DYNAMICS OF THE FIBONACCI ORDER OF APPEARANCE MAP

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Abstract. The *order of appearance* $z(n)$ of a positive integer n in the Fibonacci sequence is defined as the smallest positive integer j such that n divides the j-th Fibonacci number. A *fixed point* arises when, for a positive integer n, we have that the n^{th} Fibonacci number is the smallest Fibonacci that n divides. In other words, $z(n) = n$.

In 2012, Marques proved that fixed points occur only when n is of the form 5^k or $12 \cdot 5^k$ for all non-negative integers k. It immediately follows that there are infinitely many fixed points in the Fibonacci sequence. We prove that there are infinitely many integers that iterate to a fixed point in exactly k steps. In addition, we construct infinite families of integers that go to each fixed point of the form $12 \cdot 5^k$. We conclude by providing an alternate proof that all positive integers n reach a fixed point after a finite number of iterations.

1. Introduction

In 1202, the Italian mathematician Leonardo Fibonacci introduced the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$, defined recursively as $F_n = F_{n-1} + F_{n-2}$ with initial conditions $F_0 = 0$ and $F_1 = 1$. By reducing $\{F_n\}_{n=0}^{\infty}$ modulo m, we obtain a periodic sequence $\{F_n \mod m\}_{n=0}^{\infty}$. This new sequence and its divisibility properties have been extensively studied, see for example [M1, M5]. To see why the reduced sequence is periodic, note that by the pigeonhole principle if we look at $n^2 + 1$ pairs (F_k, F_{k-1}), at least two are identical (mod n) and the recurrence relation generates the same future terms.

Definition 1.1. The *order (or rank) of appearance* $z(n)$ for a natural number n in the Fibonacci sequence is the smallest positive integer ℓ such that $n \mid F_\ell$.

Observe that the function $z(n)$ is well defined for all n since the Fibonacci sequences begins with $0, 1, \ldots$ and when reduced by modulo n, a 0 will appear again in the periodic sequence. Thus, there will always be a Fibonacci number that is congruent to $0 \mod n$ for each choice of n. The upper bound of $n^2 + 1$ on $z(n)$ is improved in [S], which states $z(n) \leq 2n$ for all $n \geq 1$. This is the sharpest upper bound on $z(n)$. In [M2], sharper upper bounds for $z(n)$ are provided for some positive integers n. Additional results on $z(n)$ include explicit formulae for the order of appearance of some n relating to sums containing Fibonacci numbers [M3] and products of Fibonacci numbers [M4]. We study repeated applications of z on n and denote the k^{th} application of z on n as $z^k(n)$. We are interested in the following quantity.

Definition 1.2. The *fixed point order* for a natural number n is the smallest positive integer k such that $z^k(n)$ is a fixed point. If n is a fixed point, then we say the *fixed point order* of n is 0.

Table 1 shows which values occur after repeated iterations of z on the first 12 positive integers. We further the study of repeated iterations of z on n.

In Section 2, we provide some useful properties of the order of appearance in Fibonacci numbers. In the remaining sections, we prove our main results, found below.

Theorem 1.3. *For all positive integers k, there exist infinitely many n with fixed point order k.*

Theorem 1.4. *Infinitely many integers n iterate to each fixed point of the form $12 \cdot 5^k$.*

Theorem 1.5. *All positive integers n have finite fixed point order.*

Theorem 1.5 was proved first in [LT] by showing that within finite k, $z^k(n) = 2^a 3^b 5^c$ where $a, b, c \in \mathbb{Z}_{\geq 0}$ and then proving that $2^a 3^b 5^c$ iterates to a fixed point in a finite number of steps. It was later proved in [Ta1] using a relationship between the Pisano period of n and $z(n)$. We provide an alternate proof using a minimal counterexample argument.

$n \setminus k$	1	2	3	4	5
1	**1**				
2	3	4	6	**12**	
3	4	6	**12**		
4	6	**12**			
5	5				
6	**12**				
7	8	6	**12**		
8	6	**12**			
9	**12**				
10	15	20	30	**60**	
11	10	15	20	30	**60**
12	**12**				

Table 1. Iterations of z on n, numbers in bold are fixed points.

2. Auxiliary Results

Here we include some needed results from previous papers.

Lemma 2.1. *Let n be a positive integer. Then $z(n) = n$ if and only if $n = 5^k$ or $n = 12 \cdot 5^k$ for some $k \geq 0$.*

A proof of Lemma 2.1 can be found in [M1, SM].

Lemma 2.2. *For all $a \in \mathbb{Z}, a \geq 3$, $z(2^a) = 2^{a-2} \cdot 3$. For all $b \in \mathbb{Z}^+, z(3^b) = 4 \cdot 3^{b-1}$.*

Lemma 2.2 is Theorem 1.1 of [M2].

Lemma 2.3. *Let $n \geq 2$ be an integer with prime factorization $n = p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m}$ where $p_1, p_2, \ldots, p_m$ are prime and $e_1, e_2, \ldots, e_m$ are positive integers. Then*

$$z(n) \ = \ z(p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m}) \ = \ \operatorname{lcm}\left(z(p_1^{e_1}), z(p_2^{e_2}), \ldots, z(p_m^{e_m})\right). \tag{2.1}$$

A proof of Lemma 2.3 can be found in Theorem 3.3 of [R]. Lemma 2.3 has been generalized as follows.

Lemma 2.4. *Let $n \geq 2$ be an integer with prime factorization $n = p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m}$ where $p_1, p_2, \ldots, p_m$ are prime and $e_1, e_2, \ldots, e_m$ are positive integers. Then*

$$z\left(\operatorname{lcm}(m_1, m_2, \ldots, m_n)\right) \ = \ \operatorname{lcm}\left(z(m_1), z(m_2), \ldots, z(m_n)\right). \tag{2.2}$$

A proof of Lemma 2.4 can be found in Lemma 4 of [Ty].

Lemma 2.5. *For all primes p, $z(p) \leq p + 1$.*

A proof of Lemma 2.5 can be found in Lemma 2.3 of [M1].

Lemma 2.6. *For all positive integers n, $z(n) \leq 2n$, with equality if and only if $n = 6 \cdot 5^k$ for some $k \in \mathbb{Z}_{\geq 0}$*

Lemma 2.6 is proven in [S].

Lemma 2.7. *For all primes $p \neq 5$, we have that* $\gcd(p, z(p)) = 1$.

Lemma 2.7 is proven in Lemma 2.3 of [M1].

Lemma 2.8. *If $n|F_m$, where F_m is the m^{th} number in the Fibonacci sequence, then $z(n)|m$.*

Lemma 2.8 is Lemma 2.2 of [M1].

Lemma 2.9. *For all odd primes p, we have $z(p^e) = p^{\max(e-a,0)}z(p)$ where a is the number of times that p divides $F_{z(p)}$, $a \geq 1$. In particular, $z(p^e) = p^r z(p)$ for some $0 \leq r \leq e-1$.*

For a proof of Lemma 2.9, see Theorem 2.4 of [FM].

3. Infinitely many integers take a given number of iterations to reach a fixed point

In this section we first prove Lemma 3.1, which helps us show that when $z^i(n)$ is written as the product of a constant relatively prime to 5 and a power of 5, then $z^i(n \cdot 5^a)$ can be written as the product of that same constant and another power of 5. Table 2 lists the smallest n that takes exactly k iterations to reach a fixed point for positive integers k up to 10.

k	n	FP
1	1	1
2	4	12
3	3	12
4	2	12
5	11	60
6	89	60
7	1069	60
8	2137	60
9	4273	60
10	59833	60

Table 2. First n that takes k iterations to reach a fixed point.

Lemma 3.1. *Let $z^i(5^a \cdot n) = c_{(i,a,n)}5^{a_i}$, where $c_{(i,a,n)}$ is a constant that is relatively prime to 5 and depends on i and n, and $a_i \in \mathbb{Z}^+$. Fix $i, n \in \mathbb{Z}_{\geq 0}$. Then $c_{(i,a,n)}$ remains the same for all choices of a.*

Proof. Let the prime factorization of an integer n be $n = 5^{e_1}p_2^{e_2}\cdots p_m^{e_m}$ where $e_1 \geq 0$ and each of the $e_2, e_3, \ldots, e_m \geq 1$.

We proceed by induction on the number of iterations of z. First suppose $i = 1$ and let the prime factorization of $\operatorname{lcm}(z(p_2^{e_2}), \ldots, z(p_m^{e_m}))$ equal $5^{f_1}q_2^{f_2}\cdots q_r^{f_r}$ where $f_1 \geq 0$ and each of the $f_2, \ldots, f_r \geq 1$. Observe that

$$\begin{aligned} z(5^a \cdot n) &= \operatorname{lcm}\left(z(5^{e_1+a}), z(p_2^{e_2}), \ldots, z(p_m^{e_m})\right) \\ &= \operatorname{lcm}\left(z\left(5^{e_1+a}\right), \operatorname{lcm}\left(z(p_2^{e_2}), \ldots, z(p_m^{e_m})\right)\right) \\ &= \operatorname{lcm}\left(5^{e_1+a}, 5^{f_1}q_2^{f_2}\cdots q_r^{f_r}\right) \\ &= q_2^{f_2}\cdots q_r^{f_r} \cdot 5^{\max(e_1+a, f_1)}. \end{aligned} \tag{3.1}$$

Thus, $c_{(1,a,n)} = q_2^{f_2}\cdots q_r^{f_r}$ for any non-negative integer a when n is not a power of 5 or when $\operatorname{lcm}(z(p_2^{e_2}), \ldots, z(p_m^{e_m}))$ is not a power of 5, and $c_{(1,a,n)} = 1$ otherwise.

Next, assume that for some $i, n \in \mathbb{Z}^+$, we have $z^i(5^a \cdot n) = c_{(i,a,n)}5^{a_i}$ where $c_{(i,a,n)}$ is the same for all choices of $a \in \mathbb{Z}_{\geq 0}$. First suppose $c_{(i,a,n)} = 1$. Then

$$\begin{aligned} z^{i+1}(5^a \cdot n) &= z(z^i(5^a \cdot n)) \\ &= z\left((c_{(i,a,n)}) \cdot 5^{a_i}\right) \qquad \text{where } a_i \in \mathbb{Z}_{\geq 0} \\ &= \operatorname{lcm}\left(z(1), z(5^{a_i})\right) \\ &= 5^{a_i}. \end{aligned} \tag{3.2}$$

Therefore, for any choice of a, $c_{(i+1,a,n)} = 1$.

Now suppose $c_{(i,a,n)} \neq 1$. Then let the prime factorization of $c_{(i,a,n)} = q_1^{f_1} \cdots q_r^{f_r}$, where $q_1, \dots, q_r \neq 5$ since $\gcd\left(c_{(i,a,n)}, 5\right) = 1$. Let $\operatorname{lcm}\left(z(q_1^{f_1}), \dots, z(q_r^{f_r})\right) = 5^{g_1}h_2^{g_2} \cdots h_j^{g_j}$ where $h_2, \dots, h_j$ are primes not equal to 5. Then

$$\begin{aligned} z^{i+1}(5^a \cdot n) &= z(z^i(5^a \cdot n)) \\ &= z\left((c_{(i,a,n)}) \cdot 5^{a_i}\right) \qquad \text{where } a_i \in \mathbb{Z}_{\geq 0} \\ &= \operatorname{lcm}\left(z(q_1^{f_1}), \dots, z(q_r^{f_r}), z(5^{a_i})\right) \\ &= \operatorname{lcm}\left(\operatorname{lcm}\left(z(q_1^{f_1}), \dots, z(q_r^{f_r})\right), z(5^{a_i})\right) \\ &= \operatorname{lcm}\left(5^{g_1}h_2^{g_2} \cdots h_j^{g_j}, 5^{a_i}\right) \\ &= h_2^{g_2} \cdots h_j^{g_j} \cdot 5^{\max(g_1,t)} && (3.3) \\ &= c\,(i,a,n) \cdot 5^{\max(g,t)}. && (3.4) \end{aligned}$$

□

We use Lemma 3.1 in our proof of Theorem 1.3 to show that if there exists an integer n that takes exactly k iterations of z to reach a fixed point, then there are infinitely many integers that take exactly k iterations of z to reach a fixed point. The following lemma provides us with information on the k^{th} iteration of z on powers of 10, enabling us to find integers that require exactly k iterations of z to reach a fixed point for any positve integer k.

Lemma 3.2. *For all $k, m \in \mathbb{Z}, k \geq 0, m \geq 4$ and $2k + 2 \leq m$, $z^k(10^m) = 3 \cdot 5^m \cdot 2^{m-2k}$.*

Proof. We proceed by induction on the number of iterations of z. Observe that when $k = 1$,

$$\begin{aligned} z\left(10^m\right) &= \operatorname{lcm}\left(z\left(2^m\right), z\left(5^m\right)\right) \\ &= \operatorname{lcm}\left(3 \cdot 2^{m-2}, 5^m\right) \\ &= 3 \cdot 5^m \cdot 2^{m-2}. \end{aligned} \tag{3.5}$$

Now suppose that $z^k(10^m) = 3 \cdot 5^m \cdot 2^{m-2k}$ for some positive integer k. Then we have

$$\begin{aligned} z^{k+1}(10^m) &= z\left(z^k\left(10^m\right)\right) \\ &= z\left(3 \cdot 5^m \cdot 2^{m-2k}\right) \\ &= \operatorname{lcm}\left(z\left(3\right), z\left(5^m\right), z(2^{m-2k})\right) \\ &= \operatorname{lcm}\left(4, 5^m, 2^{m-2k-2} \cdot 3\right). \end{aligned} \tag{3.6}$$

By assumption, $m \geq 2(k+1) + 2 = 2k + 4$, thus we have $z^{k+1}(10^m) = 3 \cdot 5^m \cdot 2^{m-2(k+1)}$. □

Using Lemmas 3.1 and 3.2, we now prove Theorem 1.3:

For all $k \in \mathbb{Z}_{\geq 0}$, there exist infinitely many n with fixed point order k.

Proof of Theorem 1.3. Let $g, h \in \mathbb{Z}^+, g > h$. Then $g = h + \ell$ for some $\ell \in \mathbb{Z}^+$. Suppose that $z^h(n)$ is a fixed point. Then $z^g(n) = z^\ell\left(z^h(n)\right)$, so $z^g(n)$ is also a fixed point. Similarly, if $z^g(n)$ is not a fixed point, then $z^h(n)$ cannot be a fixed point for any $h < g$.

Note that by Lemma 3.2

$$z^k(10^{2k+2}) \;=\; 3 \cdot 5^{2k+2} \cdot 2^{(2k+2)-2k} \;=\; 12 \cdot 5^{2k+2} \tag{3.7}$$

and

$$z^{k-1}(10^{2k+2}) \;=\; 3 \cdot 5^{2k+2} \cdot 2^{(2k+2)-2(k-1)} \;=\; 12 \cdot 5^{2k+2} \cdot 2^2. \tag{3.8}$$

Thus, 10^{2k+2} takes exactly k iterations of z to reach a fixed point, as $z^{k-1}(10^{2k+2})$ is not a fixed point. We prove that we can find infinitely many integers that take exactly k iterations to reach a fixed point once one such integer is identified (which we have just done).

We first consider the case where an integer n goes to a fixed point of the form $12 \cdot 5^{a'}$, where $a' \in \mathbb{Z}_{\geq 0}$, in exactly k iterations of z. Thus, $z^k(n) = 12 \cdot 5^{a'}$ and $z^{k-1}(n) = c \cdot 5^{b'}$ for some non-negative integer b' and positive integer $c \neq 1, 12$. Let r be an arbitrary positive integer. By Lemma 3.1, we have $z^k\left(5^r \cdot n\right) = 12 \cdot 5^{a''}$ and $z^{k-1}\left(5^r \cdot n\right) = c \cdot 5^{b''}$ for non-negative integers a'' and b''. Thus, $5^r \cdot n$ requires exactly k iterations to reach a fixed point. Next we consider the case where n goes to a fixed point of the form $5^{a'}$ in exactly k steps. Then, $z^k(n) = 5^{a'}$ and $z^{k-1}(n) = c \cdot 5^{b'}$ for some non-negative integer b' and positive integer $c \neq 1, 12$. Let r be an arbitrary positive integer. By Lemma 3.1, we have $z^k\left(5^r \cdot n\right) = 5^{a''}$ and $z^{k-1}\left(5^r \cdot n\right) = c \cdot 5^{b''}$ for non-negative integers a'' and b''. Thus, $5^r \cdot n$ requires exactly k iterations to reach a fixed point. As r is arbitrary, there are infinitely many integers with fixed point order k for any positive integer k. □

4. Infinitely many integers go to each fixed point

We begin this section with a proof about the k^{th} iteration of z on powers of 2.

Lemma 4.1. *For all $k, a \in \mathbb{Z}$ such that $2 \leq k$ and $4 \leq a$, $z^k(2^a) = \operatorname{lcm}\left(2^{a-2k} \cdot 3, 4\right)$.*

Proof. We induct on k; we use Lemma 2.2 to note that $z(2^a) = z^{a-2} \cdot 3$ (valid as $a \geq 3$) with base case $k = 2$:

$$\begin{aligned} z^2(2^a) &= z\left(z(2^a)\right) \\ &= z(2^{a-2} \cdot 3) \\ &= \operatorname{lcm}\left(z(2^{a-2}), z(3)\right) \\ &= \operatorname{lcm}\left(2^{a-4} \cdot 3, 4\right). \end{aligned} \tag{4.1}$$

For the inductive step, assume that $z^k(2^a) = \text{lcm}\left(2^{a-2k} \cdot 3, 4\right)$ for some k. We show that $z^{k+1}(2^a) = \text{lcm}\left(2^{a-2(k+1)} \cdot 3, 4\right)$. First suppose that $a > 2k+2$. Then

$$\begin{aligned} z^{k+1}(2^a) &= z\left(z^k(2^a)\right) \\ &= z\left(\text{lcm}\left(2^{a-2k} \cdot 3, 4\right)\right) \\ &= z\left(2^{a-2k} \cdot 3\right) \\ &= \text{lcm}\left(z(2^{a-2k}), z(3)\right) \\ &= \text{lcm}\left(2^{a-2k-2} \cdot 3, 4\right) \\ &= \text{lcm}\left(2^{a-2(k+1)} \cdot 3, 4\right). \end{aligned} \tag{4.2}$$

Now suppose that $a \leq 2k+2$. Then

$$\begin{aligned} z^{k+1}(2^a) &= z\left(z^k\left(2^a\right)\right) \\ &= z\left(\text{lcm}\left(2^{a-2k} \cdot 3, 4\right)\right) \\ &= z\left(12\right) \\ &= 12 \\ &= \text{lcm}\left(2^{a-2(k+1)} \cdot 3, 4\right). \end{aligned} \tag{4.3}$$

□

We now use Lemma 4.1 in our proof of Lemma 4.2, which proves that all powers of 2 go to the fixed point 12 and determines how many iterations of z it takes for a power of 2 to reach 12.

Lemma 4.2. *For all $a \in \mathbb{Z}^+$, 2^a reaches the fixed point 12 in finitely many iterations of z. For $a \geq 4$, exactly $\lceil \frac{a}{2} \rceil - 1$ iterations of z are required to reach 12.*

Proof. When $a \leq 4$, the claim follows from straightforward computation. Notice that $z^4(2) = 12, z^2(2^2) = 12, z^2(2^3) = 12, z(2^4) = 12$. We prove for $a > 4$ using Lemma 4.1.

Note that if a is even, then $\lceil \frac{a}{2} \rceil = \frac{a}{2}$. Thus, in the case where a is even,

$$z^{\lceil \frac{a}{2} \rceil - 1}(2^a) \;=\; \text{lcm}(2^{a-2(\frac{a}{2}-1)} \cdot 3, 4) \;=\; \text{lcm}(2^{a-a+2} \cdot 3, 4) \;=\; \text{lcm}(2^2 \cdot 3, 4) \;=\; 12. \tag{4.4}$$

So 2^a takes at most $\lceil \frac{a}{2} \rceil - 1$ iterations of z to reach a fixed point when a is even. We next show that it takes exactly $\lceil \frac{a}{2} \rceil - 1$ by showing that $z^{(\lceil \frac{a}{2} \rceil - 1) - 1}(2^a)$ is not a fixed point:

$$z^{(\lceil \frac{a}{2} \rceil - 1) - 1}(2^a) = \text{lcm}(2^{a-2(\frac{a}{2}-2)} \cdot 3, 4) \;=\; \text{lcm}(2^{a-a+4} \cdot 3, 4) = \text{lcm}(2^4 \cdot 3, 4) = 12 \cdot 2^2, \tag{4.5}$$

which is not a fixed point. When a is odd, $\lceil \frac{a}{2} \rceil - 1 = \frac{a-1}{2}$, giving us

$$z^{\lceil \frac{a}{2} \rceil - 1}(2^a) \;=\; \text{lcm}(2^{a-2(\frac{a-1}{2})} \cdot 3, 4) \;=\; \text{lcm}(2^{a-a+1} \cdot 3, 4) \;=\; \text{lcm}(2 \cdot 3, 4) \;=\; 12. \tag{4.6}$$

However

$$z^{(\lceil \frac{a}{2} \rceil - 1) - 1}(2^a) \;=\; \text{lcm}(2^{a-2(\frac{a-1}{2}-1)} \cdot 3, 4) \;=\; \text{lcm}(2^{a-a+1+2} \cdot 3, 4) \;=\; \text{lcm}(2^3 \cdot 3, 4) \;=\; 12 \cdot 2, \tag{4.7}$$

which is not a fixed point. □

Lemma 4.2 and Lemma 3.1 now yield Theorem 1.4:

Infinitely many integers n go to each fixed point of the form $12 \cdot 5^k$.

Proof of Theorem 1.4. Using Lemma 4.2, we know that $z^{\lceil \frac{a}{2} \rceil -1}(2^a \cdot 5^0) = 12$. Thus, by Lemma 3.1, $z^{\lceil \frac{a}{2} \rceil -1}(2^a \cdot 5^b) = 12 \cdot 5^{b'}$ for some nonnegative integer b'. We show that $b = b'$ by inducting on t to show that $z^t(2^a \cdot 5^b) = 2^{a'} \cdot 3 \cdot 5^b, a' \in \mathbb{Z}^+$, for all $a > t, a > 2$. When $t = 1$,

$$\begin{aligned} z(2^a \cdot 5^b) &= \operatorname{lcm}\left(z(2^a), z(5^b)\right) \\ &= \operatorname{lcm}\left(2^{a-2} \cdot 3, 5^b\right) \\ &= 2^{a-2} \cdot 3 \cdot 5^b. \end{aligned} \tag{4.8}$$

Now suppose that $z^t(2^a \cdot 5^b) = 2^{a'} \cdot 3 \cdot 5^b$ for some positive integer a'. Then

$$\begin{aligned} z^{t+1}(2^a \cdot 5^b) &= z\left(z^t(2^a \cdot 5^b)\right) \\ &= z\left(2^{a'} \cdot 3 \cdot 5^b\right) \\ &= \operatorname{lcm}\left(z(2^{a'}), z(3), z(5^b)\right). \end{aligned} \tag{4.9}$$

If $a' \leq 3$, then $\operatorname{lcm}\left(z(2^{a'}), z(3), z(5^b)\right) = 2^2 \cdot 3 \cdot 5^b$. If $a' > 3$, then

$$\begin{aligned} \operatorname{lcm}\left(z(2^{a'}), z(3), z(5^b)\right) &= \operatorname{lcm}\left(2^{a'-2} \cdot 3, 4, 5^b\right) \\ &= 2^{a'-2} \cdot 3 \cdot 5^b. \end{aligned} \tag{4.10}$$

A straightforward calculation shows that $2 \cdot 5^b$ and $2^2 \cdot 5^b$ iterate to the fixed point $12 \cdot 5^b$ (see Appendix 1 for a proof). Therefore $2^a \cdot 5^b$ iterates to the fixed point $12 \cdot 5^b$ for all $a \in \mathbb{Z}^+$. □

5. All integers have finite fixed point order

We now prove that when a, b are relatively prime, $z^k(ab) = \operatorname{lcm}(z^k(a), z^k(b))$. We will use this in the proof of Theorem 1.5.

Lemma 5.1. *Let $n = ab$ where $\gcd(a, b) = 1$. Then $z^k(n) = \operatorname{lcm}(z^k(a), z^k(b))$.*

Proof. We first consider the case where n has only one prime in its prime factorization. Without loss of generality, suppose $a = 1$ and $b = n$ and $z^k(n) = \operatorname{lcm}(1, z^k(n))$. If $n = 1$, then $a = b = 1$ and $z^k(1) = 1 = \operatorname{lcm}\left(z^k(1), z^k(1)\right)$.

Next consider when n has at least two distinct primes in its prime factorization. Let the prime factorization of $n = p_1^{e_1} \cdots p_m^{e_m}$ and let $a = p_1^{e_1} \cdots p_r^{e_r}$, $b = p_{r+1}^{e_{r+1}} \cdots p_m^{e_m}$ where $1 \leq r < m$. Note that the primes are not necessarily in increasing order. We proceed by induction. In the base case $k = 1$, and using Lemma 2.3 we have:

$$\begin{aligned} z(n) &= \operatorname{lcm}\left(z\left(p_1^{e_1}\right), \ldots, z(p_r^{e_r}), z(p_{r+1}^{e_{r+1}}), \ldots, z(p_m^{e_m})\right) \\ &= \operatorname{lcm}\left(\operatorname{lcm}\left(z(p_1^{e_1}), \ldots, z(p_r^{e_r})\right), \operatorname{lcm}\left(z\left(p_{r+1}^{e_{r+1}}\right), \ldots, z(p_m^{e_m})\right)\right) \\ &= \operatorname{lcm}\left(z(p_1^{e_1} \cdots p_r^{e_r}), z(p_{r+1}^{e_{r+1}} \cdots p_m^{e_m})\right) \\ &= \operatorname{lcm}\left(z(a), z(b)\right). \end{aligned} \tag{5.1}$$

For the inductive step, assume that for some $k \geq 1$, $z^k(n) = \text{lcm}\left(z^k(a), z^k(b)\right)$. We show that $z^{k+1}(n) = \text{lcm}\left(z^{k+1}(a), z^{k+1}(b)\right)$. We have

$$\begin{aligned} z^{k+1}(n) &= z\left(z^k(n)\right) \\ &= z\left(\text{lcm}(z^k(a), z^k(b))\right) \\ &= \text{lcm}\left(z(z^k(a)), z(z^k(b))\right) \qquad \text{by Lemma 2.4} \\ &= \text{lcm}\left(z^{k+1}(a), z^{k+1}(b)\right). \end{aligned} \tag{5.2}$$

□

Now we are ready to prove Theorem 1.5.

Proof of Theorem 1.5. Suppose that n is the smallest positive integer with undefined fixed point order. We prove first the case where $n = ab$ where $\gcd(a, b) = 1$ and $a, b \geq 2$, then prove cases where n is a power of a prime.

Case 1: First suppose that n has at least two distinct primes in its prime factorization, so n can be written $n = ab$ where $\gcd(a, b) = 1$ and $a, b > 1$. Since $a, b < n$, we know that a, b have finite fixed point order. Suppose the fixed point order of a is c and the fixed point order of b is d. Let $k = \max(c, d)$. Then by Lemma 5.1 we have $z^k(n) = \text{lcm}\left(z^k(a), z^k(b)\right)$ telling us $z^k(n)$ is a fixed point, which is a contradiction.

Case 2: Now suppose that $n = p$ for some prime p. Notice that $p \neq 2$ since we prove in Lemma 4.1 that powers of 2 reach the fixed point 12 in finitely many iterations of z. By Lemma 2.5, $z(p) \leq p + 1$. Note that $z(p) \neq p$, or else p would have fixed point order of 0. Thus, $z(p) = p + 1$ or $z(p) < p$

Since we are assuming p does not iterate to a fixed point, neither does $z(p)$. Thus $z(p)$ is not a power of 2 since powers of 2 iterate to a fixed point by Lemma 4.1. Thus if $z(p) = p + 1$ then $z(p) = 2^r \cdot t$ where $r \in \mathbb{Z}^+$ (since $p + 1$ is even) and $t \in \mathbb{Z}, t \geq 3$, $\gcd(2, t) = 1$ and $2^r, t < p$. So by the same argument as in Case 1, $z(p)$ has finite fixed point order, so p also has finite fixed point order since it reaches a fixed point after one more iteration of z than $z(p)$.

If $z(p) < p$ then $z(p)$ has finite fixed point order by the assumption that p is the smallest integer with undefined fixed point order. Thus p also has finite fixed point order.

Case 3: Now suppose $n = p^e$ where p is prime and $e \geq 2$. From Lemmas 2.9 and 2.6 we know that for some $r \in \mathbb{Z}_{\geq 0}, r < e$, we have $z(p^e) = p^r z(p)$. As $e > 1$, we have $z(p) \leq p + 1 < p^e$ by Lemma 2.5. Thus, $z(p)$ has finite fixed point order. Notice that $p^r < p^e$, so p^r also has finite fixed point order. Let $h = \max(\text{fixed point order of } z(p), \text{fixed point order of } p^r)$.

Note that $\gcd(z(p), p^r) = 1$ since $z(p)$ is relatively prime to p by Lemma 2.7. Then by Lemma 5.1,

$$z^{h+1}(p^e) \;=\; z^h\left(z\left(p^e\right)\right) \;=\; z^h\left(p^r z(p)\right) \;=\; \text{lcm}\left(z^h(p^r), z^h(z(p))\right). \tag{5.3}$$

Thus, p^e iterates to a fixed point within $h + 1$ iterations of z, so p^e has finite fixed point order. □

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Appendix

(1) We first prove that $2^2 \cdot 5^b$ iterates to the fixed point $12 \cdot 5^b$. Observe:

$$\begin{aligned} z^2\left(4 \cdot 5^b\right) &= z\left(z\left(4 \cdot 5^b\right)\right) \\ &= z\left(\operatorname{lcm}\left(z(4), z(5^b)\right)\right) \\ &= z\left(\operatorname{lcm}\left(6, 5^b\right)\right) \\ &= z\left(\operatorname{lcm}\left(6, 5^b\right)\right) \\ &= z\left(6 \cdot 5^b\right) \\ &= z\left(6 \cdot 5^b\right) \\ &= \operatorname{lcm}\left(z(2), z(3), z(5^b)\right) \\ &= \operatorname{lcm}\left(3, 4, 5^b\right) \\ &= 12 \cdot 5^b. \end{aligned} \tag{6.1}$$

(2) Next we prove that $2 \cdot 5^b$ iterates to the fixed point $12 \cdot 5^b$. Observe:

$$\begin{aligned} z^4(2 \cdot 5^b) &= z^3\left(z(2 \cdot 5^b\right) \\ &= z^3\left(\operatorname{lcm}\left(z(2), z(5^b)\right)\right) \\ &= z^3\left(\operatorname{lcm}\left(3, 5^b\right)\right) \\ &= z^3\left(3 \cdot 5^b\right) \\ &= z^2\left(z\left(3 \cdot 5^b\right)\right) \\ &= z^2\left(\operatorname{lcm}\left(z(3), z(5^b)\right)\right) \\ &= z^2\left(\operatorname{lcm}\left(4, 5^b\right)\right) \\ &= z^2\left(4 \cdot 5^b\right) \\ &= 12. \end{aligned} \tag{6.2}$$

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