0)$ and $(0, 1)$ both win trivially. As shown in the proof of Theorem 4.2, any player who places $(3\alpha, 1)$ or $(1, 3\beta)$ can reduce to $(3(\alpha-1), 1)$ and $(1, 3(\beta-1))$, respectively, so $(3\alpha, 1)$ or $(1, 3\beta)$ win by induction. \square

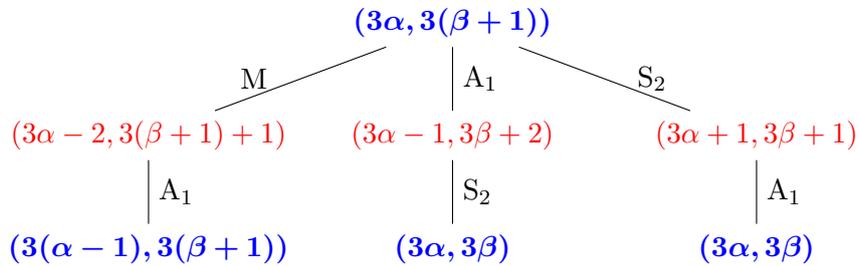
4.2. General Winning Setups

We now consider general winning setups of the form (a, b) . We show any case can be reduced modulo 3, and since both players have winning strategies in the lower cases, both players have winning strategies for higher cases.

4.2.1. P Positions

Theorem 4.4. *Let (a, b) be a setup for an F_3 Black Hole Zeckendorf game. If $a \equiv b \equiv 0 \pmod{3}$, then (a, b) is a P position.*

Proof. We proceed by induction on β . First, with the base case of $\beta = 0$, $(3\alpha, 0)$ is a P position by Theorem 4.1. Then, let our inductive hypothesis be that $(3\alpha, 3\beta)$ is a P position for all α and some fixed β . Then, consider the following game tree on $(3\alpha, 3(\beta+1))$.

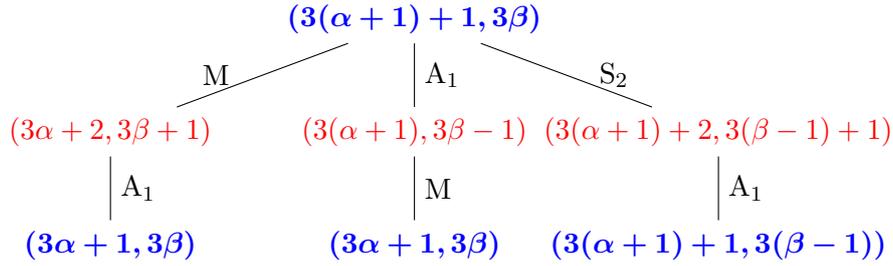


In the right and center columns, it is possible to reduce to a game of the form $(3\alpha, 3\beta)$, which is a P position by the inductive hypothesis. The left column is not immediately a P position by induction but the game options are the same, as before, so it can be reduced to a P position

anytime the next player adds or splits. If the next player always chooses to merge, then the game eventually reduces to $(0, 3(\beta + 1))$ which is a P position by Theorem 4.1. Thus, (a, b) is a P position for the F_3 Black Hole Zeckendorf game if $a \equiv b \equiv 0 \pmod{3}$. \square

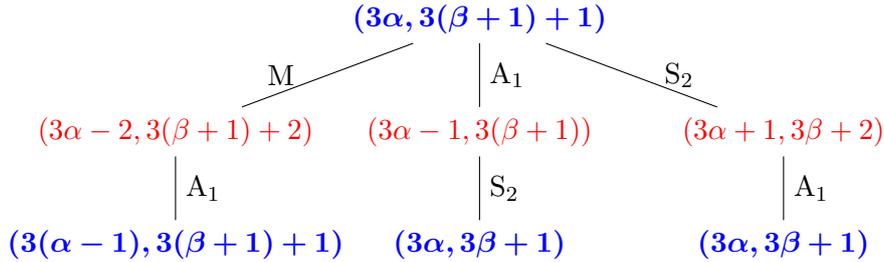
Theorem 4.5. *Let (a, b) be a setup for an F_3 Black Hole Zeckendorf game. If $a \equiv 1 \pmod{3}$ and $b \equiv 0 \pmod{3}$, or vice versa, (a, b) is a P position.*

Proof. We proceed by induction, first considering the game $(3\alpha + 1, 3\beta)$. First, fixing $\alpha = 0$, we have that $(1, 3\beta)$ is a P position by Corollary 4.3. Then, for our inductive hypothesis, suppose any setup of the form $(3\alpha + 1, 3\beta)$ is a P position for fixed α and all β . Then, consider the following game tree on $(3(\alpha + 1) + 1, 3\beta)$.



In the left and center columns, it is possible to reduce to a game of the form $(3\alpha + 1, 3\beta)$, which is a P position by the inductive hypothesis. The right column is not immediately a P position by inductive hypothesis, but the game options are the same as before so they will reduce to a P position by inductive hypothesis anytime the next player to move merges or adds. If the next player always splits, then the game eventually reduces to $(3(\alpha + 1) + 1, 0)$ which is a P position by Theorem 4.1.

Similarly, consider the initial setup $(3\alpha, 3\beta + 1)$. Fixing $\beta = 0$, $(3\alpha, 1)$ is a P position by Corollary 4.3. Then, as our inductive hypothesis, we assume that $(3\alpha, 3\beta + 1)$ is a P position for some fixed β and all α . We consider the game on $(3\alpha, 3(\beta + 1) + 1)$.



The center and right columns are P positions by inductive hypothesis. Again, $(3(\alpha - 1), 3(\beta + 1) + 1)$ is not immediately a P position by induction. However, it is either a P position by induction once the next player chooses to add or split, or once the game is reduced to $(0, 3(\beta + 1) + 1)$, as a result of Theorem 4.1.

Hence, (a, b) is a P position if $a \equiv 1 \pmod{3}$ and $b \equiv 0 \pmod{3}$ or vice versa. \square

4.2.2. N Positions

Theorem 4.6. *Let (a, b) be a setup for an F_3 Black Hole Zeckendorf game. The possibilities listed below are all N positions.*

- (1) $a \equiv b \equiv 1 \pmod{3}$,
- (2) $a \equiv b \equiv 2 \pmod{3}$,
- (3) $a \equiv 2 \pmod{3}$ and $b \equiv 1 \pmod{3}$,
- (4) $a \equiv 1 \pmod{3}$ and $b \equiv 2 \pmod{3}$,
- (5) $a \equiv 0 \pmod{3}$ and $b \equiv 2 \pmod{3}$,
- (6) $a \equiv 2 \pmod{3}$ and $b \equiv 0 \pmod{3}$.

Proof. For each statement, we show that each position leads directly to a position that we showed is P above.

First, suppose the board is set as $(3\alpha + 1, 3\beta + 1)$. Then, the next player can add from columns F_1 and F_2 to place $(3\alpha, 3\beta)$ which is a P position as shown in Theorem 4.4.

Similarly, suppose the board is set as $(3\alpha + 2, 3\beta + 2)$. Then the next player can merge from F_1 to place the board as $(3\alpha, 3(\beta + 1))$ which is a P position by Theorem 4.4 again.

Then, suppose the board is set as $(3\alpha + 2, 3\beta + 1)$ or $(3\alpha + 1, 3\beta + 2)$. The next player can add from columns F_1 and F_2 to place $(3\alpha + 1, 3\beta)$ and $(3\alpha, 3\beta + 1)$ respectively, which by Theorem 4.5 are both P positions.

Then, suppose the board is set as $(3\alpha, 3\beta + 2)$. The next player can split from column F_2 , to place the board as $(3\alpha + 1, 3\beta)$, which is a P position by Theorem 4.5.

Lastly, suppose the board is set as $(3\alpha + 2, 3\beta)$. The next player can merge from column F_1 to place the board as $(3\alpha, 3\beta + 1)$, which is a P position by Theorem 4.5. \square

4.3. Empty Board Game

We now continue to the Empty Board F_3 Black Hole Zeckendorf game and determine which player has a winning strategy for any given $n \in \mathbb{Z}^{>0}$. We find that Player 1 has a constructive winning strategy for $n \equiv 1, 2, 3, 6, 8 \pmod{9}$ and Player 2 has a constructive winning strategy for $n \equiv 0, 4, 5, 7 \pmod{9}$.

Theorem 4.7. *Let $(0, 0)$ be the beginning board for an Empty Board F_3 Black Hole Zeckendorf game with $n \in \mathbb{Z}^{>0}$ pieces. Players can force certain game setups as outlined below.*

- (1) For any $n \equiv \pm 3 \pmod{9}$, Player 1 can force the game into a setup $(n/3, n/3)$.
- (2) For any $n \equiv 0 \pmod{9}$, Player 2 can force the game into a setup $(n/3, n/3)$.
- (3) For any $n \equiv 1 \pmod{9}$, Player 1 can force the game into a setup $((n-1)/3+1, (n-1)/3)$.
- (4) For any $n \equiv 4, 7 \pmod{9}$, Player 2 can force the game into a setup $((n-1)/3+1, (n-1)/3)$.
- (5) For any $n \equiv 2 \pmod{9}$, Player 1 can force the game into a setup $((n-2)/3, (n-2)/3+1)$.
- (6) For any $n \equiv 8 \pmod{9}$, Player 1 can force the game into a setup $((n-2)/3+2, (n-2)/3)$.
- (7) For any $n \equiv 5 \pmod{9}$, Player 2 can force the game into a setup that is either $((n-2)/3+2, (n-2)/3)$ or $((n-2)/3, (n-2)/3+1)$.

Proof. First, consider $n \equiv \pm 3 \pmod{9}$. Player 1 should place their first piece in the F_2 column. Then, they should act opposite of Player 2, until there is one piece left, which Player 2 will be forced to place in the F_1 column, setting up the board as $(n/3, n/3)$.

If $n \equiv 0 \pmod{9}$, Player 2 can force the setup $(n/3, n/3)$ by placing opposite Player 1, resulting in Player 2 placing down $(n/3, n/3)$.

Then, consider $n \equiv 1 \pmod{9}$. Player 1 should place their first piece in the F_1 column. For every round following, Player 1 should place in the opposite column of Player 2. Since $n \equiv 1 \pmod{3}$, this means that Player 1 will place the last piece, setting the board as $((n-1)/3 + 1, (n-1)/3)$.

For $n \equiv 4, 7 \pmod{9}$, Player 2 should place opposite of Player 1. Again, this means that Player 1 will place the last piece, setting the board as $((n-1)/3 + 1, (n-1)/3)$.

Next, consider $n \equiv 2 \pmod{9}$. Player 1 should place their first piece in the F_2 column, and then play opposite Player 2 until Player 1 eventually sets the board as $((n-2)/3, (n-2)/3 + 1)$.

Next, consider $n \equiv 8 \pmod{9}$. Player 1 should place their first piece in the F_1 column, and then play opposite Player 2. Then, every time after Player 1 places down, the number of pieces left to place will be some $p \equiv 1 \pmod{3}$, meaning Player 2 will eventually be forced to set $((n-2)/3 + 2, (n-2)/3)$.

Lastly, for $n \equiv 5 \pmod{9}$, Player 2 should play opposite all of Player 1's moves, so that Player 2 eventually places the board $((n-2)/3, (n-2)/3)$. This forces Player 1 to either place $((n-2)/3, (n-2)/3 + 1)$ or $((n-2)/3 + 1, (n-2)/3)$, the latter allowing Player 2 to place the setup $((n-2)/3 + 2, (n-2)/3)$. \square

Note that these are the only setups that can be forced, as Player 1 has at most one more than half of the moves, while Player 2 has at most half of the moves. In the world of combinatorial game theory, this is often referred to as the "Tweedledum and Tweedledee" strategy, where one player always plays opposite the other; it can be used in games like Hackenbush [5] and Clobber [1].

Theorem 4.8. *Let $(0, 0)$ be the beginning board for an Empty Board F_3 Black Hole Zeckendorf game with n pieces. Player 1 has a constructive strategy for winning for any $n \equiv 1, 2, 3, 6, 8 \pmod{9}$. Player 2 has a constructive strategy for winning for any $n \equiv 0, 4, 5, 7 \pmod{9}$.*

Proof. We first consider when $n \equiv 3, 6 \pmod{9}$. By Theorem 4.7, Player 1 can force Player 2 to set the board as $(n/3, n/3)$. Then, $n \equiv 0 \pmod{3}$ but $n/3 \not\equiv 0 \pmod{3}$. Since $a, b \not\equiv 0 \pmod{3}$, this is an N position by Theorem 4.6, so Player 1 has a constructive solution.

Then, consider $n \equiv 0 \pmod{9}$, where Player 2 can force the game so that they place the setup $(n/3, n/3)$ by Theorem 4.7. Since $n \equiv 0 \pmod{9}$ implies $n/3 \equiv 0 \pmod{3}$, this is a P position by Theorem 4.4, so Player 2 has a constructive winning strategy.

Next, we consider when $n \equiv 1 \pmod{9}$, meaning Player 1 sets the board as $((n-1)/3 + 1, (n-1)/3)$ by Theorem 4.7. Since $(n-1)/3 + 1 \equiv 1 \pmod{3}$ and $(n-1)/3 \equiv 0 \pmod{3}$ this is a P position by Theorem 4.5, so Player 1 has a constructive strategy for winning.

Then, suppose $n \equiv 4, 7 \pmod{9}$, meaning Player 2 can force Player 1 to place the setup $((n-1)/3 + 1, (n-1)/3)$ by Theorem 4.7. For $n \equiv 4 \pmod{9}$, $(n-1)/3 + 1 \equiv 2 \pmod{3}$ and $(n-1)/3 \equiv 1 \pmod{3}$, so by Theorem 4.6, this is an N position so Player 2 has a winning strategy. Similarly, if $n \equiv 7 \pmod{9}$, then $(n-1)/3 + 1 \equiv 0 \pmod{3}$ and $(n-1)/3 \equiv 2 \pmod{3}$ so Player 2 has a winning strategy by Theorem 4.6.

Then, we consider when $n \equiv 2 \pmod{9}$, meaning that Player 1 is able to place the setup $((n-2)/3, (n-2)/3 + 1)$ by Theorem 4.7. Since $(n-2)/3 + 1 \equiv 1 \pmod{3}$ and $(n-2)/3 \equiv 0 \pmod{3}$, this is a P position by Theorem 4.5 so Player 1 wins.

If $n \equiv 8 \pmod{9}$, then Player 1 can force Player 2 to place the setup $((n-2)/3+2, (n-2)/3)$ by Theorem 4.7. Since $(n-2)/3+2 \equiv 1 \pmod{3}$ and $(n-2)/3 \equiv 2 \pmod{3}$, then Player 1 wins by Theorem 4.6.

If $n \equiv 5 \pmod{9}$, then by Theorem 4.7, Player 2 can force the game so that either they place the setup $((n-2)/3+2, (n-2)/3)$ or Player 1 places the setup $((n-2)/3, (n-2)/3+1)$. We note that $(n-2)/3 \equiv 1 \pmod{3}$, $(n-2)/3+1 \equiv 2 \pmod{3}$ and $(n-2)/3+2 \equiv 0 \pmod{3}$. It follows that Player 2 wins the setup $((n-2)/3, (n-2)/3+1)$ by Theorem 4.6, since it is an N position. Player 2 also wins the setup $((n-2)/3+2, (n-2)/3)$ as by Theorem 4.5 it is a P position. \square

This concludes our analysis of the F_3 Black Hole Zeckendorf game, both with and without the Empty Board phase of the game.

5. Game with a Black Hole on F_4

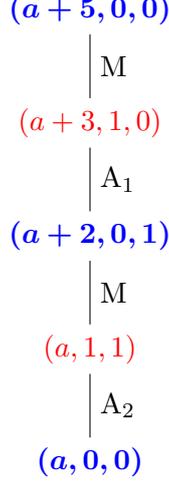
We now expand to the game with a black hole on $F_4 = 5$ with the setup (a, b, c) . This game becomes more interesting for a variety of reasons. There are more possible moves for each player and less symmetry, leading to a solution that is constructive but more intricate. Here, the winner is not solely based on the equivalence classes of a and c but also their value in relation to each other.

5.1. Single Column Winning Board Setups

We first consider winning strategies when all pieces are in either in column F_1 or column F_3 . We do not consider when all are in column F_2 for two reasons. First and foremost, we define the Empty Board game so that players are only able to place in the outermost columns, as this greatly simplifies players' abilities to force certain setups. Second, a significant strategy in the solution is forcing all pieces to the outer columns, to prevent the other player from using the "Add" move, thereby limiting their options. This strategy clearly fails when all pieces are in the F_2 column, making analysis of the game challenging. A complete analysis of the F_4 Black Hole Zeckendorf game on (a, b, c) such that $b \neq 0$ would prove interesting in future work, as would analyzing the Empty Board game without restrictions on placing in the F_2 column.

Theorem 5.1. *Let $(a, 0, 0)$ be a game state for an F_4 Black Hole Zeckendorf game. For any $a \neq 2$, $(a, 0, 0)$ is a P position.*

Proof. We proceed by induction on a . See in Appendix A that this is a P position for all base cases $a = 1, 3, 4, 5, 7$. Then, for our inductive hypothesis, suppose the starting position $(a, 0, 0)$ wins and consider the following game tree.



Since any $(a, 0, 0)$ wins by inductive hypothesis, and as shown above, it is always possible to reduce a modulo 5 until it reaches one of the base cases, then $(a, 0, 0)$ is a P position for all $a \neq 2$. \square

Corollary 5.2. *Let (a, b, c) be a game state for an F_4 Black Hole Zeckendorf game. The position $(a, 0, 1)$ is a P position for all $a \in \mathbb{Z}^{\geq 0}$. The position $(a, 0, 2)$ is an N position for all $a \neq 1 \in \mathbb{Z}^{\geq 0}$. The positions $(a, 0, 3)$, $(a, 1, 1)$, and $(a, 1, 2)$ are N positions for all $a \in \mathbb{Z}^{\geq 0}$.*

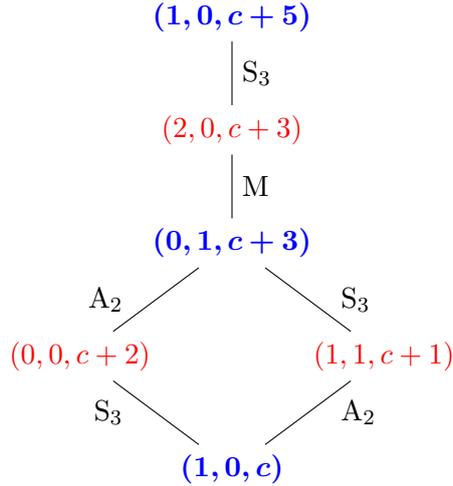
Proof. In the game tree considered in Theorem 5.1 and the base cases in Appendix A, $(a, 0, 1)$ is a P position since it reduces to $(a - 2, 0, 0)$, which is a P position for all $a \geq 2$. When $a < 2$, we have that $(0, 0, 1)$ and $(1, 0, 1)$ are trivially P positions. Note in the case $(4, 0, 1)$ the tree above reduces to $(2, 0, 0)$, which is an N position, but the base case tree for $n = 7$ as included in Appendix A reduces to $(0, 1, 0)$, which is a P position.

From $(a, 0, 2)$ and $(a, 0, 3)$, the next player can split in the F_3 column to place $(a + 1, 0, 0)$ and $(a + 1, 0, 1)$, respectively, which are P positions as shown above. The only exception to this is when the board is set as $(1, 0, 2)$, given that $(2, 0, 0)$ is an N position.

From $(a, 1, 1)$, the next player can add from columns F_2 and F_3 to place $(a, 0, 0)$, which is a P position for all $a \neq 2$. When $a = 2$, the board position is $(2, 1, 1)$ which is shown to be an N position in Appendix A. Similarly, from $(a, 1, 2)$, the next player can add from columns F_2 and F_3 to place $(a, 0, 1)$, which is a P position for all a as shown above. \square

Theorem 5.3. *Let $(0, 0, c)$ be a game state for an F_3 Black Hole Zeckendorf game. For any $c \neq 0, 1, 5 \in \mathbb{Z}^{\geq 0}$, $(0, 0, c)$ is an N position.*

Proof. It is sufficient to prove that $(1, 0, c)$ is a P position for all $c \neq 3$, since the next player to move from $(0, 0, c)$ can split in the third column to place $(1, 0, c - 2)$. Note here that we can allow $c \geq 2$ given that $(0, 0, 0)$ and $(0, 0, 1)$ are trivially P positions. We proceed by induction on c . See in Appendix B that this is a P position for all base cases $c = 0, 1, 2, 4, 8$ while it is an N position for $c = 3$. Then, consider the following game tree from $(1, 0, c)$.



Since $(1, 0, c)$ wins by inductive hypothesis and, as shown above, it is always possible to reduce modulo 5 until the game reaches a base case, then $(1, 0, c)$ is a P position for all $c \neq 3$, which implies $(0, 0, c)$ is an N position for all $c \neq 0, 1, 5$. \square

Corollary 5.4. *Let (a, b, c) be a game state for an F_4 Black Hole Zeckendorf game. The position $(1, 0, c)$ is a P position for all $c \neq 3 \in \mathbb{Z}^{\geq 0}$ and the position $(0, 1, c)$ is a P position for all $c \neq 1, 2, 6 \in \mathbb{Z}^{\geq 0}$. The position $(2, 0, c)$ is an N position for all $c \neq 1 \in \mathbb{Z}^{\geq 0}$, and the positions $(1, 1, c)$ and $(2, 1, c)$ are N positions for all $c \in \mathbb{Z}^{\geq 0}$.*

Proof. Theorem 5.3 proved that $(1, 0, c)$ is a P position for all $c \neq 3 \in \mathbb{Z}^{\geq 0}$. Similarly, $(0, 1, c)$ is a P position for all $c \neq 1, 2, 6$ as it reduces to $(1, 0, c - 3)$ in the game tree of Theorem 5.3. This is true for all $c \neq 1, 2, 6$ since placing $(0, 0, c - 1)$ only wins for $c - 1 = 0, 1, 5$.

Next, see that $(2, 0, c)$ is an N position in the tree in Theorem 5.3 and in the bases case trees in Appendix B, so it is an N position for $c \neq 1$.

Lastly, see that $(1, 1, c)$ and $(2, 1, c)$ are N position for all $c \in \mathbb{Z}^{\geq 0}$. The next player to move from $(2, 1, c)$ can place $(1, 0, c + 1)$ which is a P position for all $c \neq 2$. When $c = 2$, this player can place $(2, 0, 1)$ which we also showed to be a P position in Corollary 5.2. Placing $(1, 1, c)$ is an N position, given that the next player can place $(1, 0, c - 1)$ for $c \neq 0, 4$, and can place $(0, 0, 5)$ when $c = 4$ and $(0, 0, 1)$ when $c = 0$, all of which are P positions. \square

5.2. General Winning Setups

Our proof of general winning setups for a black hole on F_4 is more involved, as the minor exceptions in the cases above prevent us from creating a solution solely based on the equivalence classes of a and c for the setup $(a, 0, c)$. To motivate this section, we remind the reader of Figure 1, which outlines the winners for the board setup $(a, 0, c)$ in an F_4 Black Hole Zeckendorf game. We construct a solution by providing a path to place a P position. However, it is not possible to do this until we can verify that the P positions as outlined in the table are in fact P positions. We do this using a mixture of constructive and non-constructive methods.

Lemma 5.5. (Non-constructive). *Let (a, b, c) be a game state for an F_4 Black Hole Zeckendorf game. For all $\alpha, \gamma, k_1, k_3 \in \mathbb{Z}^{\geq 0}$, such that $1 \leq k_1 \leq 2$, and $0 \leq k_3 \leq 3$, $(3\alpha + k_1, 1, 4\gamma + k_3)$ is an N position.*

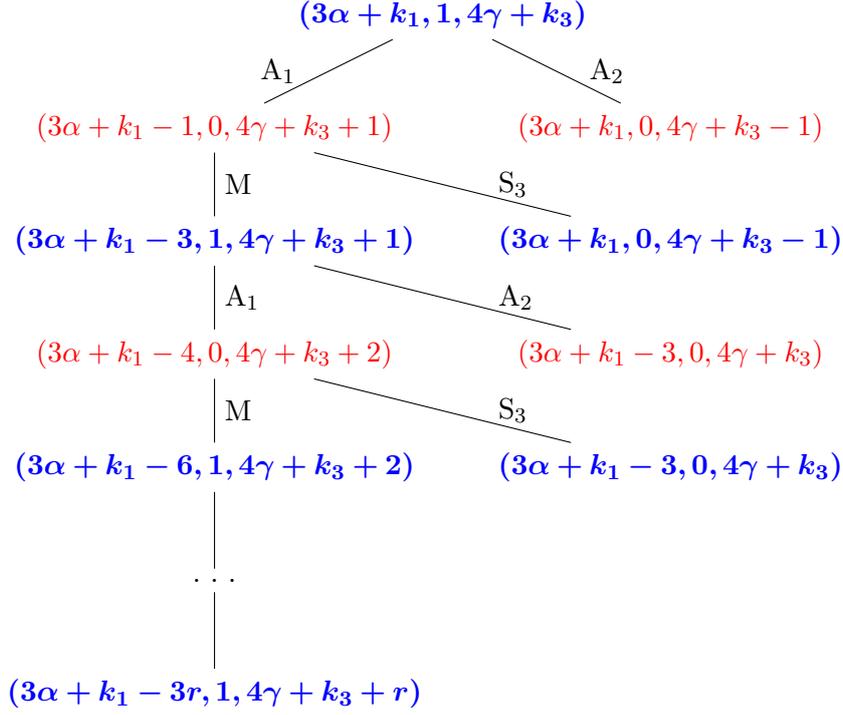


Figure 3: Game Tree for the Game State $(3\alpha + k_1, 1, 4\gamma + k_3)$

Proof. We proceed by a non-constructive proof. Recall that we assume that Red moves first from the position $(3\alpha + k_1, 1, 4\gamma + k_3)$. For contradiction's sake, suppose $(3\alpha + k_1, 1, 4\gamma + k_3)$ is a P position when $k_1 = 1, 2$. Then, consider the game tree in Figure 3, where r is the number of rounds in which both players move once. We note that Red has other possible moves, but we only consider moves relevant to the proof.

By assumption, $(3\alpha + k_1, 1, 4\gamma + k_3)$ is a P position regardless of what the other player places. Suppose Red places $(3\alpha + k_1 - 1, 0, 4\gamma + k_3 + 1)$ for their first move. Then, Blue can place either $(3\alpha + k_1 - 3, 1, 4\gamma + k_3 + 1)$ or $(3\alpha + k_1, 0, 4\gamma + k_3 - 1)$. But as shown in the tree, Red had the opportunity to place $(3\alpha + k_1, 0, 4\gamma + k_3 - 1)$ in the round before, so by assumption, placing it is a losing move. It follows that in order to win, Blue must place $(3\alpha + k_1 - 3, 1, 4\gamma + k_3 + 1)$. As shown in the tree, Red has the same options as before, so if they place $(3\alpha + k_1 - 4, 0, 4\gamma + k_3 + 2)$, then by assumption, Blue must place $(3\alpha + k_1 - 6, 1, 4\gamma + k_3 + 2)$.

Then, the game eventually reduces down to Blue placing $(3\alpha + k_1 - 3r, 1, 4\gamma + k_3 + r)$ after r rounds of each player moving once. After the α^{th} round, Blue will place $(k_1, 1, 4\gamma + k_3 + \alpha)$. If $k_1 = 1$, then Red can add from columns F_2 and F_3 to place $(1, 0, 4\gamma + \alpha + k_3 - 1)$ which is a P position by Corollary 5.4 for all $4\gamma + \alpha + k_3 - 1 \neq 3$. If $k_1 = 2$, then Red can add from columns F_1 and F_2 to place $(1, 0, 4\gamma + \alpha + k_3 + 1)$ which wins by Corollary 5.4 for all $4\gamma + \alpha + k_3 + 1 \neq 3$. Additionally, we show in Appendix C that Red also has a winning strategy for the cases when $4\gamma + \alpha + k_3 \pm 1 = 3$. Since Red moves first, this is a contradiction to the assumption that for $k_1 = 1, 2$, $(3\alpha + k_1, 1, 4\gamma + k_3)$ is a P position, thus it is instead an N position. \square

Remark 5.6. For the sake of conserving space and avoiding repetition, from this point forward we assume both players are seeking optimal strategies. Then, when $k_1 = 1, 2$, we omit any move of the form $(3\alpha + k_1, 1, 4\gamma + k_3)$, as it immediately loses.

From here, we consider when $k_1 = 0$, and show that $(3\alpha, 1, 4\gamma + 1)$ is an N position for $\alpha \geq \gamma$ and $(3\alpha, 1, 4\gamma)$ is an N position for $\alpha \geq \gamma + 1$.

Lemma 5.7. *Let (a, b, c) be a game state for an F_4 Black Hole Zeckendorf game. For all $\alpha, \gamma \in \mathbb{Z}^{\geq 0}$ such that $\alpha \geq \gamma$, $(3\alpha, 0, 4\gamma)$ is a P position.*

Proof. We proceed by strong induction on γ , letting α be an arbitrary integer such that $\alpha \geq \gamma$. As a base case, note that $(3\alpha, 0, 0)$ is a P position by Theorem 5.1, since $3\alpha \neq 2$ for any integer α . Then, consider the game on $(3\alpha, 0, 4(\gamma + 1))$ as played out in Figure 4, with $\alpha \geq \gamma + 1$, assuming as our inductive hypothesis that there exists a fixed γ such that placing $(3\alpha, 0, 4x)$ wins for all $\alpha \geq x$ and $x \leq \gamma$.

Note $\alpha \geq \gamma + 1$ implies that $\alpha - 1 \geq \gamma$ and $\alpha + 2 \geq \gamma - 2$ so B.4.A₂ and B.4.S₃ both win by inductive hypothesis. Some special consideration is needed here for the case when $\gamma = 1$, since B.4.S₃ = $(3(\alpha + 2), 0, -4)$ is not a valid game state. Here, B.3.S₃ = $(3(\alpha + 1) + 1, 0, 0)$ is a P position by Theorem 5.1 because $3(\alpha + 1) + 1 \neq 2$ for $\alpha \in \mathbb{Z}^{\geq 0}$.

For the center branch of the tree, if both players always choose to split once the tree is of the form $(3\alpha + 1, 0, 4(\gamma - 1) + 1)$, then Blue always places a board of the form $(3\alpha + 1 + 2r, 0, 4(\gamma - 1 - r) + 1)$, where r is a round of each player moving once. Then, after the $(\gamma - 1)^{\text{st}}$ round, Blue will place down $(3\alpha + 1 + 2(\gamma - 1), 0, 1)$ which is a P position by Corollary 5.2.

However, it is necessary to consider that every time Blue places $(3\alpha + 1 + 2r, 0, 4(\gamma - 1 - r) + 1)$ with $2r \equiv 1 \pmod{3}$, Red has the opportunity to place $(3\alpha + 2r - 1, 1, 4(\gamma - 1 - r) + 1)$ without immediately losing by Lemma 5.5, since here $3\alpha + 2r - 1 \equiv 0 \pmod{3}$. However, from here Blue can add from columns F_2 and F_3 to place $(3\alpha + 2r - 1, 0, 4(\gamma - 1 - r))$ which wins by inductive hypothesis, since $\alpha \geq \gamma - 1 - r$. An example of this can be seen in the leftmost column.

Therefore, $(3\alpha, 0, 4\gamma)$ is a P position for all $\alpha \geq \gamma$.

□

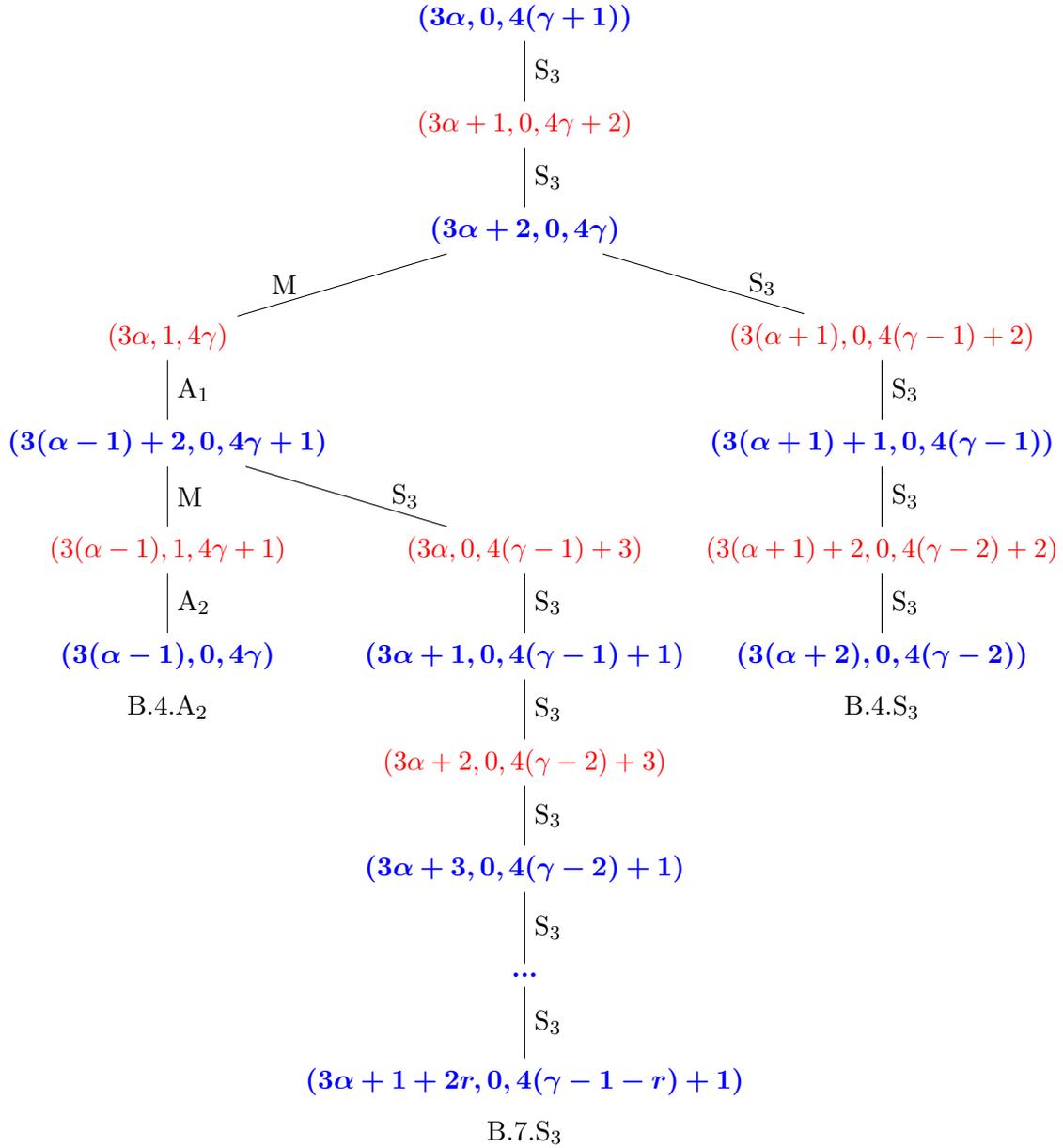
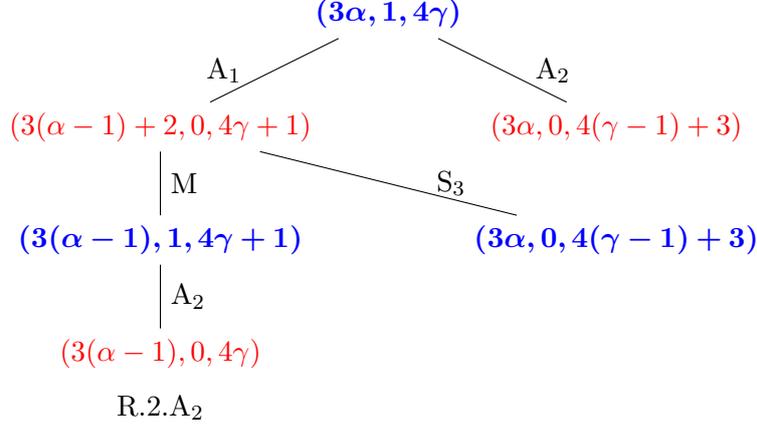


Figure 4: Game Tree for the Game State $(3\alpha, 0, 4(\gamma + 1))$

Corollary 5.8. *Let (a, b, c) be a game state for an F_4 Black Hole Zeckendorf game. For all $\alpha, \gamma \in \mathbb{Z}^{\geq 0}$, such that $\alpha \geq \gamma$, $(3\alpha, 1, 4\gamma + 1)$ is an N position. For all $\alpha, \gamma \in \mathbb{Z}^{\geq 0}$ such that $\alpha \geq \gamma + 1$, $(3\alpha, 1, 4\gamma)$ is an N position.*

Proof. $(3\alpha, 0, 4\gamma)$ can be placed immediately after $(3\alpha, 1, 4\gamma + 1)$ and since $(3\alpha, 0, 4\gamma)$ is a P position for all $\alpha \geq \gamma$ as shown in Lemma 5.7, this implies $(3\alpha, 1, 4\gamma + 1)$ is an N position. Similarly, $(3\alpha, 1, 4\gamma)$ is an N position for all $\alpha \geq \gamma + 1$. For contradiction's sake, assume it is a P position. Then, consider the game tree below.



By assumption, $(3\alpha, 0, 4(\gamma - 1) + 3)$ is an N position, so if Red places $(3(\alpha - 1) + 2, 0, 4\gamma + 1)$, Blue must place $(3(\alpha - 1), 1, 4\gamma + 1)$. Since $\alpha \geq \gamma + 1$, Red wins at R.2.A₂ by Lemma 5.7. Therefore, it must also be true that Red has a winning strategy for $(3\alpha, 1, 4\gamma)$ for all $\alpha \geq \gamma + 1$. Then, $(3\alpha, 1, 4\gamma)$ is an N position for all $\alpha \geq \gamma + 1$. \square

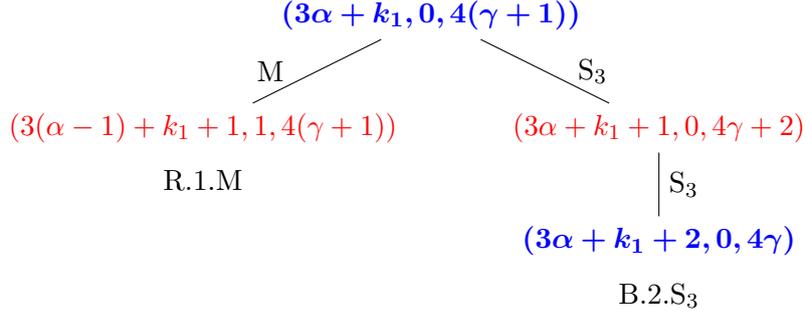
Corollary 5.9. *Let (a, b, c) be a game state for an F_4 Black Hole Zeckendorf game, with $\alpha, \gamma, k_1 \in \mathbb{Z}^{\geq 0}$ such that $0 \leq k_1 \leq 2$. Then, the following positions are P positions*

- (1) $(3\alpha + k_1, 0, 4\gamma)$ such that $\alpha \geq \gamma + 1$ and
- (2) $(3\alpha + k_1, 0, 4\gamma + 1)$ such that $\alpha \geq \gamma$.

The following positions are N positions

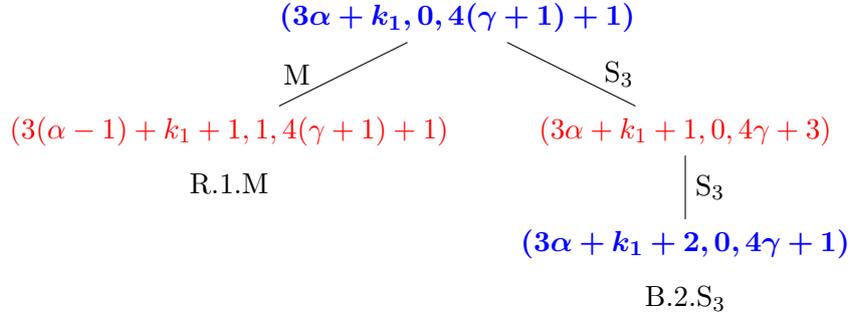
- (3) $(3\alpha + k_1, 0, 4\gamma + 2)$ such that $\alpha \geq \gamma + 1$ and
- (4) $(3\alpha + k_1, 0, 4\gamma + 3)$ such that $\alpha \geq \gamma$.

Proof. We prove Part 1 by inducting on γ , letting $\alpha \geq \gamma + 1$ be arbitrary. As a base case, note that for $\gamma = 0$, $(3\alpha + k_1, 0, 0)$ is a P position by Theorem 5.1. As our inductive hypothesis, assume $(3\alpha + k_1, 0, 4\gamma)$ is a P position for all $\alpha \geq \gamma + 1$ and consider the tree below, on $(3\alpha + k_1, 0, 4(\gamma + 1))$, letting $\alpha \geq \gamma + 2$.



By inductive hypothesis, B.2.S₃ is a *P* position, given that $\alpha \geq \gamma + 2$ implies $\alpha \geq \gamma + 1$. If $k_1 = 0, 1$ then R.1.M is an *N* position by Lemma 5.5. If $k_1 = 2$, then R.1.M is $(3\alpha, 1, 4(\gamma + 1))$ which is an *N* position by Corollary 5.8 since $\alpha \geq \gamma + 2$.

Similarly, we prove Part 2 by inducting on γ , letting $\alpha \geq \gamma$ be arbitrary. As a base case, for $\gamma = 0$, $(3\alpha + k_1, 0, 1)$ is a *P* position by Corollary 5.2. Then, assume $(3\alpha + k_1, 0, 4\gamma + 1)$ is a *P* position for all $\alpha \geq \gamma$ and consider the game tree on $(3\alpha + k_1, 0, 4(\gamma + 1) + 1)$, with $\alpha \geq \gamma + 1$.



By inductive hypothesis, B.2.S₃ is a *P* position given that $\alpha \geq \gamma + 1$ implies $\alpha \geq \gamma$. Again, if $k_1 = 0, 1$ then R.1.M is an *N* position by Lemma 5.5. If $k_1 = 2$, then R.1.M is $(3\alpha, 1, 4(\gamma + 1) + 1)$, which is an *N* position by Corollary 5.8 given that $\alpha \geq \gamma + 1$.

As a direct result, $(3\alpha + k_1, 0, 4\gamma + 2)$ is an *N* position for all $\alpha \geq \gamma + 1$ as claimed in Part 3, since the player to move next can place $(3\alpha + k_1 + 1, 0, 4\gamma)$, which is a *P* position as proven above.

Likewise, $(3\alpha + k_1, 0, 4\gamma + 3)$ is an *N* position for all $(3\alpha + k_1, 0, 4\gamma + 3)$ when $\alpha \geq \gamma$ as claimed in Part 4, since the player to move next can place $(3\alpha + k_1 + 1, 0, 4\gamma + 1)$, which is a *P* position as proven above. \square

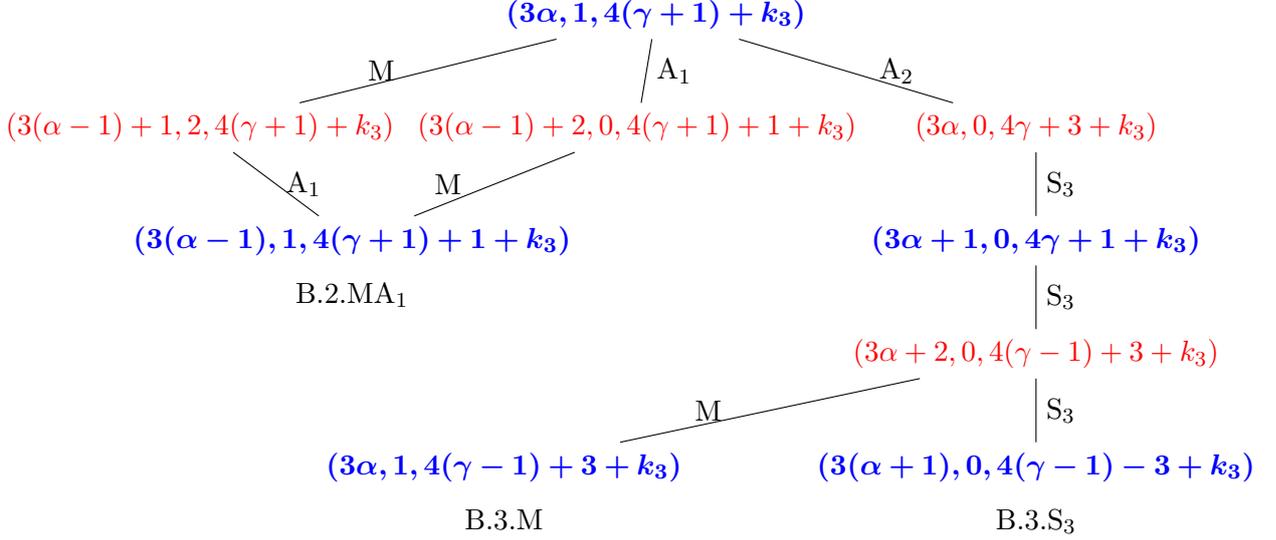
Corollary 5.10. *Let (a, b, c) be a game state for an F_4 Black Hole Zeckendorf game. For all $\alpha \geq \gamma$, $(3\alpha, 1, 4\gamma + 2)$ is an *N* position. For all $\alpha \geq \gamma + 3$, $(3\alpha, 1, 4\gamma + 3)$ is an *N* position.*

Proof. The first player to move from $(3\alpha, 1, 4\gamma + 2)$ can add the second two columns to place $(3\alpha, 0, 4\gamma + 1)$, which is a *P* position by Corollary 5.9 Part 1 for all $\alpha \geq \gamma$.

The first player to move from $(3\alpha, 1, 4\gamma + 3)$ can add the first two columns to place $(3(\alpha - 1) + 2, 0, 4(\gamma + 1))$, which is a *P* position by Corollary 5.9 Part 1 for all $\alpha - 1 \geq \gamma + 2$ which is equivalent to $\alpha \geq \gamma + 3$. \square

Lemma 5.11. *Let (a, b, c) be a game state for an F_4 Black Hole Zeckendorf game, with $\alpha, \gamma, k_3 \in \mathbb{Z}^{\geq 0}$ such that $0 \leq k_3 \leq 3$. $(3\alpha, 1, 4\gamma + k_3)$ is a P position for all $\alpha \leq \gamma - 2$.*

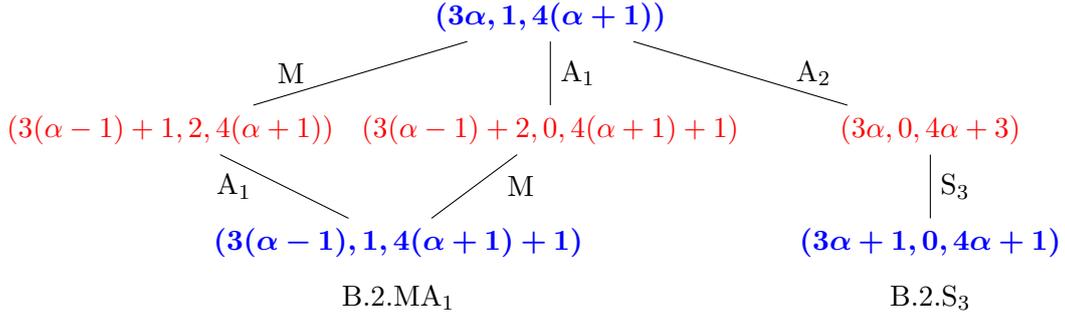
Proof. We proceed by strong induction on γ , letting $\gamma \geq 2$. We then assume as our hypothesis that there exists a fixed γ such that $(3\alpha, 1, 4x + k_3)$ is a P position for all $\alpha \leq x - 2$ and all $\gamma \geq x$. Then, consider the game on $(3\alpha, 1, 4(\gamma + 1) + k_3)$, letting $\alpha \leq \gamma - 1$.



While it is not immediately clear that B.2.MA₂ is a P position, note that the possible moves from here are the same as the possibilities from $(3\alpha, 1, 4(\gamma + 1))$ and that the necessary inequalities still hold, given that $\alpha - 1 < \alpha$. If Red always chooses to merge or add the first two columns, the game reduces to Blue placing $(0, 1, 4(\gamma + 1) + k_3 + 1)$, which is a P position for all $4(\gamma + 1) + k_3 + 1 \neq 0, 1, 6$ by Corollary 5.4. Since we require $\gamma - 1 \geq \alpha \geq 0 \implies \gamma \geq 1$, we do not need to consider these exceptions. Showing that one of B.3.M and B.3.S₃ is a P position takes a bit more effort, as which position wins depends on the values of α, γ and k_3 .

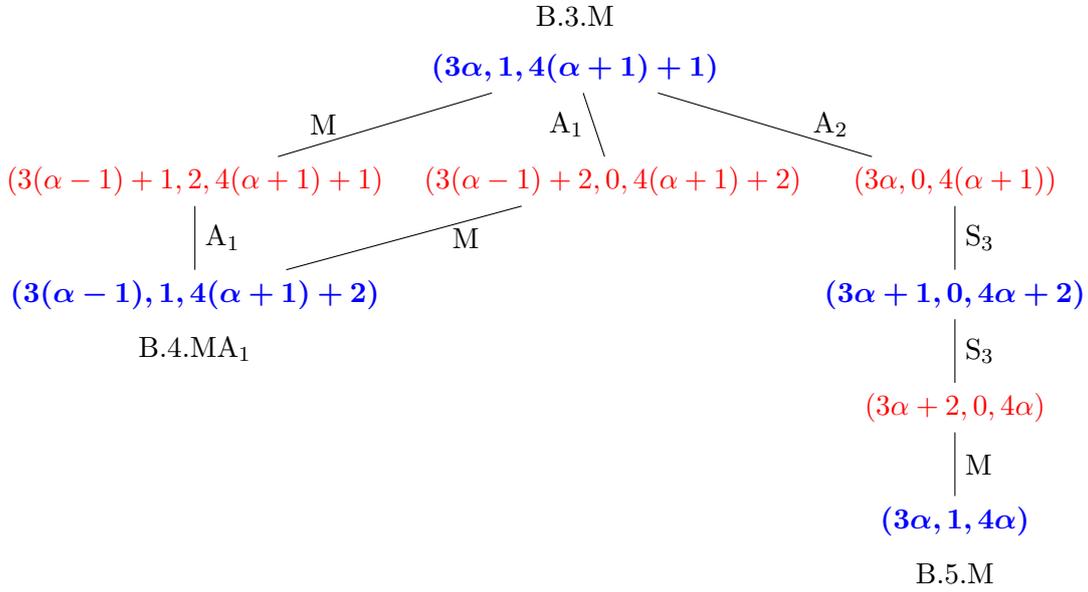
First, suppose $k_3 = 0$. Then, B.3.M = $(3\alpha, 1, 4(\gamma - 1) + 3)$ and B.3.S₃ = $(3(\alpha + 1), 0, 4(\gamma - 2) + 1)$. B.3.M is a P position by inductive hypothesis for all $\alpha \leq (\gamma - 1) - 2$, and we assumed $\alpha \leq \gamma - 1$. Therefore, we only need to show that $(3\alpha, 1, 4(\gamma + 1))$ is also a P position when $\gamma = \alpha + 1$ and $\gamma = \alpha + 2$. Note that when $\gamma = \alpha + 1$, B.3.S₃ = $(3(\alpha + 1), 0, 4(\alpha - 1) + 1)$ which is a P position by Corollary 5.9 Part 2. Similarly, when $\gamma = \alpha + 2$, B.3.S₃ = $(3(\alpha + 1), 0, 4\alpha + 1)$, which is a P position by Corollary 5.9 Part 2.

Next, suppose $k_3 = 1$, implying that B.3.M = $(3\alpha, 1, 4\gamma)$. B.3.M is a P position by inductive hypothesis for all $\alpha \leq \gamma - 2$, and since we let $\alpha \leq \gamma - 1$ so we only need to consider when $\gamma = \alpha + 1$. We consider this in the game tree below.



B.2.MA₂ is a P position by the inductive hypothesis and B.2.S₃ is a P position by Corollary 5.9 Part 2.

Then, suppose $k_3 = 2$, implying that B.3.M = $(3\alpha, 1, 4\gamma + 1)$. This is a P position by inductive hypothesis for all $\alpha \leq \gamma - 2$, and we assume $\alpha \leq \gamma - 1$, so again, we only need to consider when $\gamma = \alpha + 1$ as we do in the game tree below.



B.4.MA₂ is a P position by the inductive hypothesis. We continue the tree from B.5.M in Figure 5.

In Figure 5, B.7.MA₁ is a P position by inductive hypothesis and B.7.S₃ is a P position by Corollary 5.9 Part 2. Otherwise, the game reduces to $(3(\alpha - r), 1, 4(\alpha - r))$ at B.9.M, so the game tree repeats for r rounds, until Blue places a state which we have shown to win by our inductive hypothesis, wins by Corollary 5.9 Part 2, or places $(0, 1, 0)$ which trivially wins. This concludes our proof of the statement that $(3\alpha, 1, 4\gamma + 2)$ is a P position for all $\alpha \leq \gamma - 2$.

Lastly, we consider when $k_3 = 3$ so that B.3.M = $(3\alpha, 1, 4\gamma + 2)$. By the inductive hypothesis, this is a P position for all $\alpha \leq \gamma - 2$, and since we let $\alpha \leq \gamma - 1$, we again must only consider $\gamma = \alpha + 1$. Here, B.3.S₃ = $(3(\alpha + 1), 0, 4\alpha)$, which is a P position by Lemma 5.7.

Thus, Player 2 has a winning strategy for $(3\alpha, 1, 4\gamma + k_3)$ for all $\alpha \leq \gamma - 2$. \square

Corollary 5.12. Let (a, b, c) be a game state for an F_4 Black Hole Zeckendorf game, with $\alpha, \gamma \in \mathbb{Z}^{\geq 0}$. For all $\alpha = \gamma$, $(3\alpha, 1, 4\gamma)$ is a P position.

Proof. We showed in Lemma 5.11 that $(3\alpha, 1, 4\alpha)$ is a P position under the assumption that $(3\alpha, 1, 4\gamma + k_3)$ is a P position for all $\alpha \leq \gamma - 2$. Since we proved that Lemma 5.11 is in fact true, then it follows that $(3\alpha, 1, 4\gamma)$ is a P position for all $\alpha = \gamma$. \square

We now know that $(3\alpha, 1, 4\gamma + k_3)$ is a P position for all $\alpha \leq \gamma - 2$, and from Corollary 5.8 and Corollary 5.10, we know that $(3\alpha, 1, 4\gamma + k_3)$ is a P position for all $\alpha \geq \gamma + 3$. By considering each k_3 individually, we determine the positions which are P and N for $(3\alpha, 1, 4\gamma + k_3)$ for any $\alpha, \gamma, k_3 \in \mathbb{Z}^{\geq 0}$, such that $0 \leq k_3 \leq 3$.

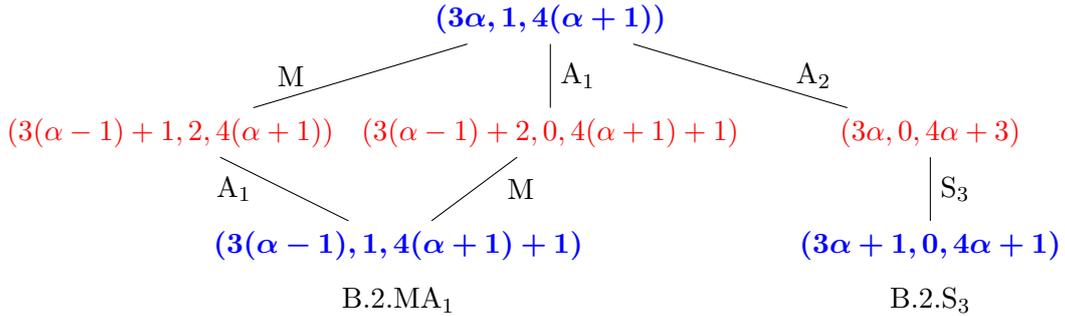
Theorem 5.13. Let (a, b, c) be a game state for an F_4 Black Hole Zeckendorf game, with $\alpha, \gamma \in \mathbb{Z}^{\geq 0}$.

- (1) $(3\alpha, 1, 4\gamma)$ is a P position for all $\alpha \leq \gamma$ and $(3\alpha, 1, 4\gamma)$ is an N position for all $\alpha \geq \gamma + 1$.
- (2) $(3\alpha, 1, 4\gamma + 1)$ is a P position for all $\alpha \leq \gamma - 1$ and $(3\alpha, 1, 4\gamma + 1)$ is an N position for all $\alpha \geq \gamma$.
- (3) $(3\alpha, 1, 4\gamma + 2)$ is a P position for all $\alpha \leq \gamma - 2$ and $(3\alpha, 1, 4\gamma + 2)$ is an N position for all $\alpha \geq \gamma - 1$.
- (4) $(3\alpha, 1, 4\gamma + 3)$ is a P position for all $\alpha \leq \gamma + 2$ and $(3\alpha, 1, 4\gamma + 3)$ is an N position for all $\alpha \geq \gamma + 3$.

Proof. We proceed by proving each part of the theorem individually.

Proof when $k_3 = 0$.

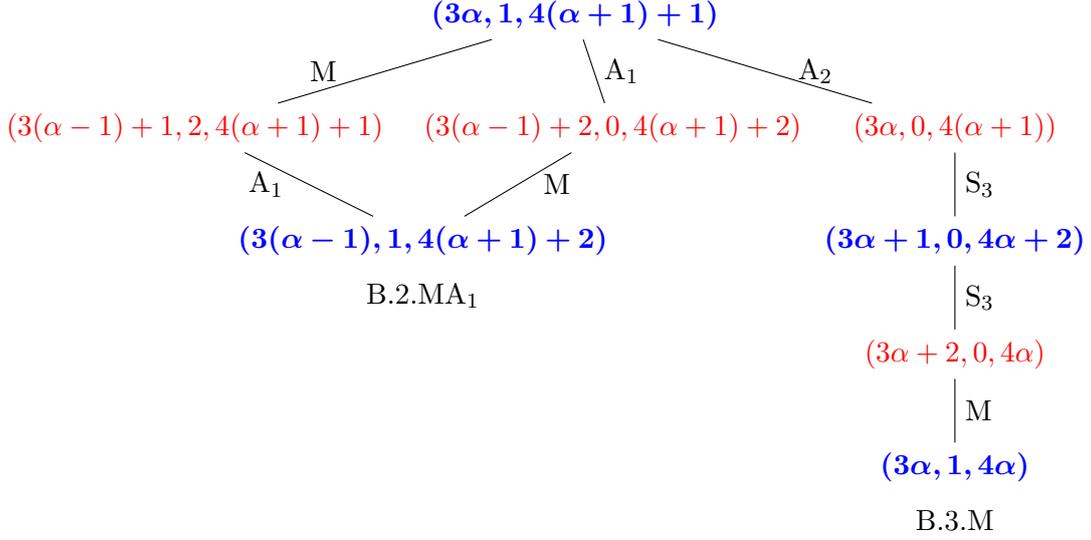
First, we consider when $k_3 = 0$. From Corollary 5.8, we know that $(3\alpha, 1, 4\gamma)$ is an N position for all $\alpha \geq \gamma + 1$, and from Lemma 5.11, we know that $(3\alpha, 1, 4\gamma)$ is a P position for all $\alpha \leq \gamma - 2$. Moreover, we showed that $(3\alpha, 1, 4\alpha)$ is a P position in Corollary 5.12. Therefore, it is only necessary to show that $(3\alpha, 1, 4\gamma)$ is a P position for $\gamma = \alpha + 1$, which we show in the game tree below.



B.2.MA₂ is a P position by Lemma 5.11 and B.2.S₃ is a P position by Corollary 5.9 Part 2. Thus, $(3\alpha, 1, 4\gamma)$ is a P position for all $\alpha \leq \gamma$ and $(3\alpha, 1, 4\gamma)$ is an N position for all $\alpha \geq \gamma + 1$.

Proof when $k_3 = 1$.

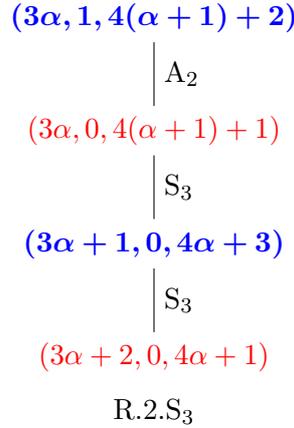
Next, we consider when $k_3 = 1$. From Corollary 5.8, we know that $(3\alpha, 1, 4\gamma + 1)$ is an N position for all $\alpha \geq \gamma$, and from Lemma 5.11, $(3\alpha, 1, 4\gamma + 1)$ is a P position for all $\alpha \leq \gamma - 2$. Thus, we only need to consider $\gamma = \alpha + 1$, which we do in the game tree below.



B.2.MA₁ is a *P* position by Lemma 5.11 and B.3.M is a *P* position by Corollary 5.12. Hence, $(3\alpha, 1, 4\gamma + 1)$ is a *P* position for all $\alpha \leq \gamma - 1$ and $(3\alpha, 1, 4\gamma + 1)$ is an *N* position for all $\alpha \geq \gamma$.

Proof when $k_3 = 2$.

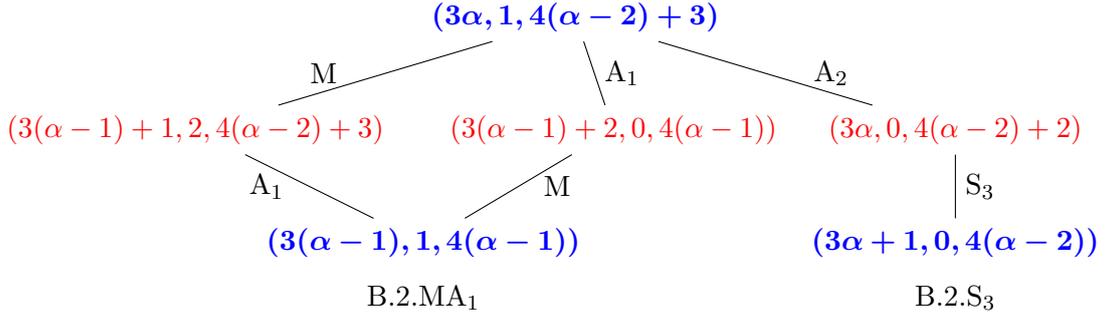
Then, we consider when $k_3 = 2$. From Corollary 5.10, we know that $(3\alpha, 1, 4\gamma + 2)$ is an *N* position for all $\alpha \geq \gamma$, and from Lemma 5.11, $(3\alpha, 1, 4\gamma + 2)$ is a *P* position for all $\alpha \leq \gamma - 2$, so again, we only need to consider when $\gamma = \alpha + 1$, as we do in the game tree below.



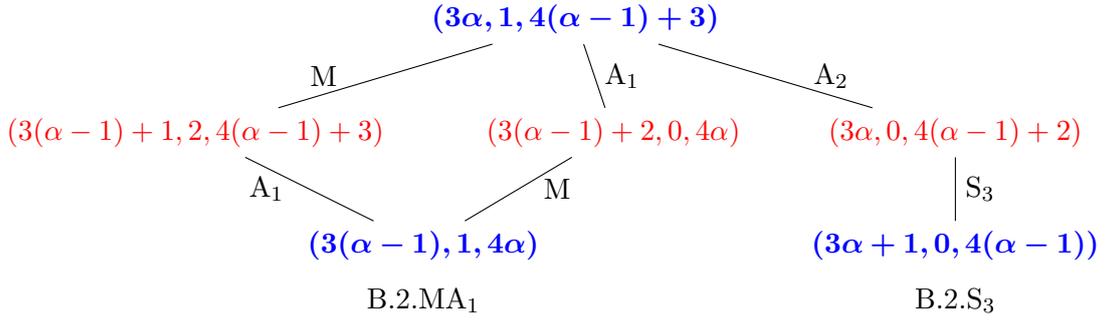
R.2.S₃ is a *P* position, by Corollary 5.9 Part 2, so $(3\alpha, 1, 4\gamma + 2)$ is a *P* position for all $\alpha \leq \gamma - 2$ and $(3\alpha, 1, 4\gamma + 2)$ is an *N* position for all $\alpha \geq \gamma - 1$.

Proof when $k_3 = 3$.

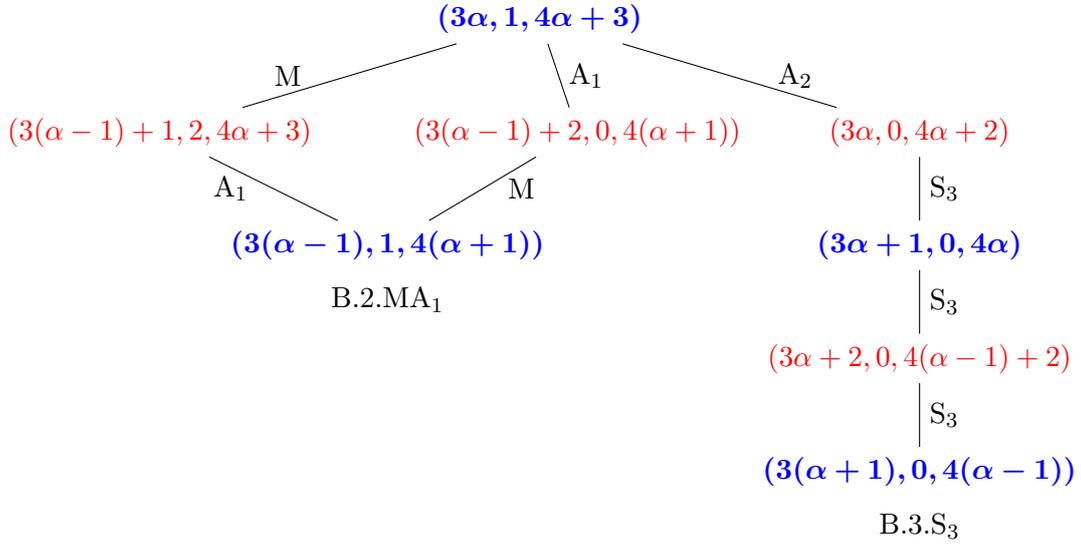
Finally, we consider when $k_3 = 3$. By Lemma 5.11, $(3\alpha, 1, 4\gamma + 3)$ is a *P* position for all $\alpha \leq \gamma - 2$ and by Corollary 5.10, $(3\alpha, 1, 4\gamma + 3)$ is an *N* position for all $\alpha \geq \gamma + 3$. Thus, we must consider when $\alpha - 2 \leq \gamma \leq \alpha + 1$. We begin with the game tree such that $\gamma = \alpha - 2$.



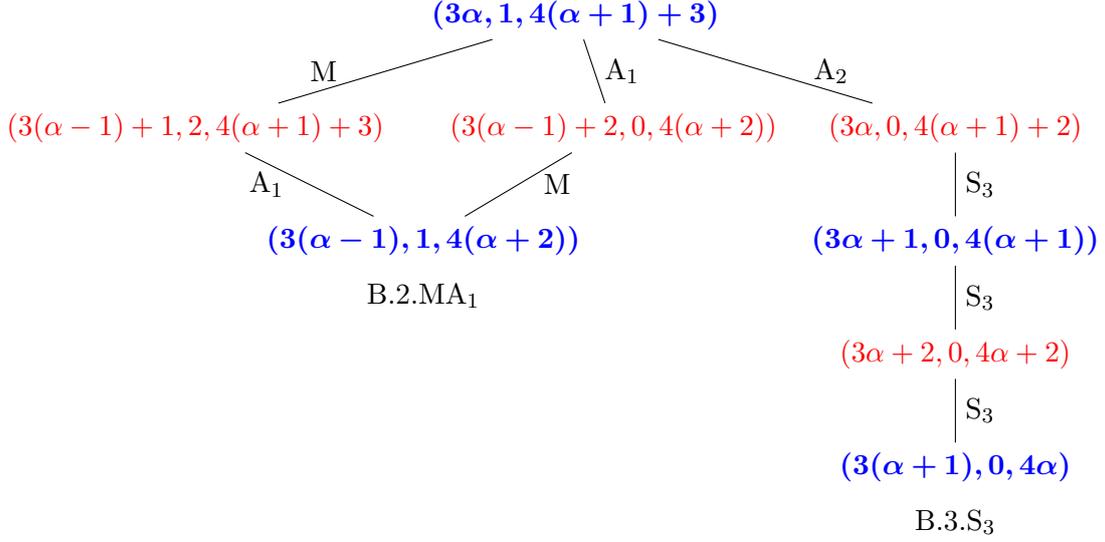
B.2.MA₁ is a *P* position by Corollary 5.12. Additionally, B.2.S₃ is a *P* position by Corollary 5.9 Part 1. Next, we consider the game tree such that $\gamma = \alpha - 1$.



B.2.MA₁ is a *P* position by the proof of Part 1 of this Theorem. Additionally, B.2.S₃ is a *P* position by Corollary 5.9 Part 1. Then, we continue to $\gamma = \alpha$.



When $\alpha = \gamma = 0$, the game starts at $(0, 1, 3)$ which is a *P* position by Corollary 5.4. For $\alpha \in \mathbb{Z}^{>0}$ B.2.MA₁ is a *P* position by Lemma 5.11 and B.3.S₃ is a *P* position by Corollary 5.9 Part 1. Then, we conclude with the case where $\gamma = \alpha + 1$, which is essentially identical to the case for $\gamma = \alpha$.



Again, B.2.MA_1 is a P position by Lemma 5.11 and B.3.S_3 is a P position by Corollary 5.9 Part 1.

This concludes our proof of Theorem 5.13. \square

From here, we can begin to prove some of the statements claimed in Figure 1. For a solution to be truly constructive, we want a path from any game position to a P position. In Lemma 5.14 we prove that certain states are indeed P positions below, and then provide constructive strategies from P positions in Theorem 5.15.

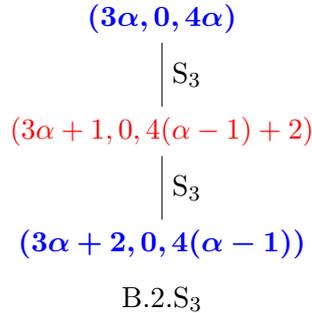
Lemma 5.14. *Let $(a, 0, c)$ be a game state for a F_4 Black Hole Zeckendorf game $\alpha, \gamma \in \mathbb{Z}^{\geq 0}$. The following positions are P positions:*

- (1) $(3\alpha, 0, 4\gamma)$ such that $\alpha \geq \gamma$, and
- (2) $(3\alpha + 1, 0, 4\gamma)$ for all α, γ .

Proof. We prove each statement individually.

Proof of Part 1.

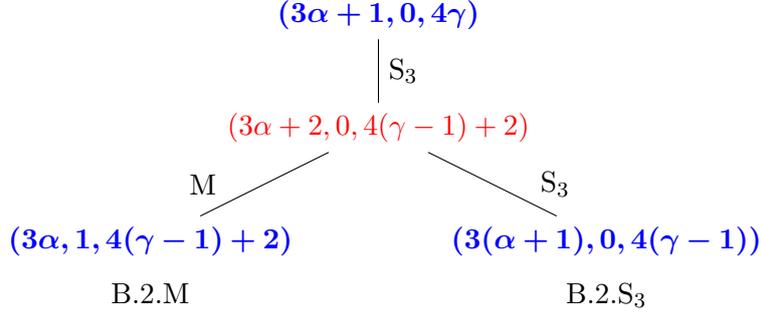
We first show that $(3\alpha, 0, 4\gamma)$ is a P position for all $\alpha \geq \gamma$, noting that when $\alpha = \gamma = 0$, $(0, 0, 0)$ is trivially a P position. By Corollary 5.9 Part 1, $(3\alpha, 0, 4\gamma)$ is a P position for all $\alpha \geq \gamma + 1$, so it is only necessary to consider the game when $\gamma = \alpha$, as we do below.



B.2.S_3 is a P position by Corollary 5.9 Part 1, so $(3\alpha, 0, 4\gamma)$ is a P position for all $\alpha \geq \gamma$.

Proof of Part 2.

To conclude this lemma, we show that $(3\alpha + 1, 0, 4\gamma)$ is a P position for all α, γ , noting that when $\alpha = \gamma = 0$, $(1, 0, 0)$ is trivially a P position. From Corollary 5.9 Part 1, we already know it is a P position for all $\alpha \geq \gamma + 1$. Then, consider the tree below.



B.2.M is a P position for all $\alpha \leq \gamma - 3$ by Theorem 5.13 Part 3, and B.2.S₃ is a P position for all $\alpha \geq \gamma - 2$, as established by the proof of Part 1. Therefore, $(3\alpha + 1, 0, 4\gamma)$ is a P position for all $\alpha, \gamma \in \mathbb{Z}^{\geq 0}$. \square

Lemma 5.14 is sufficient information for us to now construct a winning strategy from any P position. Here, we aim to show that if a player has placed a P position, they can reduce the game to another P position.

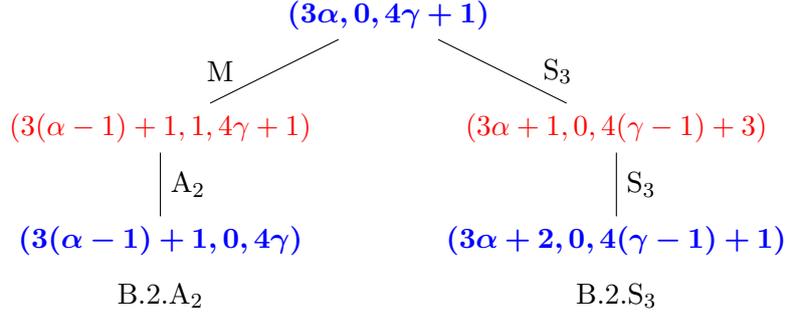
Theorem 5.15. *Let $(a, 0, c) = (3\alpha + k_1, 0, 4\gamma + k_3)$ be a board setup for an F_4 Black Hole Zeckendorf game. For $\alpha, \gamma, k_1, k_3 \in \mathbb{Z}^{\geq 0}$, $0 \leq k_1 \leq 2$, and $0 \leq k_3 \leq 3$, the following positions are P positions.*

- (1) $c \equiv 1 \pmod{4}$:
 - (a) $(3\alpha, 0, 4\gamma + 1)$ for all $\alpha \geq \gamma - 1$.
 - (b) $(3\alpha + 1, 0, 4\gamma + 1)$ for all α, γ .
 - (c) $(3\alpha + 2, 0, 4\gamma + 1)$ for all $\alpha \geq \gamma$.
- (2) $c \equiv 2 \pmod{4}$:
 - (a) $(3\alpha + 1, 0, 4\gamma + 2)$ for all $\alpha \leq \gamma$.
- (3) $c \equiv 3 \pmod{4}$:
 - (a) $(3\alpha + 1, 0, 4\gamma + 3)$ for all $\alpha \leq \gamma - 1$.
- (4) $c \equiv 0 \pmod{4}$:
 - (a) $(3\alpha, 0, 4\gamma)$ for all $\alpha \geq \gamma$.
 - (b) $(3\alpha + 1, 0, 4\gamma)$ for all α, γ .
 - (c) $(3\alpha + 2, 0, 4\gamma)$ for all $\alpha \geq \gamma + 1$.

Proof. We work through each section of the theorem, referencing a base case as an example, and then demonstrating the winning strategy. Since we aim to create a constructive solution, we assume that both players are unfamiliar with Lemma 5.5, and play out the game when players choose to set the board as $(3\alpha + 1, 1, 4\gamma + k_3)$ and $(3\alpha + 2, 1, 4\gamma + k_3)$, although we already know that game state loses.

Proof for $c \equiv 1 \pmod{4}$.

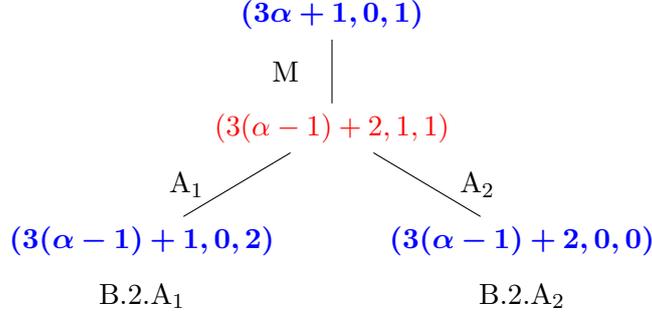
Proof of Part 1a: We first consider the game on $(3\alpha, 0, 4\gamma + 1)$ where $\alpha \geq \gamma - 1$. As base cases, we see that $(0, 0, 1)$ is trivially a P position and $(0, 0, 5)$ is a P position by Theorem 5.3. Then consider the game tree below.



B.2.A₂ is a P position by Lemma 5.14 Part 2, and B.2.S₃ is a P position, by Corollary 5.9 Part 2, given that $\alpha \geq \gamma - 1$. Therefore, $(3\alpha, 0, 4\gamma + 1)$ is a P position for all $\alpha \geq \gamma - 1$.

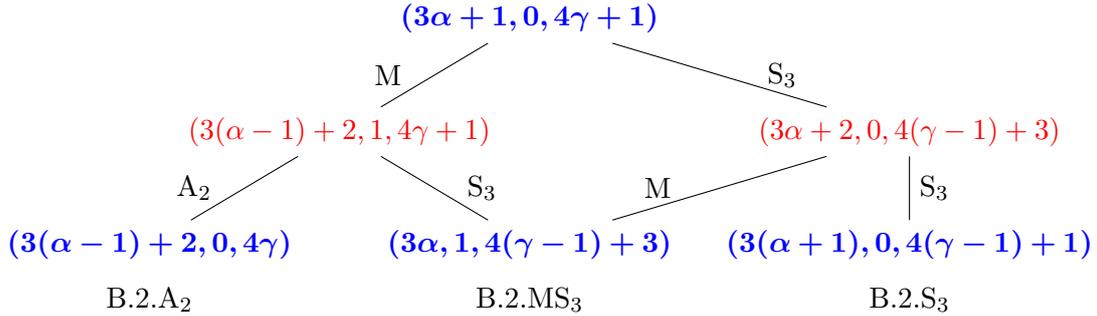
Proof of Part 1b: Next, we consider the game on $(3\alpha + 1, 0, 4\gamma + 1)$, noting that as a base case, $(1, 0, 1)$ is trivially a P position.

When $\gamma = 0$, the game tree is as follows.



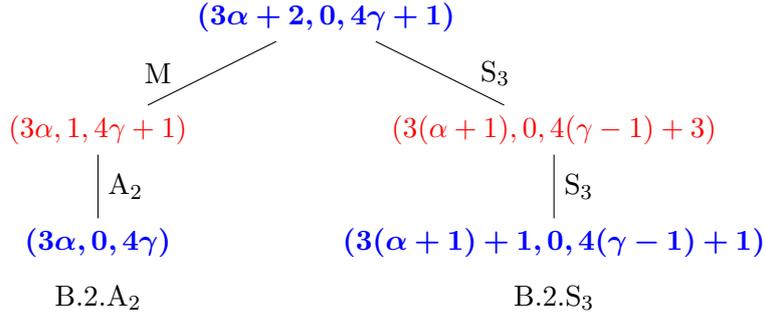
When $\alpha > 1$, B.2.A₂ is a P position by Theorem 5.1. When $\alpha = 1$, B.2.A₁=(1, 0, 2) which is a P position by Corollary 5.2.

We show in the game tree below that $(3\alpha + 1, 0, 4\gamma + 1)$ is a P position for $\gamma \geq 1$.



By Corollary 5.9 Part 1, B.2.A₂ is a P position for all $\alpha \geq \gamma + 2$ and by Corollary 5.9 Part 2, B.2.S₃ is a P position for all $\alpha \geq \gamma - 2$. By Theorem 5.13 Part 4, B.2.MS₃ is a P position for all $\alpha \leq \gamma + 1$. Thus, $(3\alpha + 1, 0, 4\gamma + 1)$ is a P position for all $\alpha, \gamma \in \mathbb{Z}^{\geq 0}$.

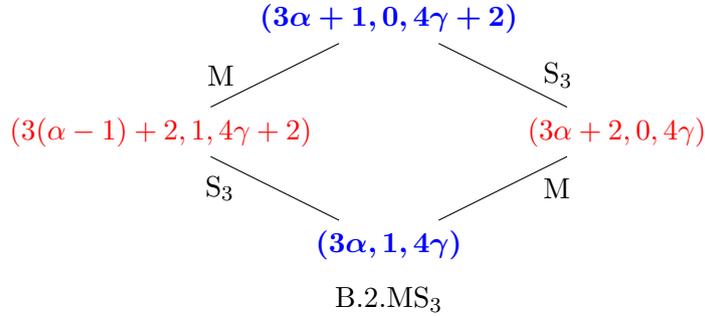
Proof of Part 1c: Next, we consider the game on $(3\alpha + 2, 0, 4\gamma + 1)$, with $\alpha \geq \gamma$, noting as a base case that $(2, 0, 1)$ is a P position by Corollary 5.2. Now, consider the game on an arbitrary $(3\alpha + 2, 0, 4\gamma + 1)$ with $\alpha \geq \gamma$.



Since, $\alpha \geq \gamma$, B.2.A₂ is a P position as proven in Lemma 5.14 Part 1, and B.2.S₃ is a P position by Corollary 5.9 Part 2.

Proof for $c \equiv 2 \pmod{4}$.

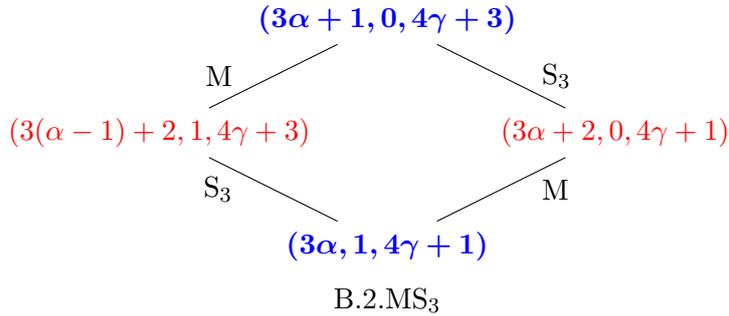
Proof of Part 2a: Next, we show that $(3\alpha + 1, 0, 4\gamma + 2)$ is a P position for all $\alpha \leq \gamma$, noting as a base case that $(1, 0, 2)$ is a P position by Corollary 5.4. We consider the general case below.



We showed that $(3\alpha, 1, 4\gamma)$ is a P position for all $\alpha \leq \gamma$ in Theorem 5.13 Part 1, so $(3\alpha + 1, 0, 4\gamma + 2)$ is a P position for all $\alpha \leq \gamma$.

Proof for $c \equiv 3 \pmod{4}$.

Proof of Part 3a: We show that $(3\alpha + 1, 0, 4\gamma + 3)$ is a P position for all $\alpha \leq \gamma - 1$, noting as a base case that $(1, 0, 7)$ is a P position by Corollary 5.4. We consider the general case below.

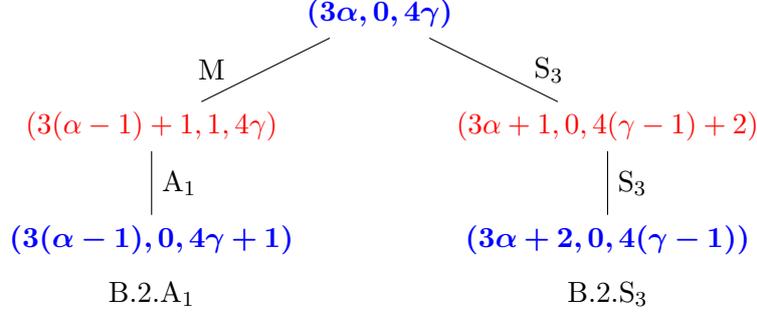


We showed that $(3\alpha, 1, 4\gamma + 1)$ is a P position for all $\alpha \leq \gamma - 1$ in Theorem 5.13 Part 2, so $(3\alpha + 1, 0, 4\gamma + 3)$ is a P position for all $\alpha \leq \gamma - 1$.

Proof for $c \equiv 0 \pmod{4}$.

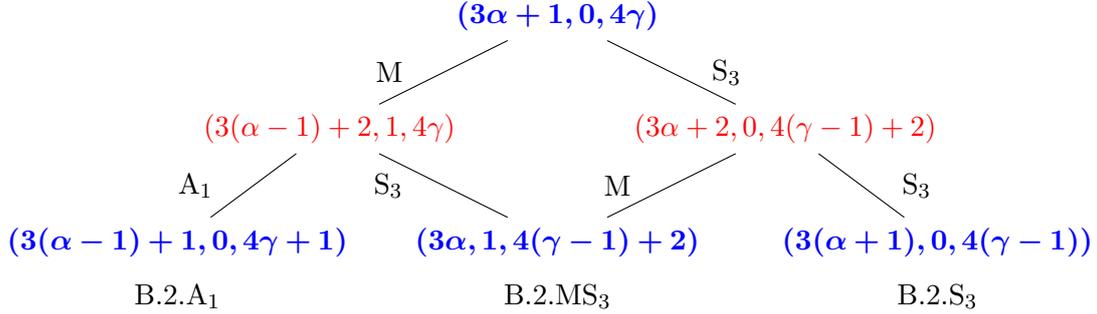
We already proved that a significant portion of Part 4 is true in Lemma 5.14. However, these proofs did not consider the possibility of placing $(3\alpha + 1, 1, 4\alpha)$ and $(3\alpha + 2, 1, 4\alpha)$ so were not fully constructive. Here, we provide a constructive strategy.

Proof of Part 4a: We first consider the game on $(3\alpha, 0, 4\gamma)$ where $\alpha \geq \gamma$. As a base case, see that $(0, 0, 0)$ is trivially a P position. Then consider the game tree below.



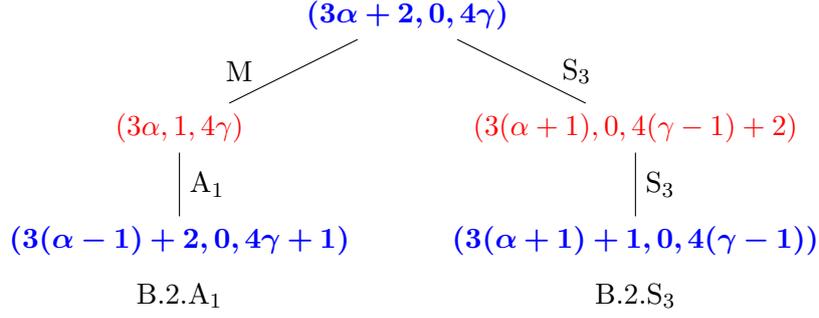
B.2.A₁ is a P position by the proof of Part 1a since $\alpha \geq \gamma$. B.2.S₃ is a P position by Corollary 5.9 Part 1 since $\alpha \geq \gamma$. Thus, $(3\alpha, 0, 4\gamma)$ is a P position for all $\alpha \geq \gamma$.

Proof of Part 4b: Next, we consider the game on $(3\alpha + 1, 0, 4\gamma)$, noting that as a base case, $(1, 0, 0)$ is trivially a P position. We show below that $(3\alpha + 1, 0, 4\gamma)$ is a P position for all α, γ .



We showed that B.2.A₁ is a P position for all α, γ by the proof of Part 1b. By the proof of Part 4a, B.2.S₃ is a P position for all $\alpha \geq \gamma - 2$, and by Theorem 5.13 Part 3, B.2.M is a P position for all $\alpha \leq \gamma - 3$. Thus, $(3\alpha + 1, 0, 4\gamma)$ is a P position for all $\alpha, \gamma \in \mathbb{Z}^{\geq 0}$.

Proof of Part 4c: Next, we consider the game on $(3\alpha + 2, 0, 4\gamma)$, with $\alpha \geq \gamma + 1$, noting as a base case that $(5, 0, 0)$ is a P position by Theorem 5.1. Now, consider the game on an arbitrary $(3\alpha + 2, 0, 4\gamma)$ with $\alpha \geq \gamma + 1$.



Since $\alpha \geq \gamma + 1$, B.2.A₁ is a P position by the proof of Part 1c and B.2.S₃ is a P position as shown in the proof of Part 4b. Then, $(3\alpha + 2, 0, 4\gamma)$ is a P position for all $\alpha \geq \gamma + 1$.

This concludes our proof of Theorem 5.15. □

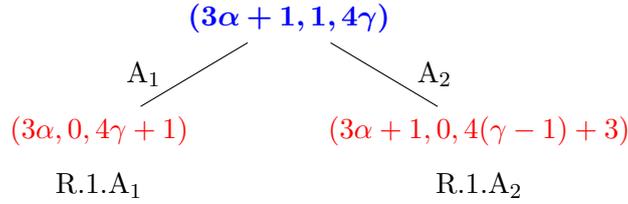
From here, we give a constructive proof of Lemma 5.5 and Corollary 5.8 with the information we have built up so far. This ensures that the strategy we outline for N positions in Theorem 5.17 is in fact constructive.

Lemma 5.16. (Constructive.) *Let (a, b, c) be a game state for an F_4 Black Hole Zeckendorf game. For all $\alpha, \gamma, k_1, k_3 \in \mathbb{Z}^{\geq 0}$, such that $1 \leq k_1 \leq 2$, and $0 \leq k_3 \leq 3$, $(3\alpha + k_1, 1, 4\gamma + k_3)$ is an N position. $(3\alpha, 1, 4\gamma)$ is an N position for all $\alpha \geq \gamma + 1$.*

Proof. We proceed by demonstrating what the opening move should be from $(3\alpha + k_1, 1, 4\gamma + k_3)$ for $k_1 = 1, 2$. For clarity, we consider each possible combination of k_1 and k_2 individually.

Proof for $k_1 = 1$.

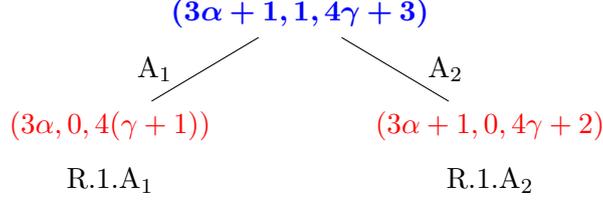
First, consider the game on $(3\alpha + 1, 1, 4\gamma)$ in the game tree below. R.1.A₁ is a P position for all $\alpha \geq \gamma - 1$ by Theorem 5.15 Part 1a, and R.1.A₂ is a P position for all $\alpha \leq \gamma - 2$ by Theorem 5.15 Part 3a. Therefore, $(3\alpha + 1, 1, 4\gamma)$ is an N position for all α, γ .



Next, consider the game on $(3\alpha + 1, 1, 4\gamma + 1)$. The next player to move can add in the second and third column to place $(3\alpha + 1, 0, 4\gamma)$, which is a P position for all α, γ , by Theorem 5.15 Part 4b. Therefore, $(3\alpha + 1, 1, 4\gamma + 1)$ is an N position for all α, γ .

Then, consider the game on $(3\alpha + 1, 1, 4\gamma + 2)$. The next player to move can add in the second and third column to place $(3\alpha + 1, 0, 4\gamma + 1)$, which is a P position for all α, γ by Theorem 5.15 Part 1b. Therefore, $(3\alpha + 1, 1, 4\gamma + 2)$ is an N position for all α, γ .

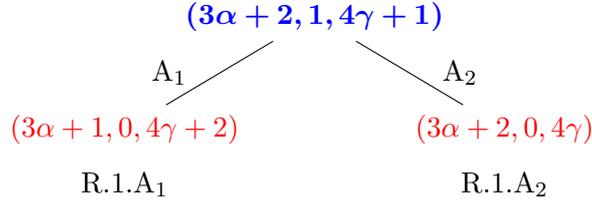
Lastly, consider the game on $(3\alpha + 1, 1, 4\gamma + 3)$ in the game tree below. R.1.A₁ is a P position for all $\alpha \geq \gamma + 1$ by Theorem 5.15 Part 4a, and R.1.A₂ is a P position for all $\alpha \leq \gamma$ by Theorem 5.15 Part 2a. Therefore, $(3\alpha + 1, 1, 4\gamma)$ is an N position for all α, γ .



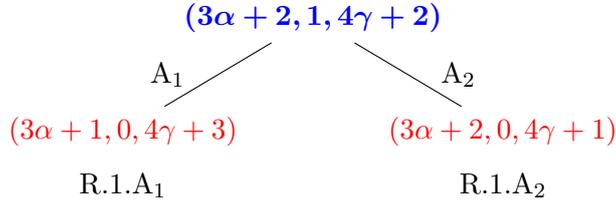
Proof for $k_1 = 2$.

The strategy here is essentially the same as the one outlined above, with the optimal strategy shifted for the value of k_3 . First, consider the game on $(3\alpha + 2, 1, 4\gamma)$. The next player to move can add in the first and second column to place $(3\alpha + 1, 0, 4\gamma + 1)$ which is a P position for all α, γ by Theorem 5.15 Part 1b. Therefore, $(3\alpha + 2, 1, 4\gamma)$ is an N position for all α, γ .

Next, consider the game on $(3\alpha + 2, 1, 4\gamma + 1)$ in the game tree below. $R.1.A_1$ is a P position for all $\alpha \leq \gamma$ by Theorem 5.15 Part 2a, and $R.1.A_2$ is a P position for all $\alpha \geq \gamma + 1$ by Theorem 5.15 Part 4c. Therefore, $(3\alpha + 2, 1, 4\gamma + 1)$ is an N position for all α, γ .



Then, consider the game on $(3\alpha + 2, 1, 4\gamma + 2)$ in the game tree below. $R.1.A_1$ is a P position for all $\alpha \leq \gamma - 1$ by Theorem 5.15 Part 3a, and $R.1.A_2$ is a P position for all $\alpha \geq \gamma$ by Theorem 5.15 Part 1c. Therefore, $(3\alpha + 2, 1, 4\gamma + 2)$ is an N position for all α, γ .



Then, consider the game on $(3\alpha + 2, 1, 4\gamma + 3)$. The next player to move can add in the first and second column to place $(3\alpha + 1, 0, 4(\gamma + 1))$ which is a P position for all α, γ by Theorem 5.15 Part 4b. Therefore, $(3\alpha + 2, 1, 4\gamma + 3)$ is an N position for all α, γ .

Finally, see that $(3\alpha, 1, 4\gamma)$ is an N position for all $\alpha \geq \gamma + 1$ as the next player to move can place $(3(\alpha - 1) + 2, 0, 4\gamma + 1)$ which is a P position for all $\alpha \geq \gamma + 1$ by Theorem 5.15 Part 1c. \square

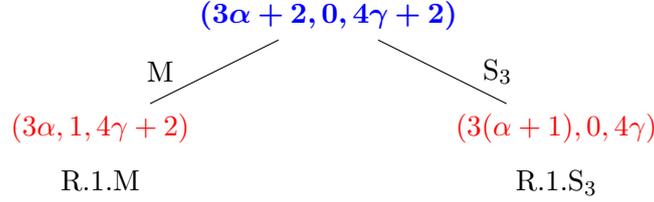
From here, we outline the winning strategies for the next player to move following an N position.

Theorem 5.17. *Let $(a, 0, c) = (3\alpha + k_1, 0, 4\gamma + k_3)$ be a board setup for an F_4 Black Hole Zeckendorf game. For $\alpha, \gamma, k_1, k_3 \in \mathbb{Z}^{\geq 0}$, $0 \leq k_1 \leq 2$, and $0 \leq k_3 \leq 3$, the following positions are N positions:*

- (1) $c \equiv 1 \pmod{4}$:
 - (a) $(3\alpha, 0, 4\gamma + 1)$ for all $\alpha \leq \gamma - 2$.

$(3\alpha + 1, 0, 4\gamma + 2)$ can always split in the F_3 column to place $(3\alpha + 2, 0, 4\gamma)$, which is a P position for all $\alpha \geq \gamma + 1$ by Theorem 5.15 Part 4c.

Proof of Part 2c: We conclude this section by showing that $(3\alpha + 2, 0, 4\gamma + 2)$ is an N position for all α, γ , noting that as a base case $(2, 0, 2)$ is an N position by Corollary 5.4. We consider the general case below.



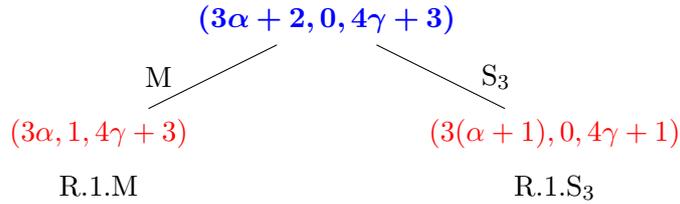
By Theorem 5.13 Part 3, R.1.M is a P position for all $\alpha \leq \gamma - 2$, and by Lemma 5.14 Part 1 R.1.S₃ is a P position for all $\alpha \geq \gamma - 1$. Thus, $(3\alpha + 2, 0, 4\gamma)$ is an N position for all α, γ .

Proof for $c \equiv 3 \pmod{4}$.

Proof of Part 3a: We begin this section by showing that $(3\alpha, 0, 4\gamma + 3)$ is an N position for all α, γ , noting as a base case that $(0, 0, 3)$ is an N position by Theorem 5.3. For the general case, see that from $(3\alpha, 0, 4\gamma + 3)$, it is possible to split in the third column to place $(3\alpha + 1, 0, 4\gamma + 1)$, which is a P position for all α, γ as shown in Theorem 5.15 Part 1b. Therefore, $(3\alpha, 0, 4\gamma + 3)$ is an N position for all α, γ .

Proof of Part 3b: We follow by showing that $(3\alpha + 1, 0, 4\gamma + 3)$ is an N position for all $\alpha \geq \gamma$, noting as a base case that $(1, 0, 3)$ is an N position by Corollary 5.4. In the general case, the first player to move from $(3\alpha + 1, 0, 4\gamma + 3)$ can split in the third column to place $(3\alpha + 2, 0, 4\gamma + 1)$, which is a P position by Theorem 5.15 Part 1c. Therefore, $(3\alpha + 1, 0, 4\gamma + 3)$ is an N position for all $\alpha \geq \gamma$.

Proof of Part 3c: We conclude this section by showing that $(3\alpha + 2, 0, 4\gamma + 3)$ is an N position for all α, γ , noting as a base case that $(2, 0, 3)$ is an N position by Corollary 5.4. We consider the general case below.



By Theorem 5.13 Part 4, R.1.M is a P position for all $\alpha \leq \gamma + 2$ and, by Corollary 5.9 Part 2, R.1.S₃ is a P position for all $\alpha \geq \gamma - 1$. Thus, $(3\alpha + 2, 0, 4\gamma + 3)$ is an N position for all α, γ .

Proof for $c \equiv 0 \pmod{4}$.

Proof of Part 4a: Consider the game on $(3\alpha, 0, 4\gamma)$ where $\alpha \leq \gamma - 1$. As a base case, note that $(0, 0, 4)$ is an N position by Theorem 5.3. In the general case, the first player to move from $(3\alpha, 0, 4\gamma)$ can split in the third column to place $(3\alpha + 1, 0, 4(\gamma - 1) + 2)$, which is a P position by Theorem 5.15 Part 2a since $\alpha \leq \gamma - 1$. Therefore, $(3\alpha, 0, 4\gamma)$ is an N position for all $\alpha \leq \gamma - 1$.

Proof of Part 4b: To conclude the proof of this theorem, we consider the game on $(3\alpha + 2, 0, 4\gamma)$ with $\alpha \leq \gamma$. As a base case, see that $(2, 0, 0)$ is an N position by Theorem 5.1. In the general case, it is possible to merge from $(3\alpha + 2, 0, 4\gamma)$ to place $(3\alpha, 1, 4\gamma)$, which is a P position for $\alpha \leq \gamma$ by Theorem 5.13 Part 1. Therefore, $(3\alpha + 2, 0, 4\gamma)$ is an N position for all $\alpha \leq \gamma$.

This concludes our proof of Theorem 5.17. □

5.3. Empty Board Game

We now continue to the Empty Board game, to determine which player has a winning strategy for any given $n \in \mathbb{Z}^{>0}$. We define the game so that players can still only place pieces in the outermost columns, meaning that no player can move to the F_2 column until all pieces have been placed on the board. Then, the initial setup is guaranteed to be in the form of some $(a, 0, c)$ above. Players are able to force certain setups, as there are only 2 options for placement, so the “Tweedledum and Tweedledee” strategy remains an option as in the F_3 Black Hole game. Again, players aim to either set the board as a P position or force their opponent to set it as an N position.

Theorem 5.18. *Let $(0, 0, 0)$ be the beginning board for an Empty Board F_4 Black Hole Zeckendorf game with $n \in \mathbb{Z}^{>0}$ pieces. Players can force certain game setups as outlined below.*

- (1) *For any $n \equiv 0 \pmod{4}$, one player can force the game into a setup $(n/4, 0, n/4)$.*
- (2) *For any $n \equiv 1 \pmod{4}$, one player can force the game into a setup $((n-1)/4 + 1, 0, (n-1)/4)$.*
- (3) *For any $n \equiv 2 \pmod{4}$, one player can force the game into a setup $((n-2)/4 + 2, 0, (n-2)/4)$.*
- (4) *For any $n \equiv 3 \pmod{4}$, Player 1 can force the game setup $((n-3)/4 + 3, 0, (n-3)/4)$ or can alternatively force the setup $((n-3)/4, 0, (n-3)/4 + 1)$. Player 2 can force the game to begin with one of these two setups, but cannot force a specific one.*

Proof. First, consider $n \equiv 0 \pmod{4}$. If $(n/4, 0, n/4)$ is an N position, then Player 1 can force Player 2 to place this setup by placing their first piece in the F_3 column, and then placing opposite Player 2 for all other moves. It will be Player 2’s turn when there is 1 piece left, so they will be forced to place down $(n/4, 0, n/4)$. If $(n/4, 0, n/4)$ is a P position, they can mirror Player 1 until they are able to place down $(n/4, 0, n/4)$.

Next, consider $n \equiv 1 \pmod{4}$. If $((n-1)/4 + 1, 0, (n-1)/4)$ is a P position, Player 1 can place their first piece in the F_1 column, and then place opposite Player 2 for all other moves. Every board Player 1 places will be of the form $(m + 1, 0, m)$ for some non-negative integer m , until all n have been placed, resulting in the setup $((n-1)/4 + 1, 0, (n-1)/4)$. If $((n-1)/4 + 1, 0, (n-1)/4)$ is an N position, Player 2 can mirror Player 1 until there is one piece left, forcing Player 1 to place $((n-1)/4 + 1, 0, (n-1)/4)$.

Then, consider $n \equiv 2 \pmod{4}$. If $((n-2)/4 + 2, 0, (n-2)/4)$ is an N position, Player 1 can place their first piece in the F_1 column, and then place opposite Player 2 for all other moves. Every board Player 1 places will be of the form $(m + 1, 0, m)$ until there is only 1 piece left, which Player 2 is forced to set down in the F_1 column, placing down $((n-2)/4 + 2, 0, (n-2)/4)$. If this is a P position, Player 2 can force it via the “Tweedledum and Tweedledee” strategy until there are only 2 pieces left. Since pieces can only be placed in the outermost columns, Player 1 is forced to place $((n-2)/4 + 1, 0, (n-2)/4)$, allowing Player 2 to place $((n-2)/4 + 2, 0, (n-2)/4)$.

Lastly, consider $n \equiv 3 \pmod{4}$. If Player 1 places their first piece in the F_1 column, and then places opposite Player 2 for all other moves, then every board Player 1 places will be of the form $(m+1, 0, m)$ until there are only 2 pieces left; Player 2 is forced to set down one in the F_1 column, and Player 1 sets the other, placing down $((n-3)/4+3, 0, (n-3)/4)$. Player 1 can also force the setup $((n-3)/4, 0, (n-3)/4+1)$ by placing their first piece in the F_3 column, and then mirroring Player 2, until the board results in Player 1 placing $((n-3)/4, 0, (n-3)/4+1)$. Player 2 can force the game to be one of these positions by mirroring Player 1 until there are 3 pieces left. However, at this point Player 1 determines whether the setup is $((n-3)/4+3, 0, (n-3)/4)$ or $((n-3)/4, 0, (n-3)/4+1)$ as noted above. \square

Theorem 5.19. *Player 2 has a constructive strategy for winning an Empty Board F_4 Black Hole Zeckendorf game for any $n \equiv 0, 2, 4, 6, 9, 11, 13 \pmod{16}$ such that $n \neq 2, 32$, in which case Player 1 has the winning strategy. Player 1 has a constructive strategy for winning an Empty Board F_4 Black Hole Zeckendorf game for any $n \equiv 1, 3, 5, 7, 8, 10, 12, 14, 15 \pmod{16}$, such that $n \neq 17, 47$, in which case Player 2 has the winning strategy.*

Proof. We split this proof into sections based on the value of n modulo 4.

Proof for $n \equiv 0 \pmod{4}$.

Player 2 sets down the board as $(n/4, 0, n/4)$ by Theorem 5.18, so they win when this is a P position and lose when it is an N position. For $n \equiv 0, 4 \pmod{16}$, we have $n/4 \equiv 0, 1 \pmod{4}$ while $n \equiv 8, 12 \pmod{16}$ implies $n/4 \equiv 2, 3 \pmod{4}$.

For all $n/4 \geq 9$, with $0 \leq k_1 \leq 2$, $0 \leq k_3 \leq 3$ there is no $\alpha \leq \gamma$ such that $3\alpha + k_1 = n/4 = 4\gamma + k_3$. Thus, when $n/4 \equiv 0, 1 \pmod{4}$, Player 2 wins for all $n/4 \geq 9$ by Theorem 5.15 Part 4 and 1, respectively. When $n/4 \equiv 2, 3 \pmod{4}$, Player 1 wins for all $n/4 \geq 9$ by Theorem 5.17 Part 2 and 3, respectively.

Then, it is only necessary to explicitly consider the cases when $n/4 \leq 8$. We outline the value of n , the cases, the winners, and the parts of Theorem 5.15 and Theorem 5.17 that determine the winner below.

Value of n	Board Setup	Winner	Theorem
$n = 4$	$(1, 0, 1) = (3(0) + 1, 0, 4(0) + 1)$	Player 2	5.15 Part 1b
$n = 8$	$(2, 0, 2) = (3(0) + 2, 0, 4(0) + 2)$	Player 1	5.17 Part 2c
$n = 12$	$(3, 0, 3) = (3(1) + 0, 0, 4(0) + 3)$	Player 1	5.17 Part 3a
$n = 16$	$(4, 0, 4) = (3(1) + 1, 0, 4(1) + 0)$	Player 2	5.15 Part 4b
$n = 20$	$(5, 0, 5) = (3(1) + 2, 0, 4(1) + 1)$	Player 2	5.15 Part 1c
$n = 24$	$(6, 0, 6) = (3(2) + 0, 0, 4(1) + 2)$	Player 1	5.17 Part 2a
$n = 28$	$(7, 0, 7) = (3(2) + 1, 0, 4(1) + 3)$	Player 1	5.17 Part 3b
$n = 32$	$(8, 0, 8) = (3(2) + 2, 0, 4(2) + 0)$	Player 1	5.17 Part 4b

Thus, Player 2 wins for all $n \equiv 0, 4 \pmod{16}$ such that $n \neq 32$ and Player 1 wins for all $n \equiv 8, 12 \pmod{16}$, as well as $n = 32$.

Proof for $n \equiv 1 \pmod{4}$.

Player 1 sets down the board as $((n-1)/4+1, 0, (n-1)/4)$ by Theorem 5.18. For $n \equiv 1, 5 \pmod{16}$, we have $(n-1)/4 \equiv 0, 1 \pmod{4}$ while $n \equiv 9, 13 \pmod{16}$ implies $(n-1)/4 \equiv 2, 3$

(mod 4).

For all $(n-1)/4 \geq 5$ such that $0 \leq k_1 \leq 2$, $0 \leq k_3 \leq 3$ there is no $\alpha \leq \gamma$ such that $3\alpha + k_1 = (n-1)/4 + 1$ and $4\gamma + k_3 = (n-1)/4$. Therefore, when $(n-1)/4 \geq 5$, Player 1 wins for all $(n-1)/4 \equiv 0, 1 \pmod{4}$ by Theorem 5.15 Parts 1 and 4. When $(n-1)/4 \equiv 2, 3 \pmod{4}$, Player 2 wins for all $(n-1)/4$ by Theorem 5.17 Parts 2 and 3.

Then, it is only necessary to explicitly consider the cases when $(n-1)/4 \leq 4$. We outline the value of n , the cases, the winners, and the part of Theorems 5.15 and 5.17 that determines the winner below.

Value of n	Board Setup	Winner	Theorem
$n = 1$	$(1, 0, 0) = (3(0) + 1, 0, 4(0) + 0)$	Player 1	5.15 Part 4b
$n = 5$	$(2, 0, 1) = (3(0) + 2, 0, 4(0) + 1)$	Player 1	5.15 Part 1c
$n = 9$	$(3, 0, 2) = (3(1) + 0, 0, 4(0) + 2)$	Player 2	5.17 Part 2a
$n = 13$	$(4, 0, 3) = (3(1) + 1, 0, 4(0) + 3)$	Player 2	5.17 Part 3b
$n = 17$	$(5, 0, 4) = (3(1) + 2, 0, 4(1) + 0)$	Player 2	5.17 Part 4b

Therefore, Player 1 wins for all $n \equiv 1, 5 \pmod{16}$ such that $n \neq 17$ and Player 2 wins for all $n \equiv 9, 13 \pmod{16}$, as well as $n = 17$.

Proof for $n \equiv 2 \pmod{4}$.

Player 2 sets down the board as $((n-2)/4 + 2, 0, (n-2)/4)$ by Theorem 5.18. For $n \equiv 2, 6 \pmod{16}$, we have $(n-2)/4 \equiv 0, 1 \pmod{4}$ while $n \equiv 10, 14 \pmod{16}$ implies $(n-2)/4 \equiv 2, 3 \pmod{4}$. For all $(n-2)/4 \geq 1$, and $0 \leq k_1 \leq 2$, $0 \leq k_3 \leq 3$ there is no $\alpha \leq \gamma$ such that $3\alpha + k_1 = (n-2)/4 + 2$ and $4\gamma + k_3 = (n-2)/4$. When $(n-2)/4 \equiv 0, 1 \pmod{4}$, Player 2 wins for all $(n-2)/4 \geq 1$ by Theorem 5.15 Parts 1 and 4. When $(n-2)/4 \equiv 2, 3 \pmod{4}$, Player 1 wins for all $(n-1)/4 \geq 1$ by Theorem 5.17 Parts 2 and 3.

Then, it is only necessary to explicitly consider the case $(2, 0, 0)$, which Player 1 wins by Theorem 5.1. Hence, Player 2 wins for all $n \equiv 2, 6 \pmod{16}$ such that $n \neq 2$ and Player 1 wins for all $n \equiv 10, 14 \pmod{16}$, as well as $n = 2$.

Proof for $n \equiv 3 \pmod{4}$.

By Theorem 5.18, Player 1 either sets down $((n-3)/4 + 3, 0, (n-3)/4)$ or $((n-3)/4, 0, (n-3)/4 + 1)$.

For $n \equiv 3, 7 \pmod{16}$, we have $(n-3)/4 \equiv 0, 1 \pmod{4}$. Here, Player 1 should set down $((n-3)/4 + 3, 0, (n-3)/4)$. Then, for all $(n-3)/4 \geq 0$, and $0 \leq k_1 \leq 2$, $0 \leq k_3 \leq 3$ there is no $\alpha \leq \gamma$ such that $3\alpha + k_1 = (n-3)/4 + 3$ and $4\gamma + k_3 = (n-3)/4$. So, for all $n \equiv 3, 7 \pmod{16}$, Player 1 wins by Theorem 5.15 Parts 1 and 4.

Next, for $n \equiv 15 \pmod{16}$, we have $(n-3)/4 + 1 \equiv 0 \pmod{4}$. Player 1 should set down $((n-3)/4, 0, (n-3)/4 + 1)$; then, for all $(n-3)/4 \geq 12$, and $0 \leq k_1 \leq 2$, $0 \leq k_3 \leq 3$ there is no $\alpha \leq \gamma$ such that $3\alpha + k_1 = (n-3)/4$ and $4\gamma + k_3 = (n-3)/4 + 1$. Then, Player 1 wins by

Theorem 5.15 Part 4. We explicitly consider $n \equiv 15 \pmod{16}$ such that $(n-3)/4 \leq 11$ below.

Value of n	Board Setup	Winner	Theorem
$n = 15$	$(3, 0, 4) = (3(1) + 0, 0, 4(1) + 0)$	Player 1	5.15 Part 4a
$n = 31$	$(7, 0, 8) = (3(2) + 1, 0, 4(2) + 0)$	Player 1	5.15 Part 4b
$n = 47$	$(11, 0, 12) = (3(3) + 2, 0, 4(3) + 0)$	Player 2	5.17 Part 4b
$n = 47$	$(14, 0, 11) = (3(4) + 2, 0, 4(2) + 3)$	Player 2	5.17 Part 3c

Lastly, for $n \equiv 11 \pmod{16}$, we have $(n-3)/4 \equiv 2 \pmod{4}$ and $(n-3)/4 + 1 \equiv 3 \pmod{4}$. Therefore, Player 2 can force Player 1 to set the board such that $c \equiv 2, 3 \pmod{4}$. If Player 1 places $((n-3)/4 + 3, 0, (n-3)/4)$, then since there exists no $\alpha \leq \gamma$ that satisfies $3\alpha + k_1 = (n-3)/4 + 3$ and $4\gamma + k_3 = (n-3)/4$, Player 2 wins by Theorem 5.15 Part 2.

If Player 1 places $((n-3)/4, 0, (n-3)/4 + 1)$, Player 2 wins for all $(n-3)/4 \geq 12$ by Theorem 5.15 Part 3. We explicitly consider $n \equiv 11 \pmod{16}$ such that $(n-3)/4 \leq 11$ below.

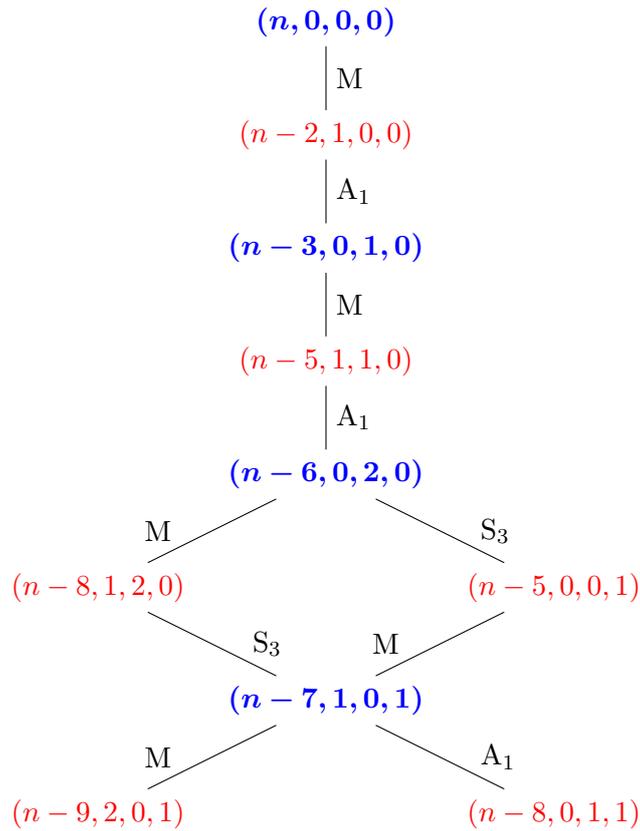
Value of n	Board Setup	Winner	Theorem
$n = 11$	$(2, 0, 3) = (3(0) + 2, 0, 4(0) + 3)$	Player 2	5.17 Part 3c
$n = 27$	$(6, 0, 7) = (3(2) + 0, 0, 4(1) + 3)$	Player 2	5.17 Part 3a
$n = 43$	$(10, 0, 11) = (3(3) + 1, 0, 4(2) + 3)$	Player 2	5.17 Part 3b

So then, we find that Player 1 has a constructive winning solution for all $n \equiv 1, 3, 5, 7, 8, 10, 12, 14, 15 \pmod{16}$ such that $n \neq 17, 47$, for which Player 2 has a winning solution. Player 2 has a constructive winning solution for all $n \equiv 0, 2, 4, 6, 9, 11, 13 \pmod{16}$ such that $n \neq 2, 32$, for which Player 1 has a winning solution. \square

6. Future Work

To determine a constructive solution for the Empty Board F_4 Black Hole Zeckendorf game as we defined it, it was only necessary to determine which player wins for any given $(a, 0, c)$. Still, a complete analysis of (a, b, c) for any value of b could give more insight into the original problem, and could be an interesting area for future work. Redefining the Empty Board F_4 Black Hole Zeckendorf game to allow players to place in the F_2 column would also be interesting, as the TweedleDee-TweedleDum strategy is no longer as applicable.

Note also that the strategy of reducing modulo F_m is not as immediately successful when the black hole is at F_m such that $m \geq 5$. See below an example of a game with a black hole on $F_5 = 8$.



In the typical Zeckendorf game with $n = 8$, Player 1 is forced to place $(n - 8, 0, 1, 1)$. When we play on 4 columns but with more than 8 pieces, Player 1 can now also place $(n - 9, 2, 0, 1)$, making it challenging for Player 2 to place a piece in the black hole without giving Player 1 the opportunity to do so first. Expanding to black holes on higher Fibonacci numbers may very well prove a fruitful step in determining a constructive solution to the original Zeckendorf game.

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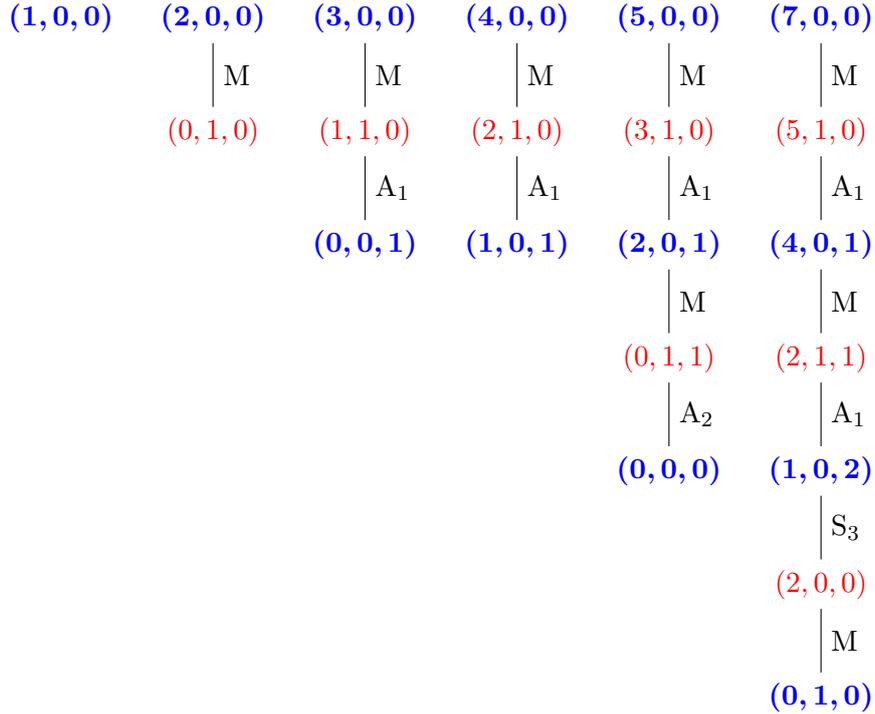
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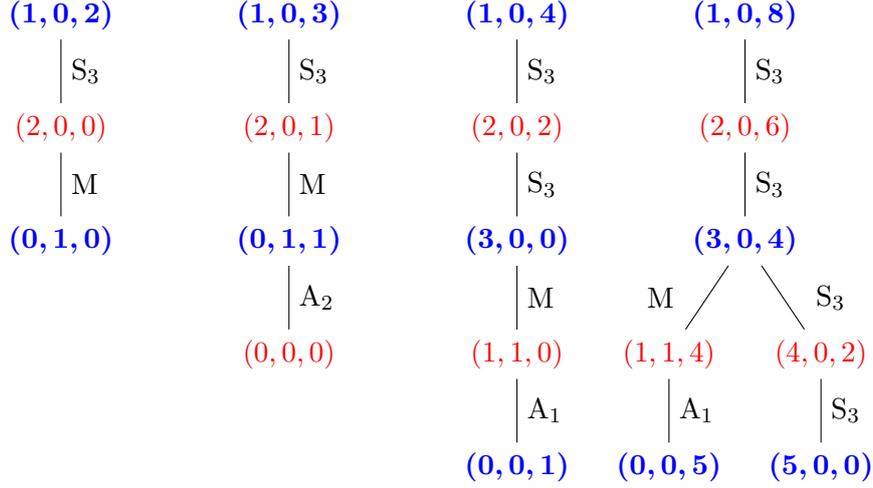
Appendix A. Base Case Games for $(a, 0, 0)$

For the base cases of Theorem 5.1, we have the following games, showing that $(a, 0, 0)$ is a P position for $a = 1, 3, 4, 5, 7$ and is an N position for $a = 2$.



Appendix B. Base Case Games for $(0, 0, c)$

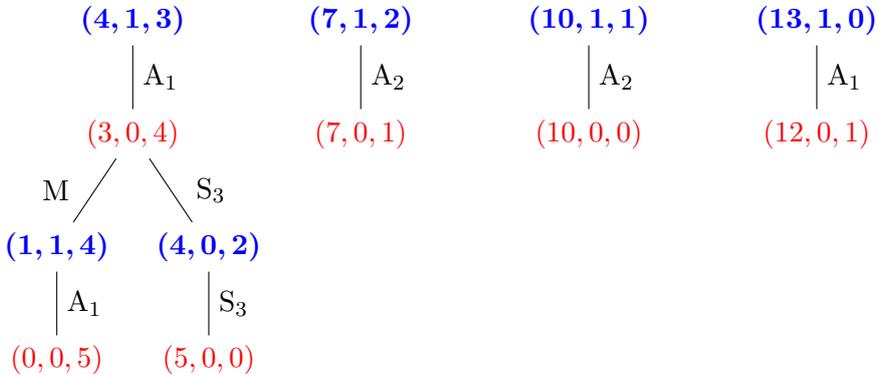
For the base cases of Theorem 5.3, we have the following games, showing that $(1, 0, c)$ is a P position for $c = 2, 4, 8$ and an N position for $c = 3$. Note that $(1, 0, 0)$ and $(1, 0, 1)$ are trivially P positions.



Appendix C. Base Case Games for Lemma 5.5

We showed in Lemma 5.5 that $(3\alpha + 1, 1, 4\gamma + k_3)$ is an N position for all $4\gamma + \alpha + k_3 - 1 \neq 3$ and that $(3\alpha + 2, 1, 4\gamma + k_3)$ is an N position for all $4\gamma + \alpha + k_3 + 1 \neq 3$. We show here that the exception cases are also N positions. Note that $\gamma \neq 0$ can only be true when in the first equation $\gamma = 1, \alpha = k_3 = 0$. Here, the board is set as $(1, 1, 4)$, from which the next player can place $(0, 0, 5)$, winning by Theorem 5.3. For all other cases, $\gamma = 0$ since $\alpha, \gamma, k_3 \in \mathbb{Z}^{\geq 0}$.

First, see that $(3\alpha + 1, 1, 4\gamma + k_3)$ is an N position for $\alpha + k_3 = 4$. The possible solutions (α, k_3) such that $\alpha, k_3 \in \mathbb{Z}^{\geq 0}$ and $0 \leq k_3 \leq 3$ are $(1, 3)$, $(2, 2)$, $(3, 1)$, and $(4, 0)$. We show that the corresponding game positions $(4, 1, 3)$, $(7, 1, 2)$, $(10, 1, 1)$, and $(12, 0, 1)$ are N positions in the game trees below.



$(5, 0, 0)$ and $(10, 0, 0)$ win by Theorem 5.1. $(0, 0, 5)$ wins by Theorem 5.3. $(7, 0, 1)$ and $(12, 0, 1)$ win by Corollary 5.2. Therefore, $(3\alpha + 2, 1, 4\gamma + k_3)$ is an N position.

Next consider $(3\alpha + 2, 1, 4\gamma + k_3)$ which must be an N position for all $\alpha + k_3 = 2$ when $\alpha, k_3 \in \mathbb{Z}^{\geq 0}$ and $0 \leq k_3 \leq 3$. The possible solutions (α, k_3) to $\alpha + k_3 = 2$ such that $\alpha, k_3 \in \mathbb{Z}^{\geq 0}$ are $(0, 2)$, $(1, 1)$, and $(2, 0)$. We show that the corresponding game positions $(2, 1, 2)$, $(5, 1, 1)$, and $(8, 1, 0)$ are all N positions.

$$\begin{array}{ccc}
(2, 1, 2) & (5, 1, 1) & (8, 1, 0) \\
\left| A_2 \right. & \left| A_2 \right. & \left| A_1 \right. \\
(2, 0, 1) & (5, 0, 0) & (7, 0, 1)
\end{array}$$

$(2, 0, 1)$ and $(7, 0, 1)$ win by Corollary 5.2 and $(5, 0, 0)$ wins by Theorem 5.1. Therefore, $(3\alpha + 2, 1, 4\gamma + k_3)$ is an N position.

Appendix D. Code

In this project, we modified code by Paul Baird-Smith for [2] to function for the Black Hole Zeckendorf game. Our code can be found here: <https://github.com/JennaShuff/ZeckendorfGameBlackHoles2024> while Baird-Smith's code is available at <https://github.com/paulbsmith1996/ZeckendorfGame/>.